Recall (results on bases of subspaces, dimension, rank, etc.)

Basis of a subspace $H$ is an LI set of vectors in $H$ that spans $H$.

Dimension of $H$, denoted by $\dim H$ or $\dim(H)$, is the # vectors in any basis of $H$.

$\dim \{\emptyset\} = 0$ as $\emptyset$ is not LI.

$\text{rank}(A) = \dim(\text{Col} A) = \# \text{ pivot columns}$
$\dim(\text{Nul} A) = \# \text{ free variables}$

A basis for $\text{Col} A$: pivot columns of $A$
A basis for $\text{Nul} A$: vectors in parametric vector form of solutions to $A\vec{x} = \vec{0}$.

\[
\text{Rank theorem}
\]

For $A \in \mathbb{R}^{m \times n}$,

$\text{rank}(A) + \dim(\text{Nul} A) = n$

# pivot columns # free variables total # columns
Invertible matrix theorem (continued...)

(a) A ∈ \( \mathbb{R}^{n \times n} \) is invertible.
(b) Columns of A form a basis for \( \mathbb{R}^n \).
(c) \( \text{Col } A = \mathbb{R}^n \).
(d) \( \dim \text{Col } A = n \).
(e) \( \text{rank } A = n \).
(f) \( \text{Nul } A = \{ 0 \} \).
(g) \( \dim (\text{Nul } A) = 0 \).

Coordinates

Let \( B = \{ \vec{b}_1, \ldots, \vec{b}_p \} \) be a basis for a subspace \( H \). Each \( \vec{x} \in H \) can be written as \( \vec{x} = c_1\vec{b}_1 + c_2\vec{b}_2 + \cdots + c_p\vec{b}_p \) for scalars \( c_1, c_2, \ldots, c_p \), which are unique to \( \vec{x} \). These scalars are called the coordinates of \( \vec{x} \) relative to the basis \( B \). Stacking these scalars into a vector

\[
[\vec{x}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{bmatrix}
\]

is the \( B \)-coordinate of \( \vec{x} \).

When \( B \) is the standard basis, i.e., \( \{ \vec{e}_1, \ldots, \vec{e}_p \} \), the \( B \)-coordinate of any \( \vec{x} \) consists of its own entries. (So, the \( B \)-coordinate of \( \vec{x} \) is \( \vec{x} \) itself here.)
In Exercises 1 and 2, find the vector $\mathbf{x}$ determined by the given coordinate vector $[\mathbf{x}]_B$ and the given basis $B$. Illustrate your answer with a figure, as in the solution of Practice Problem 2.

1. $B = \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}$, $[\mathbf{x}]_B = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$$\mathbf{x} = 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}.$$  

We just have to evaluate the linear combination given

In Exercises 3–6, the vector $\mathbf{x}$ is in a subspace $H$ with a basis $B = \{\mathbf{b}_1, \mathbf{b}_2\}$. Find the $B$-coordinate vector of $\mathbf{x}$.

5. $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -2 \\ -7 \\ 5 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 2 \\ 9 \\ -7 \end{bmatrix}$

$B = \{\mathbf{b}_1, \mathbf{b}_2\}$. Find $[\mathbf{x}]_B$. So, find $c_1, c_2$ such that

$$\mathbf{x} = c_1 \mathbf{b}_1 + c_2 \mathbf{b}_2$$

$$\begin{bmatrix} 1 & -2 \\ 4 & -7 \\ -3 & 5 \end{bmatrix} R_2 - 4R_1 \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} R_3 + 3R_1 \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_1 = \frac{4}{1}$, $C_2 = 1$

or $[\mathbf{x}]_B = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

Why study coordinates?

One often works with a nonstandard basis for a subspace. Hence we want to study how any vector is expressed in such a basis. For instance, we could start with the standard basis for an image, and do a bunch of geometric transformations, e.g., rotate $90^\circ$ CCW. After that step, we could just work with the nonstandard basis for further analyses.
In Exercises 17 and 18, mark each statement True or False. Justify each answer. Here $A$ is an $m \times n$ matrix.

17. a. If $B = \{v_1, \ldots, v_p\}$ is a basis for a space $H$ and if $x = c_1v_1 + \cdots + c_pv_p$, then $c_1, \ldots, c_p$ are the coordinates of $x$ relative to the basis $B$.

b. Each line in $\mathbb{R}^n$ is a one-dimensional subspace of $\mathbb{R}^n$.

c. The dimension of Col $A$ is the number of pivot columns in $A$.

d. The dimensions of Col $A$ and Nul $A$ add up to the number of columns in $A$.

e. If a set of $p$ vectors spans a $p$-dimensional subspace $H$ of $\mathbb{R}^n$, then these vectors form a basis for $H$.

(a) True. Definition of coordinates.
(b) False. Only if it goes through the origin.
(c) True. Definition.
(d) True. Rank theorem
(e) True. A set of $p$ vectors that spans a $p$-dimensional subspace will be LI. Hence it's a basis.

23. If possible, construct a $3 \times 5$ matrix $A$ such that $\dim \text{Nul } A = 3$ and $\dim \text{Col } A = 2$.

\[
\begin{align*}
\text{n=5, } & \quad \dim \text{Col } A + \dim \text{Nul } A = 2 + 3 = n \checkmark \\
& \text{(So, rank theorem holds)} \\
A &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ works. (has 2 pivot columns)}
\end{align*}
\]
Determinants (Section 2.1)

Recall, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, when determinant of $A$, $\det A = ad-bc \neq 0$.

In general, for $A \in \mathbb{R}^{n \times n}$, $A$ is invertible if and only if $\det A \neq 0$.

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & a & b \\ 5 & c & d \end{bmatrix}$. When is $A$ invertible?

(What condition should $a, b, c, d$ satisfy in order to make $A$ invertible?)

\[
\begin{bmatrix} 0 & 2 & 3 \\ 4 & a & b \\ 5 & c & d \end{bmatrix}
\xrightarrow{R_2-4R_1 \to R_2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & a-8 \neq 0 & b-12 \\ 0 & c-10 & d-15 \end{bmatrix}
\xrightarrow{R_3 - \frac{c-10}{a-8} R_2 \to R_3}
\]

\[
\begin{bmatrix} 1 & 2 & 3 \\ 0 & a-8 & (b-12) \\ 0 & 0 & (d-15) - \frac{(c-10)(b-12)}{a-8} \end{bmatrix}
\neq 0
\]

As we need three pivots

$(a-8)(d-15) - (c-10)(b-12) \neq 0$

$(ad-15a-8d+120) - (bc-12c-10b+120) \neq 0$

$(ad-bc) - (8d-10b) + (12c-15a) \neq 0$
i.e., we need \[1(ad-bc) - 2(4d-5b) + 3(4c-5a) \neq 0\] expanding along Row 1

\[
\begin{vmatrix} 1 & 2 & 3 \\ 4 & a & b \\ 5 & c & d \end{vmatrix} = 1 \cdot \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} - 2 \cdot \det \begin{bmatrix} 4 & b \\ 5 & d \end{bmatrix} + 3 \cdot \det \begin{bmatrix} 4 & c \\ 5 & a \end{bmatrix}
\]

We will see that one could expand along any row or any column to compute \(\det A\). More on the details in the next lecture.