Vector form of a system of linear equation:

\[ a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b \]

The system has a solution if \( \vec{b} \) is a linear combination of the vectors \( \vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n \). Equivalently, if \( \vec{b} \) is in \( \text{span} \{ \vec{a}_1, \ldots, \vec{a}_n \} \), the set of all linear combinations of \( \vec{a}_1, \ldots, \vec{a}_n \).

The parallelogram rule of vector addition in \( \mathbb{R}^2 \) — the sum \( \vec{u} + \vec{v} \) is the diagonal of the parallelogram formed by \( \vec{u}, \vec{v} \). Equivalently, \( \vec{u} + \vec{v} \) is the fourth vertex of the parallelogram formed by \( \vec{0} \) (origin), \( \vec{u} \), and \( \vec{v} \).
\( \text{Span } \{ \bar{u}, \bar{v} \} = \) ? Where
\[
\bar{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \bar{v} = \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix}?
\]
\( \{ x_1 \bar{u} + x_2 \bar{v} \} = ? \)
for all
\( x_1, x_2 \in \mathbb{R} \)

\( \text{Span } \{ \bar{u}, \bar{v} \} \) is the plane passing through \( \bar{0}, \bar{u}, \bar{v} \).

Notice that in 3D space, 3 points that are not on a single straight line determine a plane uniquely. Imagine a sheet of paper passing through the three points, but extending without limits on all of its four edges.

This illustration of \( \text{span } \{ \bar{u}, \bar{v} \} \) also demonstrates the choice of the word "span." As such, \( \text{span } \{ \bar{u}, \bar{v} \} \) is also referred to as the plane generated by \( \bar{u} \) and \( \bar{v} \) (the origin \( \bar{0} \) is understood to be included implicitly).
Problem 21.9 

\[ \bar{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad \bar{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \]  
Show that \[ \begin{bmatrix} h \\ k \end{bmatrix} \] is in \( \text{Span} \{ \bar{u}, \bar{w} \} \) for every \( h, k \).

Show that \( \bar{u}x_1 + \bar{w}x_2 = \begin{bmatrix} h \\ k \end{bmatrix} \) is consistent for all \( h, k \).

\[
\begin{bmatrix}
2 & 2 & h \\
-1 & 1 & k
\end{bmatrix} \xrightarrow{R_1 \times \frac{1}{2}} \begin{bmatrix}
1 & 1 & h_2 \\
-1 & 1 & k
\end{bmatrix} \xrightarrow{R_2 + R_1} \begin{bmatrix}
1 & 1 & h_2 \\
0 & 2 & k
\end{bmatrix}
\]

The system is consistent for every \( h \) and \( k \).

Here \( \text{Span}\{\bar{u}, \bar{w}\} = \mathbb{R}^2 \)

The span is all of \( \mathbb{R}^2 \).

Compare the span here to the \text{Span} of \( \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \) seen earlier, which was just a line through the origin. As in the case here, if the span of a set of vectors \( \bar{a}_1, \ldots, \bar{a}_n \) is all of the space in which the vectors sit, then life becomes easy. We know that the system

\[ \bar{a}_1 x_1 + \cdots + \bar{a}_n x_n = \bar{b} \]

is consistent for every \( \bar{b} \)!
\[ \overline{u} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \quad \overline{v} = \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix}, \quad \overline{y} = \begin{bmatrix} h \\ -3 \\ -5 \end{bmatrix}. \] For what values of \( h \) is \( \overline{y} \) in the plane generated by \( \overline{u} \) and \( \overline{v} \)?

Equivalently, for what values of \( h \) is the system \[ \overline{u}x_1 + \overline{v}x_2 = \overline{y} \] consistent?

\[
\begin{bmatrix}
1 & -2 & | & h \\
0 & 1 & | & -3 \\
-2 & 7 & | & -5
\end{bmatrix}
\overset{R_3 + 2R_1}{\rightarrow}
\begin{bmatrix}
1 & -2 & | & h \\
0 & 1 & | & -3 + 2h \\
0 & 3 & | & -5 + 2h
\end{bmatrix}
\overset{R_3 - 3R_2}{\rightarrow}
\begin{bmatrix}
1 & -2 & | & h \\
0 & 1 & | & -3 \\
0 & 0 & | & 4 + 2h
\end{bmatrix} = 0 \text{ for a consistent system.}
\]

So, \( h = -2 \).

The matrix form \( \overline{A} \overline{x} = \overline{b} \). (Section 1.4)

We have already seen this form! For instance,

\[
3x_1 + x_2 = 7
\]
\[
x_1 + 2x_2 = 4
\]

\[
\begin{bmatrix}
3 \\ 1
\end{bmatrix} x_1 + \begin{bmatrix}
1 \\ 2
\end{bmatrix} x_2 = \begin{bmatrix}
7 \\ 4
\end{bmatrix}
\]

vector equation

We now write it in matrix form:

\[
\begin{bmatrix}
3 & 1 \\
1 & 2
\end{bmatrix}\begin{bmatrix}
x_1 \\ x_2
\end{bmatrix} = \begin{bmatrix}
7 \\ 4
\end{bmatrix}
\]

where \( A = \begin{bmatrix}
3 & 1 \\
1 & 2
\end{bmatrix} \), \( \overline{x} = \begin{bmatrix}
x_1 \\ x_2
\end{bmatrix} \), \( \overline{b} = \begin{bmatrix}
7 \\ 4
\end{bmatrix} \).
A\bar{x} is a matrix-vector product.

Let \( A = [\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n] \) be an \( m \times n \) matrix. So, \( \bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n \) are all \( m \)-vectors.

\( \bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \) is an \( n \)-vector.

Then \( A\bar{x} = \bar{a}_1 x_1 + \bar{a}_2 x_2 + \ldots + \bar{a}_n x_n \) is the linear combination of the columns of \( A \) with the entries in \( \bar{x} \) as scalars or weights.

\( A\bar{x} = \bar{b} \) has a solution if and only if

the vector equation \( \bar{a}_1 x_1 + \bar{a}_2 x_2 + \ldots + \bar{a}_n x_n = \bar{b} \) has a solution, which happens if and only if the system represented by the augmented matrix \([A | \bar{b}]\) has a solution.

We now discuss a condition that guarantees \( A\bar{x} = \bar{b} \) has a solution, given in terms of the existence of pivots in each row of \( A \). This condition is independent of \( \bar{b} \).
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\[
A = \begin{bmatrix} 1 & -2 & -1 \\ -2 & 2 & 0 \\ 4 & -1 & 3 \end{bmatrix}, \quad \bar{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.
\]

Show that \( A\bar{x} = \bar{b} \) is not consistent for all \( \bar{b} \). Describe the collection of \( \bar{b} \) for which it is consistent.

\[
\begin{bmatrix} A | \bar{b} \end{bmatrix} = \begin{bmatrix} 1 & -2 & -1 & b_1 \\ -2 & 2 & 0 & b_2 \\ 4 & -1 & 3 & b_3 \end{bmatrix}
\]

\[
\xrightarrow{R_2 + 2R_1} \begin{bmatrix} 1 & -2 & -1 & b_1 \\ 0 & 2 & 0 & b_2 + 2b_1 \\ 4 & -1 & 3 & b_3 \end{bmatrix}
\]

\[
\xrightarrow{R_2 - 4R_1} \begin{bmatrix} 1 & -2 & -1 & b_1 \\ 0 & 2 & 0 & b_2 + 2b_1 \\ 0 & -7 & 7 & b_3 - 4b_1 \end{bmatrix}
\]

\[
\xrightarrow{R_3 + \frac{7}{2}R_2} \begin{bmatrix} 1 & -2 & -1 & b_1 \\ 0 & 2 & 0 & b_2 + 2b_1 \\ 0 & 0 & 0 & b_3 - 4b_1 + \frac{7}{2}(b_2 + 2b_1) \end{bmatrix}
\]

\[= 0 \] for system to be consistent

i.e. \( 3b_1 + \frac{7}{2}b_2 + b_3 = 0 \), i.e. \( 6b_1 + 7b_2 + 2b_3 = 0 \).

Hence, the system is not consistent for all \( \bar{b} \), but only for those \( \bar{b} \in \mathbb{R}^3 \) that satisfy \( 6b_1 + 7b_2 + 2b_3 = 0 \).

The set of all \( \bar{b} \) for which \( A\bar{x} = \bar{b} \) is consistent is a plane through the origin described by \( 6b_1 + 7b_2 + 2b_3 = 0 \).

Equivalently, \( \text{span} \{ \bar{a}_1, \bar{a}_2, \bar{a}_3 \} \) is not all of \( \mathbb{R}^3 \), but a plane through origin sitting in \( \mathbb{R}^3 \).