Subspaces (Section 2.8)

Motivation: \( \mathbb{R}^2 \) is like an infinite sheet of paper sitting on a flat "table", so to say. Now imagine the same infinite sheet sitting in 3D space. When can we guarantee that the desired properties of \( \mathbb{R}^2 \) are all retained by this sheet now sitting in \( \mathbb{R}^3 \)?

**Def**

\( H \) is a subspace of \( \mathbb{R}^n \) if:

1. The zero vector is in \( H \);
2. For \( \vec{u}, \vec{v} \) in \( H \), \( \vec{u} + \vec{v} \) is also in \( H \); and
3. For \( \vec{u} \) in \( H \) and scalar \( c \), \( c\vec{u} \) is also in \( H \).

\( H \) is closed under vector addition and scalar multiplication.

**Note:** \( \mathbb{R}^n \) as well as \( \{0\} \) are both subspaces of \( \mathbb{R}^n \).

**Prob 2, pg 151**

Exercises 1–4 display sets in \( \mathbb{R}^2 \). Assume the sets include the bounding lines. In each case, give a specific reason why the set \( H \) is not a subspace of \( \mathbb{R}^2 \). (For instance, find two vectors in \( H \) whose sum is not in \( H \), or find a vector in \( H \) with a scalar multiple that is not in \( H \). Draw a picture.)

1. \( \vec{u} \) is not in \( H \) for any \( c < 0 \).
   So \( H \) is not a subspace.

2. \( \vec{u} + \vec{v} \) is not in \( H \).
   So \( H \) is not a subspace.
What are (valid) subspaces of $\mathbb{R}^2$?

$\mathbb{R}^2$, $\{0\}$ are subspaces. So are lines passing through the origin.

$n$ot a subspace, as origin is not included.

**Def**

$\text{Span}\{\overline{v}_1, \ldots, \overline{v}_p\}$ is a subspace of $\mathbb{R}^m$, when each $\overline{v}_j \in \mathbb{R}^m$ (the set of all linear combinations of $\overline{v}_1, \ldots, \overline{v}_p$). We call this subspace the subspace generated by, or spanned by, $\overline{v}_1, \ldots, \overline{v}_p$.

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5. Let $\overline{v}_1 = \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix}$, $\overline{v}_2 = \begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix}$, and $\overline{w} = \begin{bmatrix} -3 \\ 3 \\ -10 \end{bmatrix}$. Determine if $\overline{w}$ is in the subspace of $\mathbb{R}^3$ generated by $\overline{v}_1$ and $\overline{v}_2$.

The question asks "is $\overline{w}$ in span($\overline{v}_1, \overline{v}_2$)?"

Reword: With $A = \begin{bmatrix} \overline{v}_1 & \overline{v}_2 \end{bmatrix}$, is $A\overline{x} = \overline{w}$ consistent?

Notice that we need not solve for $\overline{x}$ - we just need to determine if the system is consistent or not in order to answer the question.
\[ A = \begin{bmatrix} 1 & -2 \\ 3 & -3 \\ -4 & 7 \end{bmatrix} \quad \overline{w} = \begin{bmatrix} -3 \\ -3 \\ 10 \end{bmatrix} \]

\[
\begin{bmatrix} A | \overline{w} \end{bmatrix} = \begin{bmatrix} 1 & -2 & -3 \\ 3 & -3 & -3 \\ -4 & 7 & 10 \end{bmatrix} \xrightarrow{R_2 \rightarrow 3R_2} \begin{bmatrix} 1 & -2 & -3 \\ 0 & 3 & 6 \\ 0 & -1 & -2 \end{bmatrix} \xrightarrow{R_3 \rightarrow 1 + \frac{1}{2}R_2} \begin{bmatrix} 1 & -2 & -3 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix}
\]

System is consistent. So, \( \overline{w} \in \text{span}(\overline{v_1}, \overline{v_2}, \overline{v_3}) \).

We will now study two subspaces related to a matrix \( A \), and their relationships to various concepts we have already seen—consistency of \( A\overline{x} = \overline{b} \), one-to-one and onto LTs defined by \( A \), etc.

**Column Space and Null Space of \( A \in \mathbb{R}^{m \times n} \)**

**Def** The **column space of \( A \)** is the set of all linear combinations of the columns of \( A \). We denote it \( \text{Col} A \).

\( \text{Col} A \) is a subspace of \( \mathbb{R}^m \).

\( \overline{b} \in \mathbb{R}^m \) is in \( \text{Col} A \) if \( A\overline{x} = \overline{b} \) is consistent.

The **null space of \( A \)** is the set of all solutions to \( A\overline{x} = \overline{0} \). We denote it by \( \text{Nul} A \).

Since any \( \overline{x} \) that is a solution to \( A\overline{x} = \overline{0} \) is in \( \mathbb{R}^n \), \( \text{Nul} A \) is a subspace of \( \mathbb{R}^n \).

Since \( \text{Col} A \) is the set of all linear combinations of the columns of \( A \), it satisfies the definition of a subspace in a straightforward manner.
Let us check the definition for Nul A being a subspace now.

1. \( A\bar{0} = \bar{0} \) (trivial solution). So \( \bar{0} \in \text{Nul} A \).
2. For \( \bar{x}_1, \bar{x}_2 \) such that \( A\bar{x}_1 = \bar{0} \) and \( A\bar{x}_2 = \bar{0} \), indeed
   \[ A(\bar{x}_1 + \bar{x}_2) = A\bar{x}_1 + A\bar{x}_2 = \bar{0} + \bar{0} = \bar{0}. \]
   \( \bar{x}_1, \bar{x}_2 \in \text{Nul} A \)

   So \( \bar{x}_1 + \bar{x}_2 \) is also in \( \text{Nul} A \).

3. For \( \bar{x} \) in \( \text{Nul} A \), i.e., \( A\bar{x} = \bar{0} \), consider \( A(c\bar{x}) \).
   \[ A(c\bar{x}) = c(A\bar{x}) = c\bar{0} = \bar{0}. \]
   So \( c\bar{x} \in \text{Nul} A \).

7. Let
   \[ \mathbf{v}_1 = \begin{bmatrix} 2 \\ -8 \\ 6 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 8 \\ -7 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -4 \\ 6 \\ -7 \end{bmatrix}. \]
   \[ \mathbf{p} = \begin{bmatrix} 6 \\ -10 \\ 11 \end{bmatrix}. \]
   \[ A|\bar{p}| = \begin{bmatrix} 2 & -3 & -4 & 6 \\ -8 & 8 & 6 & -10 \\ 6 & -7 & -7 & 11 \end{bmatrix} \]

   a. How many vectors are in \( \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \} \)?
   b. How many vectors are in Col \( A \)?
   c. Is \( \mathbf{p} \) in Col \( A \)? Why or why not?

(a). Three. \( \{ \overline{\mathbf{v}_1}, \overline{\mathbf{v}_2}, \overline{\mathbf{v}_3} \} \) is just a collection of the three vectors.

(b). Infinitely many. Remember, Col \( A \) is the set of all linear combinations of the columns of \( A \).

(c) \( \bar{p} \in \text{Col} A \) if \( A\bar{x} = \bar{p} \) is consistent.
\[
\begin{bmatrix}
2 & -3 & -4 & 6 \\
8 & 8 & 6 & -10 \\
6 & -7 & 6 & 11 \\
\end{bmatrix}
\xrightarrow{R_3 \rightarrow 3R_1}
\begin{bmatrix}
2 & -3 & -4 & 6 \\
0 & 10 & -10 & 14 \\
0 & 2 & 5 & -7 \\
\end{bmatrix}
\xrightarrow{R_3 \rightarrow \frac{1}{2}R_2}
\begin{bmatrix}
2 & -3 & -4 & 6 \\
0 & 4 & -10 & 14 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

System is consistent. So \( \vec{p} \in \text{Col} \ A \).

Again, notice that we need not solve the system \( A\vec{x} = \vec{p} \).

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**Prob 9, Pg 151**

9. With \( A \) and \( \vec{p} \) as in Exercise 7, determine if \( \vec{p} \) is in \( \text{Nul} \ A \).

\( \vec{p} \in \text{Nul} \ A \) if \( A\vec{p} = \vec{0} \).

\[
A\vec{p} = \begin{bmatrix}
2 & -3 & -4 \\
-8 & 8 & 6 \\
6 & -7 & 7 \\
\end{bmatrix}
\begin{bmatrix}
6 \\
-10 \\
11 \\
\end{bmatrix}
= \begin{bmatrix}
2 \times 6 + (-3) \times (-10) + (-4) \times 11 \\
-8 \times 6 + 8 \times (-10) + 6 \times 11 \\
6 \times 6 + (-7) \times (-10) + (-7) \times 11 \\
\end{bmatrix}
= \begin{bmatrix}
-2 \\
-62 \\
29 \\
\end{bmatrix} \neq \vec{0}.
\]

So \( \vec{p} \notin \text{Nul} \ A \).

"not an element of."

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**Basis for a subspace \( H \)**

We saw that a subspace, by definition, has infinitely many vectors. Hence we try to work with a finite subset of these vectors which generates the entire subspace. It also makes sense to study such a finite set that is also minimal, i.e., has the smallest number of vectors. It turns out that such a minimal set is LI.
A linearly independent set in $H$ which spans $H$ is a **basis** for $H$.

Equivalently, a basis is a minimal subset of $H$ that generates $H$.

**Example** The unit vectors $\{\bar{e}_1, \bar{e}_2, \bar{e}_3\} = \left\{ [1, 0, 0], [0, 1, 0], [0, 0, 1] \right\}$ form a basis for $\mathbb{R}^3$.

Notice that the set $\{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ is LI, and any $\bar{x} \in \mathbb{R}^3$ can be written as a unique linear combination of $\bar{e}_1, \bar{e}_2,$ and $\bar{e}_3$.

In general, $\{\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n\}$ forms a basis for $\mathbb{R}^n$, where $\bar{e}_j$ is the $j^{th}$ unit $n$-vector. This is the **standard basis** for $\mathbb{R}^n$.

In the next lecture, we will talk about bases for $\text{Col A}$ and $\text{Nul A}$.