DIRECTIONS: Unsimplified numerical answers are okay. Show work. 10 points each part, 70 total. We’ll scale it up to 100. Extra space on back page. Some formulas:

\[
\sin^2(\theta) = \frac{1}{2}(1 - \cos(2\theta)), \quad \cos^2(\theta) = \frac{1}{2}(1 + \cos(2\theta))
\]

Polar/cylindrical coordinates:

- \(x = r \cos(\theta)\)
- \(y = r \sin(\theta)\)
- \(r^2 = x^2 + y^2\)
- \(\tan(\theta) = y/x\)
- \(dA = r \, dr \, d\theta\)
- \(dV = r \, dr \, d\theta \, dz\)

Spherical coordinates:

- \(r = \rho \sin(\phi)\)
- \(x = \rho \sin(\phi) \cos(\theta)\)
- \(y = \rho \sin(\phi) \sin(\theta)\)
- \(z = \rho \cos(\phi)\)
- \(\rho^2 = x^2 + y^2 + z^2\)
- \(dV = \rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta\)

Gradient:

- \(\nabla f(x, y) = (f_x, f_y)\)
- \(\nabla f(x, y, z) = (f_x, f_y, f_z)\)

Directional Derivative:

- \(D_u f(a, b) = \nabla f(a, b) \cdot \hat{u}\)

2nd Derivative Test:

- \(D(x, y) = (f_{xx})(f_{yy}) - (f_{xy})^2\)

1. Evaluate \(\int_0^1 \int_{2\pi}^0 x \sqrt{1 + y^3} \, dy \, dx\) by first changing the order of integration.

\[
\int_0^1 \int_{2\pi}^0 x \sqrt{1 + y^3} \, dy \, dx = \int_0^1 \int_0^{2\pi} \sqrt{1 + y^3} \, dx \, dy = \int_0^1 \left( \frac{x^2}{2} \right) \left| _0^2 \right. \, dy = \frac{1}{8} \int_0^1 (1 + y^3) - 1 \, dy = \frac{1}{3} \frac{2}{9} = \frac{13}{18}
\]

2. Evaluate \(\int_C \vec{F} \cdot d\vec{r}\), where \(\vec{F}(x, y, z) = (y, x + e^y)\) and \(C\) is given by \(\vec{r}(t) = (t^3, t^2), \quad 0 \leq t \leq 1\).

\[
\int_C \vec{F} \cdot d\vec{r} = \int_0^1 (t^6 + t^4 + e^t) \cdot (3t^2, 2t) \, dt = \int_0^1 3t^8 + 2t^6 + 2te^t \, dt = \int_0^1 (t^8 + 2t^6) \, dt
\]

\[
= (t^9 + 2t^7) \bigg| _0^1 = 1 + 2 - 1 = 2
\]
3. The function \( f(x, y) = (x - 3)(x^2 - y^2) \), which can also be written as \( f(x, y) = x^3 - xy^2 - 3x^2 + 3y^2 \), has 4 critical points. Two of them are \((0, 0)\) and \((2, 0)\).

   a) Use the Second Derivative Test to classify each of the two given critical points, \((0, 0)\) and \((2, 0)\), as a local maximum, local minimum, or saddle.

   \[
   f_x = 3x^2 - y^2 - 6x \quad f_y = -2x + 6y \quad f_{yy} = -2y
   \]

   \[
   f_{xx} = 6x - 6 \quad f_{xy} = -2x + 6
   \]

   At \((0, 0)\): \( f_{xx}(0, 0) = -6, f_{yy}(0, 0) = 6, f_{xy}(0, 0) = 0 \) so \( D = f_{xx}(0, 0)f_{yy}(0, 0) - f_{xy}(0, 0)^2 = -36 \leq 0 \) saddle

   At \((2, 0)\): \( f_{xx}(2, 0) = 0, f_{yy}(2, 0) = -2, f_{xy}(2, 0) = 0 \) so \( D = f_{xx}(2, 0)f_{yy}(2, 0) - f_{xy}(2, 0)^2 = 0 \) minimum

   b) Find the other two critical points. Do not classify them (they’re saddles). Hint: one of your equations should be \(-2xy + 6y = 0\), which is equivalent to \(-2y(x - 3) = 0\), which means that either \(y = 0\) or \(x = 3\).

   \[
   \begin{align*}
   3x^2 - y^2 - 6x & = 0 \\
   -2xy + 6y & = 0 \\
   \text{rewrite 2nd eq'n:} \\
   -y(6 - 2x) & = 0 \\
   y = 0 & \text{ or } x = 3 \\
   \end{align*}
   \]

   \[
   \begin{align*}
   \text{If } y = 0, & \text{ plug into 1st eq'n and get } 3x^2 - 6x = 0. \text{ This is } 3x(x - 2) = 0 \text{ so } x = 0, x = 2. \\
   \text{get crit. pts } (0, 0), (2, 0) & - \text{ already had these} \\
   \text{If } x = 3, & \text{ plug into 1st eq'n and get } 77 - y^2 = 18 = 0. \text{ so } y^2 = 9, y = \pm 3. \\
   \text{get crit. pts } (3, -3), (3, 3) & \\
   \end{align*}
   \]

4. Let \( f(x, y, z) = x^2 + \ln(3z + y^2) \) and let \( \mathbf{v} = (-1, 1, 4) \). Find the directional derivative of \( f \) at the point \((3, 2, 1)\) in the direction of \( \mathbf{v} \). That is, compute \( D_uf(3, 2, 1) \), where \( \mathbf{u} \) is a unit vector in the same direction as \( \mathbf{v} \).

   \[
   f_x = 2x \\
   f_y = \frac{2y}{3z + y^2} \\
   f_z = \frac{1}{3z + y^2}
   \]

   \[
   f_x(3, 2, 1) = 6 \\
   f_y(3, 2, 1) = \frac{4}{3 + 4} = \frac{4}{7} \\
   f_z(3, 2, 1) = \frac{3}{3 + 4} = \frac{3}{7}
   \]

   \[
   \mathbf{u} = \frac{(-1, 1, 4)}{\sqrt{(-1)^2 + 1^2 + 4^2}} = \left(-\frac{1}{\sqrt{18}}, \frac{1}{\sqrt{18}}, \frac{4}{\sqrt{18}}\right) = \left(-\frac{1}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}, \frac{4}{3\sqrt{2}}\right)
   \]

   \[
   \begin{align*}
   \text{So } D_uf(3, 2, 1) & = \left\langle 6, \frac{4}{7}, \frac{3}{7} \right\rangle \cdot \left\langle -\frac{1}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}, \frac{4}{3\sqrt{2}}\right\rangle \\
   & = -\frac{6}{3\sqrt{2}} + \frac{4}{7} \cdot \frac{1}{3\sqrt{2}} + \frac{3}{7} \cdot \frac{4}{3\sqrt{2}} \left(\frac{-\frac{9}{7} + \frac{16}{7}}{3\sqrt{2}}\right) = -\frac{24}{9} \cdot \frac{1}{3\sqrt{2}}
   \end{align*}
   \]
5. Evaluate \( \iiint x^2 \, dV \), where \( R = \{(x, y, z) | y \geq 0, 1 \leq x^2 + y^2 \leq 4, 1 \leq z \leq 3\} \).

\[
\begin{align*}
\iiint x^2 \, dV &= \iint_1^3 r^2 \cos^2 \theta \, dz \, r \, dr \, d\theta \\
&= \int_0^\pi \int_1^3 r^3 \cos^2 \theta \, dz \, d\theta \\
&= 2 \int_0^\pi \cos^2 \theta \left( r^4 \right) \, d\theta \\
&= 2 \int_0^\pi \cos^2 \theta \left( \frac{1}{4} \right) \, d\theta \\
&= \frac{1}{2} \left( \frac{1}{4} \right) \int_0^\pi \cos 2\theta \, d\theta \\
&= \frac{1}{4} \left( \theta + \frac{\sin 2\theta}{2} \right) \bigg|_0^\pi \\
&= \frac{1}{4} \pi 
\end{align*}
\]

6. Let \( E \) be the solid region which lies above the plane \( z = 0 \), below the cone \( z = \sqrt{x^2 + y^2} \), and inside the sphere \( x^2 + y^2 + z^2 = 9 \), as shown. Find the volume of \( E \).

Use spherical coordinates.

\[
\begin{align*}
\iiint V &= \iiint_0^\pi \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\
&= \int_0^\pi \int_0^\frac{\pi}{2} \int_0^3 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\
&= \int_0^\pi \int_0^\frac{\pi}{2} \left[ \frac{\rho^3}{3} \right]_0^3 \sin \phi \, d\phi \, d\theta \\
&= \int_0^\pi \int_0^\frac{\pi}{2} 9 \sin \phi \, d\phi \, d\theta \\
&= 9 \int_0^\pi \left( -\cos \phi \right) \bigg|_0^\frac{\pi}{2} \, d\theta \\
&= 9 \int_0^\pi \left( 1 - \frac{1}{\sqrt{2}} \right) \, d\theta \\
&= \frac{9}{\sqrt{2}} \int_0^\pi 1 \, d\theta = \frac{18 \pi}{\sqrt{2}}
\end{align*}
\]