Theorem

The following statements are equivalent for $A \in \mathbb{R}^{m \times n}$, $\bar{x} \in \mathbb{R}^n$, $\bar{b} \in \mathbb{R}^m$.

1. $A\bar{x} = \bar{b}$ has a solution for each $\bar{b} \in \mathbb{R}^m$.
2. $\bar{b}$ is in the span of the columns of $A$.
3. Columns of $A$ span $\mathbb{R}^m$.
4. Every row of $A$ has a pivot.

We use this condition to check readily whether $A\bar{x} = \bar{b}$ is consistent.

Prob 22 pg 41

$\bar{v}_1 = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$, $\bar{v}_2 = \begin{bmatrix} 0 \\ -9 \end{bmatrix}$, $\bar{v}_3 = \begin{bmatrix} -2 \\ -6 \end{bmatrix}$. Does $\mathbb{R}^3 \bar{v}_1, \bar{v}_2, \bar{v}_3$ span $\mathbb{R}^3$? Why?

Let $A = \begin{bmatrix} 0 & 0 & 4 \\ 0 & -3 & -2 \\ -3 & 9 & -6 \end{bmatrix}$ $\xrightarrow{R_2 \rightarrow R_1}$ $\begin{bmatrix} 0 & 0 & 4 \\ 0 & -3 & -2 \\ 0 & 9 & -6 \end{bmatrix}$ $\xrightarrow{R_3 \rightarrow \text{R}_2}$ $\begin{bmatrix} 0 & 0 & 4 \\ 0 & -3 & -2 \\ 0 & 0 & 0 \end{bmatrix}$

Every row of $A$ has a pivot. So $\text{Span} \{\bar{v}_1, \bar{v}_2, \bar{v}_3\} = \mathbb{R}^3$.

In more detail, since each row of $A$ has a pivot, $A\bar{x} = \bar{b}$ is consistent for every $\bar{b} \in \mathbb{R}^3$. Hence, every vector in $\mathbb{R}^3$ is in $\text{Span} \{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$.
Prob 30, pg 41

Construct a 3×3 matrix whose columns do not span \(\mathbb{R}^3\). Justify for such problems, it's best to create the "minimal", or simplest, example that works.

e.g., \[ A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \] works.

Similarly, \[ B = \begin{bmatrix} 3 & 5 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix} \] also works.

At least one row should not have a pivot.

Prob 26 pg 41

\[ \bar{u} = \begin{bmatrix} 7 \\ 2 \\ 5 \end{bmatrix}, \quad \bar{v} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \quad \bar{w} = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}, \] and they satisfy \( 2\bar{u} - 3\bar{v} - \bar{w} = \vec{0} \).

Find \( x_1, x_2 \) that satisfy \[ \begin{bmatrix} 7 & 3 \\ 2 & 1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}, \] without using EROs.

The given equation can be rewritten as \( 2\bar{u} + (-3)\bar{v} = \bar{w} \). Hence \( x_1 = 2, x_2 = -3 \) satisfies \( \bar{u}x_1 + \bar{v}x_2 = \bar{w} \), which is the given system.

How about finding a solution for \[ \begin{bmatrix} 5 & 7 \\ 1 & 2 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}, \] given the same relationship \( 2\bar{u} - 3\bar{v} - \bar{w} = \vec{0} \)?
The given system is \( \vec{w}x_1 + \vec{w}x_2 = \vec{0} \). To read off one solution, we rewrite \( 2\vec{w} - 3\vec{w} - \vec{w} = \vec{0} \) in this form, as follows:

\[
\left(-\vec{w} + 2\vec{w} = 3\vec{w}\right) \times \frac{1}{3}
\]

\[
\Rightarrow \quad \left(-\frac{1}{3}\vec{w} + \frac{2}{3}\vec{w} = \vec{w}\right)
\]

Hence \( x_1 = -\frac{1}{3}, \ x_2 = \frac{2}{3} \) is one solution.

We now study how to characterize when \( A\vec{x} = \vec{b} \) is consistent for all \( \vec{b} \). Naturally, we give this characterization in terms of \( A \).

To start with, we study the case when \( \vec{b} \) is simplest, i.e., \( \vec{b} = \vec{0} \).

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### Homogeneous Systems of Linear Equations (Section 1.5)

\( A\vec{x} = \vec{0} \) (all right-hand side entries are zero)

is a homogeneous system of linear equations.

\( \vec{x} = \vec{0} \) (the zero vector) is always a solution, and is called the trivial solution.

**Q:** Are there non-trivial solutions to \( A\vec{x} = \vec{0} \)?

Recall that a consistent system has either a unique solution, or has infinitely many solutions. For \( A\vec{x} = \vec{0} \), the trivial solution is always present. Hence, it has non-trivial solutions if it has infinitely many solutions, for which, it must have free variables.

**A:** There are non-trivial solutions if there is at least one free variable.
Prob 6, pg 47

\[ x_1 + 2x_2 - 3x_3 = 0 \]
\[ 2x_1 + x_2 - 3x_3 = 0 \]
\[ -x_1 + x_2 = 0 \]

Does this system have nontrivial solutions? If yes, describe all of them.

\[
\begin{bmatrix}
1 & 2 & -3 \\
2 & 1 & -3 \\
-1 & 1 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 2 & -3 \\
0 & -3 & 3 \\
0 & 3 & -3
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 2 & -3 \\
0 & 3 & 3 \\
0 & 0 & 0
\end{bmatrix}
\]

\[ A \]

To have nontrivial solutions, there must exist at least one free variable. \( x_3 \) is free here, so the system does have nontrivial solutions.

We now describe all its solutions.

\[
\begin{bmatrix}
1 & 2 & -3 \\
0 & 3 & 3 \\
0 & 0 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 2 & -3 \\
0 & 1 & -1 \\
0 & 0 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & -1 \\
0 & 0 & 0
\end{bmatrix}
\]

\[ x_1 - x_3 = 0 \]
\[ x_2 - x_3 = 0 \]

Hence, \( \left\{ \begin{array}{c}
x_1 = x_3 \\
x_2 = x_3
\end{array} \right\} \) describes all solutions.

Equivalently, we can write \( \left\{ \begin{array}{c}
x_1 = s \\
x_2 = s, s \in \mathbb{R}
\end{array} \right\} \), which is the parametric form.
We now represent the parametric form in an equivalent form involving a vector corresponding to the parameter $s$.

All solutions can be written in the vector form

\[
\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} x_3 \quad \text{where} \quad x_3 \text{ is free. Equivalently,}
\]

\[
\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \beta, \quad \beta \in \mathbb{R}.
\]

parameter that we can choose freely.

parametric vector form of all solutions.

We can visualize the solutions as follows. All solutions form a line through the origin along the vector $[1;1]$ associated with the parameter $s$.

Notice that the trivial solution corresponds to $s=0$.

It turns out that the solutions for $A\bar{x} = \bar{b}$ for a nonzero $\bar{b}$ could be described as a parallel line, obtained by just "shifting" the solution line for $A\bar{x} = \bar{0}$ by a vector. More on this picture in the next class...