1. (14) Find the equation for the plane tangent to the surface \( z = x^2 + y^2 - 1 \) at the point \( P_0(1, 2, 4) \). Also find the equation to the line normal to the given surface at \( P_0 \).

The surface can be written as \( f(x, y, z) = x^2 + y^2 - z = 1 \).
\[ \nabla f = 2x \hat{i} + 2y \hat{j} - \hat{k} \] \[ \left. \nabla f \right|_{P_0} = 2 \hat{i} + 4 \hat{j} - \hat{k} \]. Hence, the tangent plane is given by \( 2(x-1) + 4(y-2) - 1(3-4) = 0 \), i.e., \( 2x + 4y - z = 6 \).

The normal line is \( x = 1 + 2t, y = 2 + 4t, z = 4 - t, -\infty < t < \infty \).

2. (12) The area of the ellipse \( (x/a)^2 + (y/b)^2 = 1 \) is given by \( A = \pi ab \). If \( a = 10 \text{ cm} \) and \( b = 5 \text{ cm} \) as measured to the nearest millimeter, what is the percentage error in the calculated area?

\[ A = \pi ab \]
\[ \Rightarrow dA = \pi (b \, da + a \, db) \]
\[ \Rightarrow \frac{dA}{A} = \frac{\pi (b \, da + a \, db)}{A} = \frac{\pi (b \, da + a \, db)}{\pi ab} = \frac{da}{a} + \frac{db}{b} \]
\[ \Rightarrow 100 \frac{dA}{A} = 100 \left( \frac{da}{a} \right) + 100 \left( \frac{db}{b} \right) = 100 \left( \frac{1}{100} \right) + 100 \left( \frac{1}{50} \right) = 1 + 2 = 3 \% \].
3. (12) Find the parametric equation for the line tangent to the curve of intersection of the two surfaces \(2x^2 + y^2 + 3z = 6\) and \(x = 1\) at \(P_0(1,1,1)\).

\[
\begin{align*}
\nabla f &= 4x\hat{i} + 2y\hat{j} + 3\hat{k} \\
\nabla g &= \hat{i} \\
\n\nabla f \times \nabla g &= 
\begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
4 & 2 & 3 \\
1 & 0 & 0 \\
\end{vmatrix} \\
&= 3\hat{j} - 2\hat{k}
\end{align*}
\]

\(\Rightarrow\) Tangent line is \(x = 1,\ y = 1 + 3t,\ z = 1 - 2t,\ -\infty < t < \infty.\)

4. (14) Find all local minima, local maxima, and saddle points of the function given below. You should evaluate the function at each critical point.

\[f(x, y) = x^3 + 3xy + y^3.\]

The domain is all pairs of real numbers.

\[
\begin{align*}
f_x &= 3x^2 + 3y = 0 \quad (1) \\
f_y &= 3x + 3y^2 = 0 \quad (2) \\
\Rightarrow x &= -y^2. \quad \therefore (1) \Rightarrow \\
(-y^2)^2 + y &= 0, \text{ i.e., } y^4 + y^2 &= 0 \\
\Rightarrow y(y^3 + 1) &= 0 \Rightarrow y = 0, -1. \\
\text{giving } \quad x &= 0, -1.
\end{align*}
\]

The critical points are \((0,0)\) and \((-1,-1)\).

\[
\begin{align*}
f_{xx} &= 6x \\
f_{xy} &= 3y \\
f_{yy} &= 6y
\end{align*}
\]

\[
\begin{align*}
(0,0) \\
H &= f_{xx}f_{yy} - f_{xy}^2 = 36y^2 - 9 \\
H &= -9 < 0 \Rightarrow \text{saddle point} \\
f(0,0) &= 0. \text{ Hence} \\
(0,0,0) &\text{ is a saddle point of the surface.}
\end{align*}
\]

\[
\begin{align*}
(-1,-1) \\
H &= 36(-1)(-1) - 9 = 27 > 0 \\
f_{xx} &= 6(-1) = -6 < 0 \Rightarrow \text{local maximum} \\
f(-1,-1) &= (-1)^3 + 3(-1)(-1) + (-1)^3 = 1. \\
\text{Hence } (-1,-1,1) \text{ is a local maximum of the surface.}
\end{align*}
\]
5. (16) Find the absolute maximum and minimum values of \( f(x, y) = 4xy - 3x^3 - 2y^2 \) on the region \( R \) that is the part of the \( x \)-axis connecting the points \((1, 0)\) and \((4, 0)\).

There are no interior critical points, as \( R \) is one-dimensional.

On \( AB \), \( y = 0 \) \( \Rightarrow f(x, 0) = -3x^3 \)
\( \Rightarrow f'(x, 0) = -9x^2 = 0 \) \( \Rightarrow x = 0 \). Hence \((0, 0)\) could be considered, but \((0, 0)\) is outside of \( R \). So we just consider the end points \( A(1, 0) \) and \( B(4, 0) \).

\( A: f(1, 0) = -3(1)^3 = -3 \) \( \leftarrow \) absolute maximum

\( B: f(4, 0) = -3(4)^3 = -192 \) \( \leftarrow \) absolute minimum

\((1, 0, -3)\) is the absolute maximum and \((4, 0, -192)\) is the absolute minimum.

6. (12) Evaluate the double integral over the given region \( R \).

\[
I = \iint_R \frac{xy^3}{x^2 + 1} \, dA, \quad R: \ 0 \leq x \leq 1, \ 0 \leq y \leq 2.
\]

\[
I = \int_0^2 \int_0^1 \frac{xy^3}{x^2 + 1} \, dx \, dy
\]

\[
= \left[ \frac{1}{2} \ln(1 + 1) \right]_0^1 y^3 \, dy = \frac{1}{2} \left( \ln(1 + 1) - \ln(1 + 1) \right) y^3 \, dy
\]

\[
= 0
\]

\[
= \frac{1}{2} \left( \frac{1}{2} \frac{y^4}{2} \right)_0^2 = \frac{1}{2} \frac{1}{4} (2^4 - 0^4)
\]

\[
= \frac{\ln 2}{2} \cdot 16 = 2 \ln 2.
\]
7. (14) Sketch the region of integration, and write an equivalent integral with the order of integration reversed. Then evaluate this reverse ordered integral.

\[ I = \int_{0}^{3} \int_{\sqrt[x/3]}^{1} e^{y^3} dy \, dx. \]

I uses vertical cross sections.

y varies from \( \sqrt[x/3] \) to 1
x varies from 0 to 3.

\( y = \sqrt[x/3] \) gives \( x = 3y^2 \)

Reversing the order of integration, we get

\[
I = \int_{0}^{1} \int_{0}^{3y^2} e^{x^3} \, dx \, dy = \int_{0}^{1} \left[ e^{x^3} \right]_{0}^{3y^2} \, dy
\]

\[
= \int_{0}^{1} 3y^2 e^{3y^2} \, dy = e^{y^3} \bigg|_{0}^{1} = e - e^0 = e - 1.
\]

8. (6) Decide whether each of the following statements is True or False. Justify your answer.

(a) A saddle point of a function cannot be on the boundary of its domain.

\[ \text{FALSE. The definition for every disk centered at the point could hold on the boundary.} \]

(b) Reversing the order of integration of a double integral is equivalent to swapping x and y in the integral, i.e., replace every occurrence of x in the integral with y, and vice versa.

\[ \text{FALSE. The function, i.e., the integrand, could change. For instance, } \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} x \, dy \, dx \neq \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} x \, dx \, dy \]