PARTIAL DERIVATIVES

Notation and Terminology: given a function $f(x, y)$;

- partial derivative of $f$ with respect to $x$ is denoted by
  \[ \frac{\partial f}{\partial x}(x, y) \equiv f_x(x, y) \equiv D_x f(x, y) \equiv f_1; \]

- partial derivative of $f$ with respect to $y$ is denoted by
  \[ \frac{\partial f}{\partial y}(x, y) \equiv f_y(x, y) \equiv D_y f(x, y) \equiv f_2. \]

Definitions: given a function $f(x, y)$;

- definition for $f_x(x, y)$:
  \[ f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}; \]

- definition for $f_y(x, y)$:
  \[ f_y(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}. \]

Determination of $f_x$ and $f_y$:

- to find $f_x(x, y)$: keeping $y$ constant, take $x$ derivative;
- to find $f_y(x, y)$: keeping $x$ constant, take $y$ derivative.

Graphical Interpretation of $f_x$ and $f_y$:

- $f_x(a, b)$ is slope of tangent line in $x$ direction for the surface $z = f(x, y)$ at $f(a, b)$;
- $f_y(a, b)$ is slope of tangent line in $y$ direction for the surface $z = f(x, y)$ at $f(a, b)$. 
Applications to Implicit Differentiation

Functions of \( n > 2 \) Variables: given \( f(x) = f(x_1, \ldots, x_n) \)

- notation and terminology: the 
  partial derivative of \( f \) with respect to \( x_i \) is denoted by 
  \[
  \frac{\partial f}{\partial x_i}(x) \equiv f_{x_i}(x) \equiv D_{x_i}f(x) \equiv f_i(x);
  \]
  
  definition:
  \[
  f_{x_i}(x) = \lim_{h \to 0} \frac{f(x_1, \ldots, x_{i-1}, x_i + h, x_{i+1}, \ldots, x_n) - f(x)}{h}.
  \]

Higher Derivatives: given \( f(x, y) \) defined on a domain \( D \); 

- notation: second partials are denoted by
  \[
  \frac{\partial^2 f}{\partial x^2}(x, y) \equiv \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) \equiv (f_x)_x \equiv f_{11};
  \]
  \[
  \frac{\partial^2 f}{\partial y \partial x}(x, y) \equiv \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) \equiv f_{xy} \equiv (f_x)_y \equiv f_{12};
  \]
  \[
  \frac{\partial^2 f}{\partial x \partial y}(x, y) \equiv \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) \equiv f_{yx} \equiv (f_y)_x \equiv f_{21};
  \]
  \[
  \frac{\partial^2 f}{\partial y^2}(x, y) \equiv \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right) \equiv (f_y)_y \equiv f_{22}.
  \]

- similar notation for functions with \( > 2 \) variables.

- Clairaut’s Theorem: if \( (a, b) \in D \), and 
  \( f_{xy} \) and \( f_{yx} \) are both continuous on \( D \), then
  \[
  f_{xy}(a, b) = f_{yx}(a, b).
  \]

- Applications to Partial Differential Equations