Special cases of linear (in)dependence

1. \{\vec{u}\} is LI if \vec{u} \neq \vec{0}.

2. \{\vec{v}, \vec{z}\} is LI if one of them is not a scalar multiple of the other vector.
   
   If \vec{v} = c\vec{z} for scalar \(c\), then \(\vec{v} - c\vec{z} = \vec{0}\). So
   
   \(\vec{x} = [x_1, x_2] = [1, c]\) is a nontrivial solution to \(\vec{v}x_1 + \vec{z}x_2 = \vec{0}\),
   and hence the set of vectors is LD.

3. If \(\vec{0}\) is in the set \(\{\vec{v}_1, \ldots, \vec{v}_n\}\), the set is LD.
   
   E.g., let \(\vec{v}_2 = \vec{0}\). Then
   
   \(c\vec{v}_1 + \vec{0} + \cdots + \vec{0} = \vec{0}\) for any \(c \neq 0\).
   
   Hence \(x_1 = 0, x_2 = c, x_3 = x_4 = \cdots = x_n = 0\) is a nontrivial solution.

4. \{\vec{v}_1, \ldots, \vec{v}_n\}, where \(\vec{v}_j \in \mathbb{R}^m\), with \(n > m\) is LD.

   There are more vectors than the number of entries in each vector.
   
   E.g., \(\{[2], [0], [1]\}\) is LD.

   Notice that any two vectors out of the three are LI.
Consider \( A\vec{x} = \vec{0} \), where \( A \in \mathbb{R}^{m \times n} \) with \( A = [\vec{u}_1 \ \vec{u}_2 \ldots \ \vec{u}_n] \).

The maximum number of pivots possible is \( m \). So, there are \( n-m \) free variables.

**Prob 21, pg 61 T/F statements**

21. a. The columns of a matrix \( A \) are linearly independent if the equation \( A\vec{x} = \vec{0} \) has the trivial solution.

   b. If \( S \) is a linearly dependent set, then each vector is a linear combination of the other vectors in \( S \).

   c. The columns of any \( 4 \times 5 \) matrix are linearly dependent.

   d. If \( \vec{x} \) and \( \vec{y} \) are linearly independent, and if \( \{\vec{x}, \vec{y}, \vec{z}\} \) is linearly dependent, then \( \vec{z} \) is in \( \text{Span} \{\vec{x}, \vec{y}\} \).

**F.** The columns are LI if \( A\vec{x} = \vec{0} \) has only the trivial solution.

Recall that \( A\vec{x} = \vec{0} \) always has the trivial solution.

**F.** \( \{[2], [0]\} \) is LD, but \([1] \neq c[0] \) for any \( c \).

It is only required that one vector in \( S \) is a linear combination of the others, not each vector.

**T.** A set of \( n \) vectors each with \( m \) entries is LD if \( n > m \).

**T.** \( \exists \) can be written as a linear combination of \( \vec{x} \) and \( \vec{y} \).

\( \exists \vec{x}, \vec{y}, \vec{z} \) is LD \( \implies a\vec{x} + b\vec{y} + c\vec{z} = \vec{0} \) for \( a, b, c \in \mathbb{R} \), not all being zero. But \( c \neq 0 \), as \( \exists \vec{x}, \vec{y} \) is LI (so \( c = 0 \) would mean \( a = b = 0 \) as well). Hence

\[
\vec{z} = \left( \frac{a}{b} \right) \vec{x} + \left( \frac{b}{c} \right) \vec{y}, \text{ i.e., } \vec{z} \in \text{span}\{\vec{x}, \vec{y}\}.
\]
Linear Transformations (LT)

"mappings"

\[
A = \begin{bmatrix} 1 & -5 & -7 \\ -3 & 7 & 5 \end{bmatrix}, \quad \vec{u} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}
\]

We now talk about another context in which the matrix-vector product \(A\vec{x}\) shows up, which is somewhat different from the systems \(A\vec{x} = \vec{b}\) that we've been discussing so far.

A "acts on" \(\vec{u}\) to transform it to \([-2\, -2]\).

In general, A "acts on" \(x \in \mathbb{R}^3\) to give \(A\vec{x} \in \mathbb{R}^2\).

\[
A \vec{u} = \begin{bmatrix} 1 & -5 & -7 \\ -3 & 7 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \cdot 3 + -5 \cdot 1 + -7 \cdot 0 \\ -3 \cdot 3 + 7 \cdot 1 + 5 \cdot 0 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}
\]

\[
A \vec{v} = \begin{bmatrix} 1 & -5 & -7 \\ -3 & 7 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}
\]
Another notation is to write \( \bar{x} \mapsto A\bar{x} \). This correspondence is a function.

**Def** A transformation \( T \) from \( \mathbb{R}^n \) to \( \mathbb{R}^m \) is a rule that assigns every vector \( \bar{x} \) in \( \mathbb{R}^n \) a vector \( T(\bar{x}) \) in \( \mathbb{R}^m \).

\( T(\bar{x}) \) is the **image** of \( \bar{x} \) under \( T \).

The set of all images is the **range** of \( T \).

We are interested in **matrix transformations**, which are defined as follows.

\[
T : \mathbb{R}^n \rightarrow \mathbb{R}^m \text{ is defined as } T(\bar{x}) = A\bar{x} \text{ for } A \in \mathbb{R}^{m \times n}.
\]

\( \bar{x} \mapsto A\bar{x} \) is another notation.
Problem 2, pg 68

\[ A = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}, \quad \vec{u} = \begin{bmatrix} 3 \\ 6 \\ -9 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}. \]

Find images of \( \vec{u} \) and \( \vec{v} \) under \( T : \mathbb{R}^3 \to \mathbb{R}^3 \) given by

\[ T(\vec{x}) = A\vec{x}. \]

\[ T(\vec{u}) = A\vec{u} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ -9 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}. \]

\[ T(\vec{v}) = A\vec{v} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{1}{2} \begin{bmatrix} a \\ b \\ c \end{bmatrix}. \]

Problem 5, pg 68

\[ A = \begin{bmatrix} 1 & -5 & -7 \\ -3 & 7 & 5 \end{bmatrix}, \quad b = \begin{bmatrix} -2 \end{bmatrix}. \]

Find \( \vec{x} \) such that \( T(\vec{x}) = A\vec{x} = \vec{b}. \)

\[ 9 \text{ is } \vec{x} \text{ unique?} \]

We have seen how to solve the system \( A\vec{x} = \vec{b} \), and to decide if the system has a unique solution when it is consistent. The same results could be used to answer such questions about linear transformations.

This problem will be finished in the next lecture...