Solutions of $A\vec{x} = \vec{b}$ for nonzero $\vec{b}$ in terms of solutions to $A\vec{x} = \vec{0}$.

\[
\begin{align*}
    x_1 + 2x_2 - 3x_3 &= 3 \\
    2x_1 + x_2 - 3x_3 &= 3 \\
    -x_1 + x_2 &= 0
\end{align*}
\]

Here $\vec{b} = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}$.

Previously, we had $\vec{b} = \vec{0}$.

In Lecture 6, we solved the corresponding homogeneous system, and visualized its solutions in parametric vector form.

We now repeat the same EROs on just $\vec{b} = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}$.

\[
\begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} \xrightarrow{R_3 + R_2} \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} \xrightarrow{R_3 x (-1/3)} \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \xrightarrow{R_1 - 2R_2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}
\]

The reduced echelon form of $[A | \vec{b}]$ is hence

\[
\begin{bmatrix} 1 & 0 & -1 & | & 1 \\ 0 & 1 & -1 & | & 1 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}
\]

\[
\begin{align*}
    x_1 - x_3 &= 1 \\
    x_2 - x_3 &= 1
\end{align*}
\]

i.e., $x_1 = 1 + \delta$, $\delta \in \mathbb{R}$

\[
\begin{align*}
    x_2 &= 1 + \delta \\
    x_3 &= \text{free}
\end{align*}
\]

parametric form

\[
\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \delta, \ \delta \in \mathbb{R}
\]

\[
\overrightarrow{p} = \overrightarrow{v}
\]

parametric vector form
\( \overline{x} = \overline{v}s, s \in \mathbb{R} \) for \( \overline{v} = [1] \) is the parametric vector form of solutions to \( A\overline{x} = \overline{0} \).

\( \overline{x} = \overline{p} + \overline{v}s, s \in \mathbb{R} \) is the parametric vector form for solutions to \( A\overline{x} = \overline{b} \).

Equation for a line through \( \overline{p} \) parallel to \( \overline{v}s \).

Adding \( \overline{p} \) to \( \overline{v}s \) is equivalent to moving the vector \( \overline{v}s \) in a direction along the line through origin and \( \overline{p} \).

In fact, the above observation holds in the case of linear systems of equations in general, as long as the system in question is consistent.

**Theorem**  
If \( A\overline{x} = \overline{b} \) has a solution \( \overline{x} = \overline{p} \), then all solutions of \( A\overline{x} = \overline{b} \) are given by \( \overline{x} = \overline{p} + \overline{v}s \), where \( \overline{v}s \) is any solution of \( A\overline{x} = \overline{0} \).

Notice that the trivial solution corresponds to the choice \( s = 0 \) for the homogeneous system. For the same value of the parameter in the case of the non-homogeneous system, we get \( \overline{x} = \overline{p} \) as the solution. So, the origin gets translated to \( \overline{p} \).
Describe and compare the solution sets of
\[ x_1 - 2x_2 + 3x_3 = 0 \] and \[ x_1 - 2x_2 + 3x_3 = 4. \]

Basic: \( x_1 \), \( x_2 \)

Free: \( x_3 \)

\[ A = \begin{bmatrix} 1 & -2 & 3 \end{bmatrix} \]

\( x_1 = 2x_2 - 3x_3 \), \( x_2, x_3 \) free

\[ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}s + \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}t, \quad s, t \in \mathbb{R} \]

\[ [A | b] = \begin{bmatrix} 1 & -2 & 3 & 4 \end{bmatrix} \]

\( x_1 = 4 + 2x_2 - 3x_3 \)

\[ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}s + \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}t \]

Solutions to \( A\bar{x} = \bar{b} \) form a plane through \( \bar{b}, \bar{u}, \bar{v} \). And the solutions to \( A\bar{x} = \bar{b} \) form a parallel plane passing through \( \bar{p} \).
Linear Independence (Section 1.7)

Recall

If \( \overline{a}_1 = [3] \), \( \overline{a}_2 = [2] \), then

\[
\text{span} \{ \overline{a}_1, \overline{a}_2 \} = \mathbb{R}^2.
\]

But with \( \overline{u} = [6] \),

\[
\text{Span} \{ \overline{a}_1, \overline{u} \} \text{ is just the line through } \overline{0} \text{ and } \overline{a}_1.
\]

\( \overline{a}_1 \) and \( \overline{a}_2 \) are linearly independent here, i.e., they are not along the same line. While \( \overline{a}_1 \) and \( \overline{u} \) are linearly dependent.

We now extend this idea of being "along the same line" (or not) to arbitrary collections of vectors in high dimensions.

Def: The set \( \{ \overline{v}_1, \overline{v}_2, \ldots, \overline{v}_n \} \) with each \( \overline{v}_j \in \mathbb{R}^m \) is linearly independent (LI) if the vector equation

\[
\sum_{j=1}^{n} \overline{v}_j x_j = \overline{0}
\]

has only the trivial solution. If there is a non-trivial solution, the set of vectors is linearly dependent (LD).
Since we already know how to check if $A\vec{x} = \vec{0}$ has only the trivial solution (when there are no free variables), we can use those results to directly answer questions about whether a given set of vectors is LI or not.

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$$A = \begin{bmatrix} 0 & -3 & 9 \\ 2 & 1 & -7 \\ -1 & 4 & -5 \\ 1 & -4 & -2 \end{bmatrix}.$$ Do the columns of $A$ form a linearly independent set of vectors?

Equivalently, does $A\vec{x} = \vec{0}$ have only the trivial solution?

\[
\begin{bmatrix} 0 & -3 & 9 \\ 2 & 1 & -7 \\ -1 & 4 & -5 \\ 1 & -4 & -2 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_4} \begin{bmatrix} 1 & -4 & -2 \\ 2 & 1 & -7 \\ 0 & 4 & -5 \\ 0 & -3 & 9 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & -4 & -2 \\ 0 & 9 & -3 \\ 0 & 4 & -5 \\ 0 & -3 & 9 \end{bmatrix} \xrightarrow{R_3 + R_1} \begin{bmatrix} 1 & -4 & -2 \\ 0 & 9 & -3 \\ 0 & 0 & -7 \\ 0 & 0 & 0 \end{bmatrix}
\]

\[
\begin{bmatrix} 1 & -4 & -2 \\ 0 & 9 & -3 \\ 0 & 0 & -7 \end{bmatrix} \xrightarrow{R_4 + 3R_2} \begin{bmatrix} 1 & -4 & -2 \\ 0 & 9 & -3 \\ 0 & 0 & 24 \end{bmatrix} \xrightarrow{R_4 + \frac{24}{7}R_3} \begin{bmatrix} 1 & -4 & -2 \\ 0 & 9 & -3 \\ 0 & 0 & 0 \end{bmatrix}
\]

There are no free variables, and hence the system has only the trivial solution. So columns of $A$ are LI.
We now describe several special cases of sets of vectors, for which we can determine linear (in)dependence more directly than by performing EROs.

**Special Cases**

1. \{\vec{v}\} (Single vector).

   The set \{\vec{v}\} is LI if \(\vec{v} \neq \vec{0}\).

   To follow the definition, we are trying to find when does the system \(\vec{v}x = \vec{0}\) have only the trivial solution. Naturally, when \(\vec{v} \neq \vec{0}\), we can get the zero vector only by taking \(x = 0\).

We will discuss three more special cases in the next lecture...