This is Section 2. Although the four sections are coordinated, individual lectures, projects, etc., will be different.

Instructor: I'm Bala Krishnamoorthy, call me Bala.

I'm originally from India. If you do not understand what I say because of my accent, do let me know 😊!

My research interests are in optimization, algebraic topology, applications to biology, medicine, etc.

Discussed syllabus, and [http://www.mymathlab.com/](http://www.mymathlab.com/)

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Dude M. Major - has a Thursday problem. He has five hours in hand (say, 7pm-midnight), and $48 to spend. He has two options. The costs for each activity are listed here.

- Can get tutoring → $8/hr
- Go partying! → $16/hr

Q: How many hours to get tutored, and how many to party?

Let $x_1 = \# \text{ hours of tutoring}$
$x_2 = \# \text{ hours of partying}$

\[
\begin{align*}
1x_1 + 1x_2 &= 5 \\
8x_1 + 16x_2 &= 48
\end{align*}
\]

These coefficients are one each right-hand side (rhs) numbers coefficients
This is a system of two **linear** equations. The graph of each equation is a line

\[
\begin{align*}
2x_1 + 4x_3 &= -3 \\
-\sqrt{x_1} + 5x_6 &= -8
\end{align*}
\]

A system of **nonlinear** equations

The coefficients and the numbers could be real or complex numbers. In Math 220, we will work only with real numbers.

In general, a linear equation can have any number of terms, e.g.,

\[5x_1 + \sqrt{3} x_2 - 8x_3 \ldots + 10x_9 = -16.43\]

\[\sqrt{3}\] is just a coefficient (it's a real number) — it does not create a nonlinearity here!

A solution is a set of values for \((x_1, x_2)\) for which each equation in the system is true.

You can check that \(x_1 = 4, x_2 = 1\) is a solution.

\[
\begin{align*}
x_1 + x_2 &= 5 \\
8x_1 + 16x_2 &= 48
\end{align*}
\]

The question, of course, is to **find** a solution. In this case, since we are in 2D, we can use the graphical method.
plot the equations:

\[ x_1 + x_2 = 5 \] \[ (5,0), (0,5) \]

\[ 8x_1 + 16x_2 = 48 \] \[ (6,0), (0,3) \]

The two lines meet at exactly one point, \( (4,1) \).

Hence, the system has a unique solution, i.e., \( x_1 = 4, x_2 = 1 \).

This is the "nice" case, where the lines intersect at a single point.

But there are two other extreme cases.

* The two lines do not intersect, i.e., they are parallel.
  On this case, the system has no solutions, or is said to be inconsistent.

Say, tutoring is also \$16/hr.

\[ x_1 + x_2 = 5 \] \[ (0,3), (3,0) \text{ are two points on this line} \]

\[ 16x_1 + 16x_2 = 48 \]
The two lines coincide, i.e., they intersect at every point. Here, the system has infinitely many solutions.

Say, Dude has only 3 hours to spend now:

\[ x_1 + x_2 = 3 \]
\[ 16x_1 + 16x_2 = 48 \]

These results hold for systems of linear equations with more than two variables as well. Of course, in this case, we will not be able to use the graphical method. In this class, we will learn all about how to solve such systems with many variables.

In summary, a system of linear equations can have

1. no solution \(\rightarrow\) inconsistent system.
2. one solution \(\rightarrow\) unique solution
3. infinitely many solutions.

Notice that a system cannot have 3, 14, or 23 (or any finite number higher than 1) solutions!
Recall Dule's problem:

\[ \begin{align*}
X_1 + X_2 &= 5 \quad (1) \\
8X_1 + 16X_2 &= 48 \quad (2)
\end{align*} \]

\[ \begin{align*}
X_1 &= 4 \\
X_2 &= 1
\end{align*} \]  

This is the unique solution, but could also be viewed as a system of two linear equations.

In general, we can have any number of equations in any number of variables. To solve the system, we go to an "easier" system using operations that preserve the solutions.

Goal: Eliminate \( X_1 \) from equations (2), (3),...

eliminate \( X_2 \) from equations (1), (3),...

\[ \begin{align*}
X_1 + X_2 &= 5 \quad (1) \\
8X_1 + 16X_2 &= 48 \quad (2) \\
\end{align*} \]  

we replace equation (2) by the sum of itself and \((-8) \times (equation\ 1)\).

\[ (2')X_2 = \frac{-8}{8} \]

\[ X_1 + X_2 = 5 \quad (1) \\
8X_2 = 8 \quad (2') \\
X_1 = \frac{5 - 1}{8} = 4 \quad (1') \\
X_2 = 1 \quad (2'')
\]

To solve the system, we go to an "easier" system using operations that preserve the solutions.

This procedure of transforming the original system to an equivalent system is called Gaussian elimination.

\textbf{Def} \\
Two systems are \textit{equivalent} if they have the same set of solutions.
Matrix Notation

We present a much more compact representation of these operations — by working just with the numbers!

A matrix is a rectangular array of numbers. It has rows and columns.

e.g., \[ A = \begin{bmatrix} 1 & 1 \\ 8 & 16 \end{bmatrix} \] is the matrix of coefficients.

A vector is a single row or column of numbers.

"bar" (lower case letters with the bar are vectors, e.g., \( \vec{a}, \vec{b}, \vec{x}, \vec{y} \), etc.)

e.g., \( \vec{b} = \begin{bmatrix} 5 \\ 48 \end{bmatrix} \) is the rhs (right-hand-side) vector.

Augmented matrix for a system \( \rightarrow \) attach the rhs vector represents the entire system.

don't attach this line is a "separator" — we use it just for convenience.

\[ \begin{bmatrix} 1 & 1 & 5 \\ 8 & 16 & 48 \end{bmatrix} \] or, in general, \([A|\vec{b}]\)

We perform the permitted operations on the augmented matrix. These operations are called

elementary row operations (EROs)

do not change the solutions

Work with the rows of \([A|\vec{b}]\) or on any matrix \(A\) in general.

It is important to remember that EROs can be applied to any matrix, and not just to augmented matrices. When applied to an augmented matrix, we are working with the equations in that system.

Each row in \([A|\vec{b}]\) is one equation.
There are three types of EROs.

1. **Replacement**: Replace a row with the sum of itself and a multiple of another row.

2. **Interchange**: swap two rows.

3. **Scaling**: multiply a row by a non-zero number.

In the next (few) lecture(s), we will formalize the ideas for how to choose the EROs we would apply. For now, we will guess – the goal is to simplify, by eliminating $x$, from rows 2, 3, ..., $x_2$ from rows 1, 3, ..., and so on.

To zero out the 8 in Row 2, we could use a replacement ERO. Then we do a scaling ERO, and so on.

$$\begin{bmatrix} 1 & 1 & 5 \\ 8 & 16 & 48 \end{bmatrix} \xrightarrow{R_2 \rightarrow 8R_1} \begin{bmatrix} 1 & 1 & 5 \\ 0 & 8 & 48-8 \cdot 5 \end{bmatrix} \xrightarrow{R_2 \rightarrow \frac{1}{8}} \begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & 8 \end{bmatrix} \xrightarrow{R_1 \rightarrow R_2} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 1 \end{bmatrix}$$

For an interchange ERO, we use the following notation:

$R_1 \leftrightarrow R_2$ (for swapping rows 1 & 2, for instance).
Problem 12, Pg 10

Solve
\[
\begin{align*}
X_1 - 5X_2 + 4X_3 &= -3 \\
2X_1 - 7X_2 + 3X_3 &= -2 \\
-2X_1 + X_2 + 7X_3 &= -1
\end{align*}
\]

\[
\begin{bmatrix}
1 & -5 & 4 & | & -3 \\
2 & -7 & 3 & | & -2 \\
-2 & 1 & 7 & | & -1
\end{bmatrix}
\quad \xrightarrow{R_2 \leftrightarrow R_3} \quad
\begin{bmatrix}
1 & -5 & 4 & | & -3 \\
2 & -7 & 3 & | & -2 \\
-2 & 1 & 7 & | & -1
\end{bmatrix}
\quad \xrightarrow{R_3 + 2R_1} \quad
\begin{bmatrix}
1 & -5 & 4 & | & -3 \\
0 & -6 & 10 & | & -3 \\
0 & -9 & 15 & | & -7
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & -5 & 4 & | & -3 \\
0 & 1 & -10/6 & | & 1/2 \\
0 & -9 & 15 & | & -7/2
\end{bmatrix}
\quad \xrightarrow{R_2 \times (1/6)} \quad
\begin{bmatrix}
1 & -5 & 4 & | & -3 \\
0 & 1 & -10/6 & | & 1/2 \\
0 & 0 & 0 & | & -5/2
\end{bmatrix}
\]

For simple fractions as seen here, it's best to stick with them as is rather than go to decimal notation.

Hence the system is inconsistent, i.e., it has no solutions.

Whenever you get a row of the form [0 0 ... 0 | *,] where * is nonzero, the system is inconsistent.

So as long as you do not see such a row, the system is consistent. It can have a unique solution, or infinitely many solutions.
Two matrices are *row equivalent* if there is a series of EROs that transforms one matrix into the other.

**Note** Every ERO is reversible, i.e., for every ERO, there is a complementary ERO that reverses its effect.

E.g., consider the first ERO from the previous problem. The complementary ERO is shown here.

\[
\begin{bmatrix}
1 & -5 & 4 & -3 \\
2 & -7 & 3 & -2 \\
2 & 1 & 7 & -1
\end{bmatrix}
\begin{array}{c}
\xrightarrow{R_2 \leftrightarrow R_3}
\xrightarrow{R_2 - 2R_3}
\end{array}
\begin{bmatrix}
1 & -5 & 4 & -3 \\
0 & -6 & 10 & -3 \\
-2 & 1 & 7 & -1
\end{bmatrix}
\]

**Prob 20, pg 10**

Given the matrix \[
\begin{bmatrix}
1 & h & -5 \\
2 & -8 & 6
\end{bmatrix},
\] find the values of \(h\) so that it is the augmented matrix of a consistent system.

\[
\begin{bmatrix}
1 & h & -5 \\
2 & -8 & 6
\end{bmatrix}
\xrightarrow{R_2 - 2R_1}
\begin{bmatrix}
1 & h & -5 \\
0 & -8-2h & 16
\end{bmatrix}
\]

We need \(-8-2h \neq 0\), i.e., \(h \neq -4\).

You could write all values except \(-4\), or simply put \(h = -4\).
Recall: If the augmented matrix of a linear system has a row of the form $[0 \ldots 0 | x \neq 0]$, then the system is inconsistent.

**Problem 25 pg 11**

Find an equation connecting $g, h, k$ so that this is the augmented matrix of a consistent system.

$$
\begin{bmatrix}
1 & -4 & 7 & g \\
0 & 3 & -5 & h \\
-2 & 5 & -9 & k
\end{bmatrix}
\rightarrow
\begin{bmatrix}
0 & 3 & -5 & h \\
0 & 3 & -5 & k+2g+h
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & -4 & 7 & g \\
0 & 3 & -5 & h \\
0 & 0 & 0 & k+2g+h
\end{bmatrix}
$$

$k+2g+h = 0$ for the system to be consistent.

The "simpler" augmented matrices from which we could make conclusions about the existence (and uniqueness) of solutions have a nice, step-like structure. All entries below the "steps" are zero! We will now formalize this concept by introducing the echelon form and the reduced echelon form of a matrix.
Echelon Form of a matrix

"step-like"

We need some definitions first.

nonzero row: a row that has at least one nonzero entry
leading entry: the leftmost nonzero entry of a nonzero row.

Def A matrix is in row echelon form if
(1) all nonzero rows are above any zero rows, and
(2) the leading entry of each nonzero row is in a column
to the right of the leading entry of the row above.

A consequence of (1) & (2) is that
(3) all entries in a column below a leading entry are zero.

A matrix is in reduced row echelon form (RREF) if
(4) each leading entry is 1, and
(5) each leading 1 is the only nonzero entry in its column, i.e., all other entries in the column are zero.

The word "row" is understood in our discussions, and hence we often talk just about echelon form and reduced echelon form.
Examples

\[
\begin{bmatrix}
3 & 5 & 0 & 0 & 9 \\
0 & 1 & -2 & 6 \\
0 & 0 & 0 & 7
\end{bmatrix}
\] is in echelon form

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\] is in reduced echelon form

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\] is not in echelon form, as the zero row (2nd row) is not below all non-zero rows.

Given any matrix, we can do EROs to transform it to echelon form, and further to reduced echelon form. This process is called row reduction. The procedure is called row reduction algorithm.

A matrix is row equivalent to any of its echelon forms. A matrix can have multiple echelon forms, but its reduced echelon form is unique.

**Def** The leading entries are called pivot elements (or pivots). The columns with a pivot are called pivot columns.
Row reduce to echelon form, and to reduced echelon form. Circle pivot entries in both the final and original matrices, and mark the pivot columns.

\[
\begin{bmatrix}
1 & 2 & 4 & 5 \\
2 & 4 & 5 & 4 \\
4 & 5 & 4 & 2
\end{bmatrix}
\]

\[
\begin{align*}
R_2 & \rightarrow R_2 - 2R_1 \\
R_3 & \rightarrow R_3 - 4R_1
\end{align*}
\]

\[
\begin{bmatrix}
1 & 2 & 4 & 5 \\
0 & 0 & -3 & -6 \\
0 & -3 & -12 & -18
\end{bmatrix}
\]

\[
\begin{align*}
R_2 & \rightarrow R_2 - 3R_3 \\
R_3 & \rightarrow R_3 - 3R_3
\end{align*}
\]

\[
\begin{bmatrix}
1 & 2 & 4 & 5 \\
0 & 0 & -3 & -6 \\
0 & 0 & 1 & 2
\end{bmatrix}
\]

As illustrated above there is no harm in combining more than one EROs in one step of row reduction. Of course, one needs to be very careful to specify the particular order in which the multiple EROs in a step are to be executed.
Echelon form and reduced echelon form

We now introduce notation using which we describe matrices in the forms in general - without writing the actual numbers.

Standard notation: \( \bullet \rightarrow \text{nonzero number} \)
\( \ast \rightarrow \text{zero or nonzero} \)

\[ \begin{bmatrix} \ast & \ast & \ast & 0 \\ 0 & 0 & \ast & \ast \\ 0 & 0 & 0 & \ast \end{bmatrix} \]

is in echelon form.

For example, the above matrix is a \( 3 \times 4 \) matrix (read as 3 "by" 4), which denotes its size. The size of a matrix tells us how big it is.

In general, the size of a matrix is given as \( (# \text{ rows}) \times (# \text{ columns}) \).

Similarly,
\[ \begin{bmatrix} 1 & \ast & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \]
is in reduced echelon form.

Notice that the entry denoted by \( \ast \) is not a pivot. As such, it could be zero or nonzero - in either case, the matrix is in reduced echelon form.

\[ \begin{bmatrix} \ast & 0 & \ast & 0 \\ 0 & \ast & \ast & 0 \\ * & 0 & 0 & 1 \end{bmatrix} \]
is not in echelon form, though. When we are talking about such general forms, we consider all possible values for \( \ast \). So, when \( \ast \) in the bottom left is indeed \( \neq 0 \), the matrix is not in echelon form.
**Solution of linear systems**

We can use row reduction to solve linear systems.

* form the augmented matrix.
* reduce to echelon form.
  - if the echelon form has a row of the form [0 ... 0 0], the system is inconsistent.
  - if it does not have such a row, convert the matrix to reduced echelon form, and describe the solution(s). We illustrate this step on examples now.

**Prob 10, pg 22**

\[
\begin{bmatrix}
1 & -2 & -1 & 4 \\
2 & 4 & -5 & 6
\end{bmatrix} R_2 + 2R_1 \rightarrow \begin{bmatrix}
1 & -2 & -1 & 4 \\
0 & 0 & 3 & 14
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & -2 & -1 & 4 \\
0 & 0 & 1 & -2
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & -2 & 0 & 2 \\
0 & 0 & 1 & -2
\end{bmatrix}
\]

- \(x_1, x_2\) are basic variables
- \(x_2\) is free

\[\begin{align*}
x_1 - 2x_2 &= 2 \\
x_3 &= -2
\end{align*}\]

**Def**

The variables corresponding to pivot columns in the augmented matrix are called **basic variables**. The remaining variables are called **free variables** (also called nonbasic variables).

**Idea**: Describe (all) solution(s) by expressing the basic variables in terms of the free variables.

\[
x_1 = 2 + 2x_2, \quad x_2 \text{ free} \\
x_3 = -2
\]

Here, the value of \(x_3\) does not depend on \(x_2\).
Here is another example.

**Prob 13, pg 22**

Augmented matrix is given. Solve the corresponding system.

\[
\begin{bmatrix}
1 & -3 & 0 & -1 & 0 & | & -2 \\
0 & 1 & 0 & 0 & -4 & | & 1 \\
0 & 0 & 0 & 1 & 9 & | & 4 \\
0 & 0 & 0 & 0 & 0 & | & 0 \\
\end{bmatrix}
\]

The augmented matrix is in echelon form, and does not have a row of the form \([0 \ldots 0] | \neq 0\). Hence the system is consistent.

Notice that \(x_1, x_2, x_4\) are basic, and \(x_3, x_5\) are free.

\[
\begin{align*}
\begin{bmatrix}
1 & -3 & 0 & -1 & 0 & | & -2 \\
0 & 1 & 0 & 0 & -4 & | & 1 \\
0 & 0 & 0 & 1 & 9 & | & 4 \\
0 & 0 & 0 & 0 & 0 & | & 0 \\
\end{bmatrix} & \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix}
1 & 0 & 0 & -1 & 0 & | & -2 \\
0 & 1 & 0 & 0 & -4 & | & 1 \\
0 & 0 & 0 & 1 & 9 & | & 4 \\
0 & 0 & 0 & 0 & 0 & | & 0 \\
\end{bmatrix} & \xrightarrow{R_1 + 3R_2} \begin{bmatrix}
1 & 0 & 0 & -1 & 0 & | & 1 \\
0 & 1 & 0 & 0 & -4 & | & 1 \\
0 & 0 & 0 & 1 & 9 & | & 4 \\
0 & 0 & 0 & 0 & 0 & | & 0 \\
\end{bmatrix} \\
\end{align*}
\]

Reduced echelon form

\[
\begin{align*}
x_1 - 3x_5 &= 5 \\
x_2 - 4x_5 &= 1 \\
x_4 + 9x_5 &= 4
\end{align*}
\]

\[
\begin{align*}
x_1 &= 5 + 3x_5 \\
x_2 &= 1 + 4x_5 \\
x_3, x_5 &\text{ free} \\
x_4 &= 4 - 9x_5
\end{align*}
\]

also called the parametric solution

\(x_3\) and \(x_5\) are parameters that can be chosen freely.

One can notice that all coefficients of \(x_3\) are zero. Hence, we could just leave out \(x_3\) from the discussion, without affecting the rest of the solution. At the same time, one should not assume that \(x_3=0\), which is effectively what you are doing if you leave it out! Note that \(x_3\) can assume any value, and hence is included as a parameter along with \(x_5\), as one would do by default.
Vector Equations (Section 1.3)

\[ 3x_1 + x_2 = 7 \]
\[ x_1 + 2x_2 = 4 \]

\[
\begin{bmatrix}
3 & 1 & 7 \\
1 & 2 & 4
\end{bmatrix}
\xrightarrow{\text{Eros}}
\begin{bmatrix}
1 & 0 & 2 \\
0 & 1 & 1
\end{bmatrix}
\]

\[ x_1 = 2, \ x_2 = 1 \] is the unique solution.

This is the "row picture." We plotted each row as a line, and the unique solution is the point of intersection of these lines.

We now talk about the "column picture."

\[
\begin{bmatrix}
3 & 1 & 7 \\
1 & 2 & 4
\end{bmatrix}
\text{corresponds to}
\begin{bmatrix}
3 \\
1
\end{bmatrix}x_1 + \begin{bmatrix}
1 \\
2
\end{bmatrix}x_2 = \begin{bmatrix}
7 \\
4
\end{bmatrix}.
\]

The columns here are called vectors. In fact, they are 2-vectors. (we specify the # entries as the size).

The set of all 2-vectors is denoted by \( \mathbb{R}^2 \). ("R-two")

\( \mathbb{R}^n \): set of all \( n \) vectors with real entries.

\[ \vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \] is an \( n \)-vector.

The "bar" specifies that it's a vector!

\[ u_j \] (without the "bar") is a scalar for each \( j = 1, 2, \ldots, n \).

We can plot the vectors (in 2D and in 3D).
\[ \begin{bmatrix} 3 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 2 \end{bmatrix} x_2 = \begin{bmatrix} 7 \\ 4 \end{bmatrix} \] (with \( x_1 = 2, x_2 = 1 \)).

We scale the vector \( \begin{bmatrix} 3 \\ 1 \end{bmatrix} \) by the number \( x_1 \), and similarly scale \( \begin{bmatrix} 1 \\ 2 \end{bmatrix} \) by \( x_2 \), and add these two resulting vectors, and we should get \( \begin{bmatrix} 7 \\ 4 \end{bmatrix} \).

\[ \begin{bmatrix} 3 \\ 1 \end{bmatrix} 2 + \begin{bmatrix} 1 \\ 2 \end{bmatrix} 1 = \begin{bmatrix} 7 \\ 4 \end{bmatrix} \]

Adding scalar multiples of vectors in this fashion is called taking a linear combination.

**Def**

If \( \overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_n} \) all are \( m \)-vectors, then

\[ \overrightarrow{w} = c_1 \overrightarrow{v_1} + c_2 \overrightarrow{v_2} + \ldots + c_n \overrightarrow{v_n}, \]

where \( c_i \)'s are scalars, is a linear combination of \( \overrightarrow{v_1}, \ldots, \overrightarrow{v_n} \).

The set of all linear combinations is denoted as the span of the vectors.

Denoting \( \overrightarrow{a_1} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \), \( \overrightarrow{a_2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \), \( \text{Span} \{ \overrightarrow{a_1}, \overrightarrow{a_2} \} = \{ \text{all vectors of the form } c_1 \overrightarrow{a_1} + c_2 \overrightarrow{a_2}, \text{for scalars } c_1, c_2 \} \).

E.g., \( -3 \overrightarrow{a_1} + 4 \overrightarrow{a_2} = -3 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + 4 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \times 3 + 4 \times 1 \\ -3 \times 1 + 4 \times 2 \end{bmatrix} = \begin{bmatrix} -5 \\ 5 \end{bmatrix} \).

So, \( \begin{bmatrix} -5 \\ 5 \end{bmatrix} \) is a vector in \( \text{Span} \{ \overrightarrow{a_1}, \overrightarrow{a_2} \} \).
We have already seen questions about when a system of linear equations has solutions, or not. The same questions could be raised in the context of vectors, their span, and linear combinations. Here is an illustration.

Q: Is \([\begin{bmatrix} 8 \\ 3 \end{bmatrix}] \) in \(\text{span}\{\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\}\) ?

Equivalently, are there scalars \(x_1, x_2\) such that

\[
\begin{bmatrix} 3 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 2 \end{bmatrix} x_2 = \begin{bmatrix} 8 \\ 3 \end{bmatrix}
\]

But this question is the same as the following one:

Does the system \(\begin{bmatrix} 3 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 2 \end{bmatrix} x_2 = \begin{bmatrix} 8 \\ 3 \end{bmatrix}\) have a solution?

Or, equivalently, does the system \(\begin{cases} 3x_1 + x_2 = 8 \\ x_1 + 2x_2 = 3 \end{cases}\) have a solution?

\[
\begin{bmatrix} 3 & 1 & 8 \\ 1 & 2 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 8 \\ 0 & 5 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3/5 \end{bmatrix}
\]

\(x_1 = \frac{13}{5}, \ x_2 = \frac{1}{5}\) is the unique solution.

Hence, \(\vec{b} = \begin{bmatrix} 8 \\ 3 \end{bmatrix}\) is in \(\text{span}\{\vec{a}_1, \vec{a}_2\}\), where \(\vec{a}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}\), \(\vec{a}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}\).
Vector form of a system of linear equation:
\[ a_1x_1 + a_2x_2 + \ldots + a_nx_n = \bar{b} \]

The system has a solution if \( \bar{b} \) is a linear combination of the vectors \( \bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n \). Equivalently, if \( \bar{b} \) is in \emph{span} \( \{\bar{a}_1, \ldots, \bar{a}_n\} \) — the set of all linear combinations of \( \bar{a}_1, \ldots, \bar{a}_n \).

The parallelogram rule of vector addition in \( \mathbb{R}^2 \) — the sum \( \bar{u} + \bar{v} \) is the diagonal of the parallelogram formed by \( \bar{u}, \bar{v} \). Equivalently, \( \bar{u} + \bar{v} \) is the fourth vertex of the parallelogram formed by \( \bar{0} \) (origin), \( \bar{u} \), and \( \bar{v} \).

\[ \text{Span} \{[3], [6]\} = ? \]

The span of \([3]\) and \([6]\) is the line through the origin and \([3]\).
\[ \text{Span} \{ \vec{u}, \vec{v} \} = \text{?} \quad \text{where} \]
\[ \vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix}. \]
\[ \{ x_1 \vec{u} + x_2 \vec{v} \} = \text{?} \]
for all \( x_1, x_2 \in \mathbb{R} \)

\[ \text{Span} \{ \vec{u}, \vec{v} \} \text{ is the plane passing through } \vec{0}, \vec{u}, \vec{v}. \]

Notice that in 3D space, 3 points that are not on a single straight line determine a plane uniquely. Imagine a sheet of paper passing through the three points, but extending without limits on all of its four edges.

This illustration of \( \text{span} \{ \vec{u}, \vec{v} \} \) also demonstrates the choice of the word "span." As such, \( \text{span} \{ \vec{u}, \vec{v} \} \) is also referred to as the plane generated by \( \vec{u} \) and \( \vec{v} \) (the origin \( \vec{0} \) is understood to be included implicitly).
\( \vec{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \), \( \vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \). Show that \[ \begin{bmatrix} h \\ k \end{bmatrix} \] is in \( \text{Span}\{\vec{u}, \vec{v}\} \) for every \( h, k \).

Show that \[ \vec{u}_1 + \vec{u}_2 = \begin{bmatrix} h \\ k \end{bmatrix} \] is consistent for all \( h, k \).

\[
\begin{bmatrix}
2 & 2 & h \\
-1 & 1 & k
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1 & h/2 \\
-1 & 1 & k
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1 & h/2 \\
0 & 2 & k
\end{bmatrix}
\]

The system is consistent for every \( h \) and \( k \).

Here \( \text{Span}\{\vec{u}, \vec{v}\} = \mathbb{R}^2 \)

The span is all of \( \mathbb{R}^2 \).

Compare the span here to the span of \( \{\vec{c}^1, \vec{c}^2\} \) seen earlier, which was just a line through the origin. As in the case here, if the span of a set of vectors \( \vec{a}_1, \ldots, \vec{a}_n \) is all of the space in which the vectors sit, then life becomes easy. We know that the system

\[
\vec{a}_1 x_1 + \ldots + \vec{a}_n x_n = \vec{b}
\]

is consistent for every \( \vec{b} \)!
\[ \bar{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad \bar{v} = \begin{bmatrix} 2 \\ 1 \\ 7 \end{bmatrix}, \quad \bar{y} = \begin{bmatrix} h \\ -3 \\ -5 \end{bmatrix}. \] For what values of \( h \) is \( \bar{y} \) in the plane generated by \( \bar{u} \) and \( \bar{v} \)?

Equivalently, for what values of \( h \) is the system
\[ \bar{w}_1 \bar{x}_1 + \bar{w}_2 \bar{x}_2 = \bar{y} \] consistent?

\[
\begin{bmatrix}
1 & -2 & \hbar \\
0 & 1 & -3 \\
-2 & 7 & -5
\end{bmatrix}
\xrightarrow{R_3 + 2R_1}
\begin{bmatrix}
1 & -2 & \hbar \\
0 & 1 & -3 \\
0 & 3 & -5+2\hbar
\end{bmatrix}
\xrightarrow{R_3 - 3R_2}
\begin{bmatrix}
1 & -2 & \hbar \\
0 & 1 & -3 \\
0 & 0 & 4+2\hbar
\end{bmatrix}
= 0 \text{ for a consistent system.}
\]

So, \( \hbar = -2 \).

The matrix form \( A\bar{x} = \bar{b} \) (Section 1.4)

We have already seen this form! For instance,

\[ 3\bar{x}_1 + \bar{x}_2 = 7 \quad \text{augmented matrix } \begin{bmatrix} 3 & 1 & 7 \\ 1 & 2 & 4 \end{bmatrix} \]

\[ [3]_1 \bar{x}_1 + [1]_2 \bar{x}_2 = [7] \quad \text{vector equation} \]

We now write it in matrix form:

\[
\begin{bmatrix}
3 & 1 \\
1 & 2
\end{bmatrix}
\begin{bmatrix}
\bar{x}_1 \\
\bar{x}_2
\end{bmatrix}
= \begin{bmatrix}
7 \\
4
\end{bmatrix} \quad \text{where } A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}, \quad \bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}, \quad \bar{b} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}.
\]
$A\vec{x}$ is a matrix-vector product.

Let $A = [\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n]$ be an $m \times n$ matrix. So, $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ are all $m$-vectors.

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is an $n$-vector.

Then $A\vec{x} = \vec{a}_1x_1 + \vec{a}_2x_2 + \ldots + \vec{a}_nx_n$ is the linear combination of the columns of $A$ with the entries in $\vec{x}$ as scalars or weights.

$A\vec{x} = \vec{b}$ has a solution if and only if

the vector equation $\vec{a}_1x_1 + \vec{a}_2x_2 + \ldots + \vec{a}_nx_n = \vec{b}$ has a solution, which happens if and only if the system represented by the augmented matrix $[A | \vec{b}]$ has a solution.

We now discuss a condition that guarantees $A\vec{x} = \vec{b}$ has a solution, given in terms of the existence of pivots in each row of $A$. This condition is independent of $\vec{b}$.
Prob 16 pg 40

\[
A = \begin{bmatrix}
1 & -2 & -1 \\
-2 & 2 & 0 \\
4 & -1 & 3
\end{bmatrix}, \quad \bar{b} = \begin{bmatrix}
b_1 \\
b_2 \\
b_3
\end{bmatrix}.
\]

Show that \( A\bar{x} = \bar{b} \) is not consistent for all \( \bar{b} \). Describe the collection of \( \bar{b} \) for which it is consistent.

\[
\begin{bmatrix}
A | \bar{b}
\end{bmatrix} = \begin{bmatrix}
1 & -2 & -1 & | & b_1 \\
-2 & 2 & 0 & | & b_2 \\
4 & -1 & 3 & | & b_3
\end{bmatrix}
\]

\[
\xrightarrow{R_2 + 2R_1} \begin{bmatrix}
1 & -2 & -1 & | & b_1 \\
0 & 2 & 2 & | & b_2 + 2b_1 \\
0 & 7 & 7 & | & b_3 - 4b_1
\end{bmatrix}
\]

\[
\xrightarrow{R_3 - \frac{7}{2}R_2} \begin{bmatrix}
1 & -2 & -1 & | & b_1 \\
0 & 2 & 2 & | & b_2 + 2b_1 \\
0 & 0 & 0 & | & b_3 - \frac{3}{2}(b_2 + 2b_1)
\end{bmatrix}
\]

= 0 for system to be consistent

\( 3b_1 + \frac{7}{2}b_2 + b_3 = 0 \), i.e., \( 6b_1 + 7b_2 + 2b_3 = 0 \).

Hence, the system is not consistent for all \( \bar{b} \), but only for those \( \bar{b} \in \mathbb{R}^3 \) that satisfy \( 6b_1 + 7b_2 + 2b_3 = 0 \).

The set of all \( \bar{b} \) for which \( A\bar{x} = \bar{b} \) is consistent is a plane through the origin described by \( 6b_1 + 7b_2 + 2b_3 = 0 \).

Equivalently, \( \text{span} \bar{a}_1, \bar{a}_2, \bar{a}_3 \) is not all of \( \mathbb{R}^3 \), but a plane through origin sitting in \( \mathbb{R}^3 \).
Theorem

The following statements are equivalent for \( A \in \mathbb{R}^{m \times n} \), \( x \in \mathbb{R}^n \), \( b \in \mathbb{R}^m \):

1. \( Ax = b \) has a solution for each \( b \in \mathbb{R}^m \).
2. \( b \) is in the span of the columns of \( A \).
3. Columns of \( A \) span \( \mathbb{R}^m \).
4. Every row of \( A \) has a pivot.

We use this condition to check readily whether \( Ax = b \) is consistent.

Prob 22 pg 41

\[ \bar{v}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \bar{v}_2 = \begin{bmatrix} 0 \\ -3 \end{bmatrix}, \quad \bar{v}_3 = \begin{bmatrix} 4 \\ -2 \end{bmatrix}. \]
Does \( \mathbb{R}^3 \) span \( \bar{v}_1, \bar{v}_2, \bar{v}_3 \)? Why?

Let \( A = [\bar{v}_1 \mid \bar{v}_2 \mid \bar{v}_3] = \begin{bmatrix} 0 & 0 & 4 \\ 0 & -3 & -2 \\ -3 & 9 & -6 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 4 \\ 0 & -3 & -2 \\ 0 & 9 & -6 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \)

Every row of \( A \) has a pivot. So \( \mathbb{R}^3 \) is the span of \( \bar{v}_1, \bar{v}_2, \bar{v}_3 \).

In more detail, since each row of \( A \) has a pivot, \( Ax = b \) is consistent for every \( b \in \mathbb{R}^3 \). Hence, every vector in \( \mathbb{R}^3 \) is in the span of \( \bar{v}_1, \bar{v}_2, \bar{v}_3 \).
Prob 30, pg 41

Construct a 3x3 matrix whose columns do not span \( \mathbb{R}^3 \). Justify for such problems, it's best to create the "minimal", or simplest, example that works.

E.g.,

\[
A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

works.

Similarly,

\[
B = \begin{bmatrix} 3 & 5 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix}
\]

also works.

At least one row should not have a pivot.

Prob 26 pg 41

\[
\vec{u} = \begin{bmatrix} 7 \\ 2 \\ 5 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}, \quad \text{and they satisfy } 2\vec{u} - 3\vec{v} - \vec{w} = \vec{0}.
\]

Find \( x_1, x_2 \) that satisfy

\[
\begin{bmatrix} 7 & 3 \\ 2 & 1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}, \quad \text{without using EROS.}
\]

The given equation can be rewritten as

\[ 2\vec{u} + (-3)\vec{v} = \vec{w}. \]

Hence \( x_1 = 2, x_2 = -3 \) satisfies \( \vec{u}x_1 + \vec{v}x_2 = \vec{w} \), which is the given system.

How about finding a solution for

\[
\begin{bmatrix} 5 & 7 \\ 1 & 2 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}, \quad \text{given the}
\]

same relationship \( 2\vec{u} - 3\vec{v} - \vec{w} = \vec{0} \)?
The given system is \( \overline{w} x_1 + \overline{w} x_2 = \overline{0} \). To read off one solution, we rewrite \( 2\overline{w} - 3\overline{w} - \overline{w} = \overline{0} \) in this form, as follows:

\[
(- \overline{w} + 2\overline{w} = 3\overline{w}) x_1^\frac{1}{3}
\]

\[\implies \left( \frac{-1}{3} \overline{w} + \frac{2}{3} \overline{w} = \overline{w} \right) \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) = \overline{w} \]. Hence \( x_1 = -\frac{1}{3}, \ x_2 = \frac{2}{3} \) is one solution.

We now study how to characterize when \( A\overline{x} = \overline{b} \) is consistent for all \( \overline{b} \). Naturally, we give this characterization in terms of \( A \).
To start with, we study the case when \( \overline{b} = \overline{0} \), i.e., \( \overline{b} = \overline{0} \) is the zero vector.

---

**Homogeneous Systems of Linear Equations** (Section 1.5)

\[ A\overline{x} = \overline{0} \] (all right-hand side entries are zero)

is a **homogeneous system** of linear equations.

\[ \overline{x} = \overline{0} \] (the zero vector) is always a solution, and is called the **trivial solution**.

**Q:** Are there non-trivial solutions to \( A\overline{x} = \overline{0} \)?

Recall that a consistent system has either a unique solution, or has infinitely many solutions. For \( A\overline{x} = \overline{0} \), the trivial solution is always present. Hence, it has nontrivial solutions if it has infinitely many solutions, for which, it must have free variables.

**A:** There are nontrivial solutions if there is at least one free variable.
Prob 6, pg 47

\[ x_1 + 2x_2 - 3x_3 = 0 \]
\[ 2x_1 + x_2 - 3x_3 = 0 \]
\[ -x_1 + x_2 = 0 \]

Does this system have nontrivial solutions? If yes, describe all of them.

\[
\begin{bmatrix}
1 & 2 & -3 \\
2 & 1 & -3 \\
-1 & 1 & 0
\end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix}
1 & 2 & -3 \\
0 & -3 & 3 \\
0 & 3 & -3
\end{bmatrix} \xrightarrow{R_3 + R_2} \begin{bmatrix}
1 & 2 & -3 \\
0 & 3 & 3 \\
0 & 0 & 0
\end{bmatrix}
\]

\[ A \]

To have nontrivial solutions, there must exist at least one free variable. \( x_3 \) is free here, so the system does have nontrivial solutions.

We now describe all its solutions.

\[
\begin{bmatrix}
1 & 2 & -3 \\
0 & 3 & 3 \\
0 & 0 & 0
\end{bmatrix} \xrightarrow{R_2 \times (-\frac{1}{3})} \begin{bmatrix}
1 & 2 & -3 \\
0 & 1 & -1 \\
0 & 0 & 0
\end{bmatrix} \xrightarrow{R_1 - 2R_2} \begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & -1 \\
0 & 0 & 0
\end{bmatrix}
\]

\[ x_1 - x_3 = 0 \]
\[ x_2 - x_3 = 0 \]

Hence, \( \{ x_1 = x_3, \quad x_3 \text{ free} \} \) describes all solutions.

Equivalently, we can write \( \{ x_1 = s, \quad x_2 = s, \quad s \in \mathbb{R} \} \), which is the parametric form.
We now represent the parametric form in an equivalent form involving a vector corresponding to the parameter \( s \).

All solutions can be written in the vector form

\[
\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} x_3, \quad \text{where } x_3 \text{ is free. Equivalently,}
\]

\[
\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} s, \quad s \in \mathbb{R}.
\]

\( \bar{x} \) represents the parametric vector form of all solutions.

We can visualize the solutions as follows. All solutions form a line through the origin along the vector \([1; 1; 0]\) associated with the parameter \( s \).

Notice that the trivial solution corresponds to \( s = 0 \).

It turns out that the solutions for \( A\bar{x} = \bar{b} \) for a nonzero \( \bar{b} \) could be described as a parallel line, obtained by just “shifting” the solutions line for \( A\bar{x} = \bar{0} \) by a vector. More on this picture in the next class...
Solutions of $A\vec{x} = \vec{b}$ for nonzero $\vec{b}$ in terms of solutions to $A\vec{x} = \vec{0}$.

$\begin{align*}
    x_1 + 2x_2 - 3x_3 &= 3 \\
    2x_1 + x_2 - 3x_3 &= 3 \\
    -x_1 + x_2 &= 0
\end{align*}$

Here $\vec{b} = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}$.

Previously, we had $\vec{b} = \vec{0}$.

In Lecture 6, we solved the corresponding homogeneous system, and visualized its solutions in parametric vector form.

We now repeat the same EROs on just $\vec{b} = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}$.

$\begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix} \xrightarrow{R_3 + R_2} \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} \xrightarrow{R_2 \cdot (-\frac{1}{3})} \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \xrightarrow{R_1 - 2R_2} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

The reduced echelon form of $[A|\vec{b}]$ is hence

$\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$x_1 - x_3 = 1$, $x_2$ free

i.e., $x_1 = 1 + \delta$, $\delta \in \mathbb{R}$

$x_2 = 1 + \delta$, parametric form

$x_3 = \text{free}$

$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \delta$, $\delta \in \mathbb{R}$, parametric vector form
\( \mathbf{r} = \mathbf{v}s, s \in \mathbb{R} \) for \( \mathbf{v} = \begin{bmatrix} 1 \end{bmatrix} \) is the parametric vector form of solutions to \( A\mathbf{r} = \mathbf{0} \).

\[ \mathbf{x} = \mathbf{p} + s\mathbf{v}, s \in \mathbb{R} \]

is the parametric vector form for solutions to \( A\mathbf{x} = \mathbf{b} \).

Equation for a line through \( \mathbf{p} \) parallel to \( \mathbf{v} \).

Adding \( \mathbf{p} \) to \( s\mathbf{v} \) is equivalent to moving the vector \( s\mathbf{v} \) in a direction along the line through origin and \( \mathbf{p} \).

In fact, the above observation holds in the case of linear systems of equations in general, as long as the system in question is consistent.

**Theorem**: If \( A\mathbf{x} = \mathbf{b} \) has a solution \( \mathbf{x} = \mathbf{p} \), then all solutions of \( A\mathbf{x} = \mathbf{b} \) are given by \( \mathbf{x} = \mathbf{p} + s\mathbf{v} \), where \( \mathbf{v} \) is any solution of \( A\mathbf{x} = \mathbf{0} \).

Notice that the trivial solution corresponds to the choice \( s = 0 \) for the homogeneous system. For the same value of the parameter in the case of the non-homogeneous system, we get \( \mathbf{x} = \mathbf{p} \) as the solution. So, the origin gets translated to \( \mathbf{p} \).
Describe and compare the solution sets of
\[ x_1 - 2x_2 + 3x_3 = 0 \quad \text{and} \quad x_1 - 2x_2 + 3x_3 = 4. \]

\[ A = \begin{bmatrix} 1 & -2 & 3 \end{bmatrix} \]
\[ \text{basic:} x_1, \quad \text{free:} x_2, x_3 \]
\[ x_1' = 2x_2 - 3x_3, \quad x_2, x_3 \text{ free} \]

\[ \bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} s + \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} t, \quad s, t \in \mathbb{R} \]

\[ [A \mid \mathbf{b}] = \begin{bmatrix} 1 & -2 & 3 & 1 \end{bmatrix} \]
\[ x_1 = 4 + 2x_2 - 3x_3 \]

\[ \bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} t \]

Solutions to \( A\bar{x} = \mathbf{0} \) form a plane through \( \mathbf{0}, \bar{u}, \bar{v} \). And the solutions to \( A\bar{x} = \bar{b} \) form a parallel plane passing through \( \bar{P} \).
Recall

If $\vec{a}_1 = [3]$, $\vec{a}_2 = [2]$, then
\[ \text{span} \{ \vec{a}_1, \vec{a}_2 \} = \mathbb{R}^2. \]

But with $\vec{u} = [6]$, $\text{span} \{ \vec{a}_1, \vec{u} \}$ is just the line through $\vec{0}$ and $\vec{a}_1$.

$\vec{a}_1$ and $\vec{a}_2$ are linearly independent here, i.e., they are not along the same line. While $\vec{a}_1$ and $\vec{u}$ are linearly dependent.

We now extend this idea of being "along the same line" (or not) to arbitrary collections of vectors in high dimensions.

**Def.** The set $\{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \}$ with each $\vec{v}_i \in \mathbb{R}^m$ is linearly independent (LI) if the vector equation
\[ \vec{v}_1 x_1 + \vec{v}_2 x_2 + \ldots + \vec{v}_n x_n = \vec{0} \]
has only the trivial solution $x_1 = \ldots = x_n = 0$.

If there is a non-trivial solution, the set of vectors is linearly dependent (LD).
Since we already know how to check if \( A\vec{x} = \vec{0} \) has only the trivial solution (when there are no free variables), we can use those results to directly answer questions about whether a given set of vectors is LI or not.

Prob 5 pg 60

\[
A = \begin{bmatrix}
0 & -3 & 9 \\
2 & 1 & -7 \\
-1 & 4 & -5 \\
1 & -4 & -2
\end{bmatrix}.
\]

Do the columns of \( A \) form a linearly independent set of vectors?

Equivalently, does \( A\vec{x} = \vec{0} \) have only the trivial solution?

\[
\begin{bmatrix}
0 & -3 & 9 \\
2 & 1 & -7 \\
-1 & 4 & -5 \\
1 & -4 & -2
\end{bmatrix} \xrightarrow{R_1 \leftarrow R_4} \begin{bmatrix}
1 & -4 & -2 \\
2 & 1 & -7 \\
-1 & 4 & -5 \\
0 & -3 & 9
\end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix}
1 & -4 & -2 \\
0 & 9 & -3 \\
-1 & 4 & -5 \\
0 & -3 & 9
\end{bmatrix} \xrightarrow{R_3 + R_1} \begin{bmatrix}
1 & -4 & -2 \\
0 & 9 & -3 \\
0 & 0 & -7 \\
0 & 9 & -3
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & -4 & -2 \\
0 & 9 & -3 \\
0 & 0 & -7 \\
0 & 0 & 24
\end{bmatrix} \xrightarrow{R_4 + 3R_2} \begin{bmatrix}
1 & -4 & -2 \\
0 & 9 & -3 \\
0 & 0 & -7 \\
0 & 0 & 0
\end{bmatrix}
\]

There are no free variables, and hence the system has only the trivial solution. So columns of \( A \) are LI.
We now describe several special cases of sets of vectors, for which we can determine linear (in)dependence more directly than by performing EROs.

**Special Cases**

1. \( \{ \overrightarrow{v} \} \) (Single vector).

   The set \( \{ \overrightarrow{v} \} \) is LI if \( \overrightarrow{v} \neq \overrightarrow{0} \).

   To follow the definition, we are trying to find when does the system \( \overrightarrow{v} \cdot x = \overrightarrow{0} \) have only the trivial solution. Naturally, when \( \overrightarrow{v} \neq \overrightarrow{0} \), we can get the zero vector only by taking \( x = 0 \).

We will discuss three more special cases in the next lecture...
Special cases of linear (in)dependence

1. \( \{ \mathbf{v} \} \) is LI if \( \mathbf{v} \neq \mathbf{0} \).

2. \( \{ \mathbf{v}_1, \mathbf{v}_2 \} \) is LI if one of them is not a scalar multiple of the other vector.
   
   If \( \mathbf{v}_1 = c \mathbf{v}_2 \) for scalar \( c \), then \( \mathbf{v}_1 - c \mathbf{v}_2 = \mathbf{0} \). So \( \mathbf{x} = [x_1, x_2] = [1, -c] \) is a nontrivial solution to \( \mathbf{v}_1 x_1 + \mathbf{v}_2 x_2 = \mathbf{0} \), and hence the set of vectors is LD.

3. If \( \mathbf{0} \) is in the set \( \{ \mathbf{v}_1, \ldots, \mathbf{v}_n \} \), the set is LD.
   
   E.g., let \( \mathbf{v}_2 = \mathbf{0} \). Then
   
   \[
   0 \mathbf{v}_1 + c \mathbf{v}_2 + 0 \mathbf{v}_3 + \ldots + 0 \mathbf{v}_n = \mathbf{0}
   \]
   
   for any \( c \neq 0 \).

   Hence \( x_1 = 0, x_2 = c, x_3 = x_4 = \ldots = x_n = 0 \) is a nontrivial solution.

4. \( \{ \mathbf{v}_1, \ldots, \mathbf{v}_n \} \), where \( \mathbf{v}_j \in \mathbb{R}^m \), with \( n > m \) is LD.

There are more vectors than the number of entries in each vector.

E.g., \( \{ \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -5 \end{bmatrix}, \begin{bmatrix} 1 \end{bmatrix} \} \) is LD.

Notice that any two vectors out of the three are LI.
Consider \( A\vec{x} = \vec{0} \), where \( A \in \mathbb{R}^{m \times n} \) with \( A = [\vec{v}_1 \ \vec{v}_2 \ldots \ \vec{v}_n] \).

The maximum number of pivots possible is \( m \). So, there are \( n-m \) free variables.

**Prob 21, pg 61 T/F statements**

21. a. The columns of a matrix \( A \) are linearly independent if the equation \( A\vec{x} = \vec{0} \) has the trivial solution.

   b. If \( S \) is a linearly dependent set, then each vector is a linear combination of the other vectors in \( S \).

   c. The columns of any \( 4 \times 5 \) matrix are linearly dependent.

   d. If \( \vec{x} \) and \( \vec{y} \) are linearly independent, and if \( \{\vec{x}, \vec{y}, \vec{z}\} \) is linearly dependent, then \( \vec{z} \) is in Span \( \{\vec{x}, \vec{y}\} \).

(a) **F**. The columns are LI if \( A\vec{x} = \vec{0} \) has only the trivial solution. Recall that \( A\vec{x} = \vec{0} \) always has the trivial solution.

(b) **F**. \( \{[1], [2]\} \) is LD, but \([1] \neq c[2]\) for any \( c \). It is only required that one vector in \( S \) is a linear combination of the others, not each vector.

(c) **T**. A set of \( n \) vectors each with \( m \) entries is LD if \( n > m \).

(d) **T**. \( \vec{z} \) can be written as a linear combination of \( \vec{x} \) and \( \vec{y} \). If \( \vec{z} \in \text{Span}\{\vec{x}, \vec{y}\} \), as \( \{\vec{x}, \vec{y}\} \) is LI (so \( c = 0 \) would mean \( a = b = 0 \) as well). Hence \( \vec{z} = (\frac{a}{c})\vec{x} + (\frac{b}{c})\vec{y} \), i.e., \( \vec{z} \in \text{Span}\{\vec{x}, \vec{y}\} \).
Linear Transformations (LT) "mappings"

\[ A = \begin{bmatrix} 1 & -5 & -7 \\ -3 & 7 & 5 \end{bmatrix}, \quad \vec{u} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} \]

We now talk about another context in which the matrix-vector product \( A\vec{x} \) shows up, which is somewhat different from the systems \( A\vec{x} = \vec{b} \) that we've been discussing so far.

A "acts on" \( \vec{u} \) to transform it to \( \vec{v} \).

In general, A "acts on" \( \vec{x} \in \mathbb{R}^3 \) to give \( A\vec{x} \in \mathbb{R}^2 \).

\[ A\vec{u} = \begin{bmatrix} 1 & -5 & -7 \\ -3 & 7 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} \]

\[ A\vec{v} = \begin{bmatrix} 1 & -5 & -7 \\ -3 & 7 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]
Another notation is to write $\bar{x} \mapsto A\bar{x}$. This correspondence is a function.

**Def.** A transformation $T$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns every vector $\bar{x}$ in $\mathbb{R}^n$ a vector $T(\bar{x})$ in $\mathbb{R}^m$.

$T(\bar{x})$ is the **image** of $\bar{x}$ under $T$.

The set of all images is the **range** of $T$.

We are interested in **matrix transformations**, which are defined as follows.

$T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is defined as $T(\bar{x}) = A\bar{x}$ for $A \in \mathbb{R}^{m \times n}$.

$\bar{x} \mapsto A\bar{x}$ is another notation.
Prob 2, pg 68

\[ A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \vec{u} = \begin{bmatrix} 2 \\ 6 \\ -9 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}. \]

Find images of \(\vec{u}\) and \(\vec{v}\) under \(T: \mathbb{R}^3 \rightarrow \mathbb{R}^3\) given by \(T(\vec{x}) = A\vec{x}\).

\[ T(\vec{u}) = A\vec{u} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \\ -9 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ -9 \end{bmatrix}. \]

\[ T(\vec{v}) = A\vec{v} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}. \]

Prob 5, pg 68

\[ A = \begin{bmatrix} 1 & -5 & -7 \\ -3 & 7 & 5 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ -2 \end{bmatrix}. \]

Find \(\vec{x}\) such that \(T(\vec{x}) = A\vec{x} = \vec{b}\). Is \(\vec{x}\) unique?

We have seen how to solve the system \(A\vec{x} = \vec{b}\), and to decide if the system has a unique solution when it is consistent. The same results could be used to answer such questions about linear transformations.

This problem will be finished in the next lecture...
Review

We are going to slow down a bit, as the other sections are a bit behind. We will review some concepts in this lecture.

\[ A = \begin{bmatrix} 1 & x_2 \\ 0 & 2 \end{bmatrix}, \quad x \text{ basic, } x_2 \text{ free} \]

Trivial solution to \( Ax = 0 \) is \( x = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \).

\[ \begin{bmatrix} 1 & 2 & 0 \end{bmatrix} \text{ in reduced echelon form} \]

\[ x_1 + 2x_2 = 0 \quad x_2 \text{ free} \quad x_1 = -2s, \ s \in \mathbb{R} \]

\[ \bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}s, \ s \in \mathbb{R}. \]

E.g., \( s = 1 \), \( \bar{x} = \begin{bmatrix} -4 \\ 2 \end{bmatrix} \quad (-4) + 2(2) = 0 \)

So, pivot in every column of \( A \) means \( Ax = \bar{x} \) has only the trivial solution.

Section 14  Page 41, Prob 18

\[ B = \begin{bmatrix} 1 & 4 & 1 & 2 \\ 0 & 1 & 3 & -4 \\ 0 & 2 & 6 & 7 \\ 2 & 9 & 5 & -7 \end{bmatrix} \]

Do the columns of \( B \) span \( \mathbb{R}^4 \)?

Every vector in \( \mathbb{R}^4 \) can be written as a linear combination of the columns of \( B \), if \( B \) has a pivot in every row.

Equivalently, if \( B \) has a pivot in every row, the span of its columns is (all of) \( \mathbb{R}^4 \).
\[
B = \begin{bmatrix}
2 & 4 & 1 & 2 \\
0 & 1 & 3 & -4 \\
0 & 2 & 6 & 7 \\
2 & 9 & 5 & -7 \\
\end{bmatrix} \xrightarrow{R_4 - 2R_1} \begin{bmatrix}
1 & 4 & 1 & 2 \\
0 & 1 & 3 & -4 \\
0 & 2 & 6 & 7 \\
0 & 1 & 3 & -11 \\
\end{bmatrix} \xrightarrow{R_3 - 2R_2} \begin{bmatrix}
1 & 4 & 1 & 2 \\
0 & 1 & 3 & -4 \\
0 & 0 & 0 & 15 \\
0 & 0 & 0 & -7 \\
\end{bmatrix} \xrightarrow{R_4 + \frac{7}{15}R_3} \begin{bmatrix}
1 & 4 & 1 & 2 \\
0 & 1 & 3 & -4 \\
0 & 0 & 0 & 15 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Since every row does not have a pivot, \(-\)span\(\) columns of \(B\) \(-\neq\) \(\mathbb{R}^4\).

\(B\) has 3 pivots.

Q: Do columns of \(B\) span \(\mathbb{R}^3\)?

\(\mathbb{R}^3\)

Reword: Can you write every vector in \(\mathbb{R}^4\) as a combination of columns of \(B\)?

No! As columns of \(B\) sit in \(\mathbb{R}^4\), and not in \(\mathbb{R}^3\).

Every column of \(B\) has four entries, while any vector \(\vec{u}\) in \(\mathbb{R}^3\) has three entries. So, we cannot write \(\vec{u}\) as \(b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4\).

Similarly, if \(B\) had only 2 pivots, its columns would still not span \(\mathbb{R}^2\).
24. a. Every matrix equation $Ax = b$ corresponds to a vector equation with the same solution set.

b. If the equation $Ax = b$ is consistent, then $b$ is in the set spanned by the columns of $A$.

c. Any linear combination of vectors can always be written in the form $Ax$ for a suitable matrix $A$ and vector $x$.

d. If the coefficient matrix $A$ has a pivot position in every row, then the equation $Ax = b$ is inconsistent.

e. The solution set of a linear system whose augmented matrix is $[a_1 \ a_2 \ a_3 \ b]$ is the same as the solution set of $Ax = b$, if $A = [a_1 \ a_2 \ a_3]$.

f. If $A$ is an $m \times n$ matrix whose columns do not span $\mathbb{R}^m$, then the equation $Ax = b$ is consistent for every $b$ in $\mathbb{R}^m$.

(a) T. If $A = [\bar{a}_1 \ \bar{a}_2 \ \ldots \ \bar{a}_n]$, then $A\bar{x} = \bar{b}$ corresponds to the vector equation $\bar{a}_1x_1 + \bar{a}_2x_2 + \ldots + \bar{a}_nx_n = \bar{b}$.

(b) T. If $\bar{x}$ is a solution, we can write $\bar{b} = \bar{a}_1x_1 + \ldots + \bar{a}_nx_n$, where $\bar{a}_i$'s are the columns of $A$.

(c) T. Same reason as above.

(d) F. Pivot in every row means $A\bar{x} = \bar{b}$ is consistent for every $\bar{b}$.

(e) T. From the definition.

(f) F. If columns of $A$ span $\mathbb{R}^m$, then $A\bar{x} = \bar{b}$ has a solution for every $\bar{b} \in \mathbb{R}^m$. 

Section 1.5 pg 48, prob 26.

A is the 3x3 zero matrix. Describe solutions of $A \bar{x} = \bar{0}$

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is in reduced echelon form.

$x_1, x_2, x_3$ are all free variables.

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u, \quad s, t, u \in \mathbb{R}.$$

So, the solution set is all of $\mathbb{R}^3$.

Prob 27 (pg 48)

$A \bar{x} = \bar{b}$ is consistent. Explain why it has a unique solution precisely when $A \bar{x} = \bar{0}$ has only the trivial solution.

$A \bar{x} = \bar{0}$ has only the trivial solution if $A$ has no free variables.

We could use the same set of EROs that take $A$ to echelon form, and apply them to $[A \mid \bar{b}]$. Hence, the system $A \bar{x} = \bar{b}$ has free variables if and only if $A$ has free variables.
A similar question

Let $A\bar{x} = \bar{b}$ have nontrivial solutions. Can you guarantee that $A\bar{x} = \bar{b}$ always has infinitely many solutions?

The statement holds as long as $A\bar{x} = \bar{b}$ is consistent.

\[
A = \begin{bmatrix} 1 & x_2 \\ 0 & 0 \end{bmatrix}
\]

$x_2$ free So $A\bar{x} = \bar{b}$ has nontrivial solutions.

But $A\bar{x} = \begin{bmatrix} 1 \\ 0 & 0 & 1 \end{bmatrix}$ is inconsistent!

When answering True/False problems, try to provide the simplest counterexamples when possible, as above.
Transformations

\[ T(x) \] is the image of \( x \) under transformation \( T \). The set of all images under \( T \) is called the range of \( T \).

\[ T: \mathbb{R}^n \to \mathbb{R}^m \quad \text{or} \quad x \mapsto T(x) \]

Matrix Transformation

\[ T(x) = Ax \quad \text{or} \quad x \mapsto Ax \quad \text{for} \quad A \in \mathbb{R}^{m \times n} \]

\[ \text{Prob 5, pg 68} \quad \text{continued from Lecture 8...} \]

\[ A = \begin{bmatrix} 1 & 5 & -7 \\ -3 & 7 & 5 \end{bmatrix}, \quad \bar{b} = \begin{bmatrix} -2 \\ -2 \end{bmatrix} \]

Find \( x \) such that \( T(x) = Ax = \bar{b} \). Is this \( x \) unique?

We will use the results on solutions of systems of the form \( Ax = \bar{b} \) to answer questions of this form.
Reword: Find a solution to $A\vec{x} = \vec{b}$. Is the solution unique?

\[
\begin{bmatrix}
1 & -5 & -7 \\
-3 & 7 & 5 \\
-2 & 0 & 1
\end{bmatrix}
R_2 \leftrightarrow R_1
\begin{bmatrix}
1 & -5 & -7 \\
0 & 16 & -8 \\
0 & 1 & 2 \\
\end{bmatrix}
R_2 \times \frac{1}{8}
\begin{bmatrix}
1 & -5 & -7 \\
0 & 1 & 2 \\
0 & 0 & 1
\end{bmatrix}
\begin{array}{r}
x_1 = 3, \\
x_2 = 1 \\
x_3 \text{ is free}
\end{array}
\]

$\vec{x} = 3\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is one $\vec{x}$ such that $T(\vec{x}) = \vec{b}$. $\vec{x}$ is not unique.

$A$ is $2 \times 3$, so $T(\vec{x}) = A\vec{x} : \mathbb{R}^3 \to \mathbb{R}^2$.

$x_1 = 3 - 3x_3 \\
x_2 = 1 - 2x_3$

So $\vec{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} -3 \\ -2 \end{bmatrix} \bar{s}$, $\bar{s} \in \mathbb{R}$.

All the $\vec{x}$ (as described above) have $T(\vec{x}) = A\vec{x} = \vec{b}$.

**Linear Transformations**

$T: \mathbb{R}^n \to \mathbb{R}^m$ is a **linear transformation** if

(i) $T(\bar{u} + \bar{v}) = T(\bar{u}) + T(\bar{v})$ for all $\bar{u}, \bar{v} \in \mathbb{R}^n$, and

(ii) $T(c\bar{u}) = cT(\bar{u})$ for $\bar{u} \in \mathbb{R}^n$, $c \in \mathbb{R}$. "in" or "element of"

$c$ is a scalar

In words, we say that $T$ preserves vector addition and scalar multiplication.
(ii) immediately implies that

$T(\bar{0}) = \bar{0}$, as $T(0, \bar{u}) = 0 \cdot T(\bar{u}) = \bar{0}$ for any $\bar{u}$.

(iii) $T(c \bar{u} + d \bar{v}) = cT(\bar{u}) + dT(\bar{v})$ for $\bar{u}, \bar{v} \in \mathbb{R}^n$ and $c, d \in \mathbb{R}$ (or scalars).

We could specify (iii) alone as the definition of a linear transformation.

**Prob 17, pg 68**

$T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is linear. It is given that

for $\bar{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, $\bar{v} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$, $T(\bar{u}) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, $T(\bar{v}) = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

Find $T(2\bar{u}), T(3\bar{v})$, and $T(2\bar{u} + 3\bar{v})$.

$T(2\bar{u}) = 2T(\bar{u}) = 2 \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$.

$T(3\bar{v}) = 3T(\bar{v}) = 3 \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 9 \end{bmatrix}$.

$T(2\bar{u} + 3\bar{v}) = T(2\bar{u}) + T(3\bar{v}) = \begin{bmatrix} 8 \\ 2 \end{bmatrix} + \begin{bmatrix} -3 \\ 9 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$.

Equivalently, $T(2\bar{u} + 3\bar{v}) = 2T(\bar{u}) + 3T(\bar{v}) = 2 \begin{bmatrix} 4 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$.
Prob 15, pg 68

\[ \mathbf{u} = \begin{bmatrix} \frac{5}{2} \\ \frac{1}{2} \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -2 \\ 4 \end{bmatrix} \]

\[ T(\mathbf{x}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{5}{2} \\ \frac{1}{2} \end{bmatrix} \]

\[ T(\mathbf{u}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix} \]

Notice that \( T \) here flips the \( x \)- and \( y \)-coordinates. Equivalently, \( T \) reflects the vectors through the \( y=x \) line (45° line).

This problem illustrates the applications of linear transformations in image (and video) analysis. For instance, when you rotate a picture that you took with the camera set sideways so that the subjects in the picture are upright involves the application of a linear transformation.

We will discuss more such problems in the next lecture.

We now consider an example that illustrates how we can define the general form of an LT (linear transformation), once we have defined the images of certain canonical vectors under the same LT. Taking a lead from this example, we will define next the matrix of any linear transformation.
Let $\bar{e}_1 = [1, 0]$, $\bar{e}_2 = [0, 1]$ under $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $T(\bar{e}_1) = \bar{y}_1 = [\frac{2}{5}]$, and $T(\bar{e}_2) = \bar{y}_2 = [-\frac{1}{6}]$. Find $T([\frac{5}{3}])$ and $T([\frac{x_1}{x_2}])$. 

Since $T$ is a linear transformation, $T(c\bar{e}_1 + d\bar{e}_2) = cT(\bar{e}_1) + dT(\bar{e}_2)$.

$$[\frac{5}{3}] = 5[1, 0] + (-3)[0, 1] \quad T([\frac{5}{3}]) = 5T(\bar{e}_1) + (-3)T(\bar{e}_2)$$

$$= 5[\frac{2}{5}] + (-3)[-\frac{1}{6}] = [\frac{13}{7}]$$

$c = 5$, $d = -3$

Similarly, $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1[1, 0] + x_2[0, 1] = x_1\bar{e}_1 + x_2\bar{e}_2$.

$$T(\bar{x}) = x_1T(\bar{e}_1) + x_2T(\bar{e}_2) = x_1[\frac{2}{5}] + x_2[-\frac{1}{6}] = [\frac{2x_1 - x_2}{5x_1 + 6x_2}]$$

Hence

$$\operatorname{Check:} T([\frac{5}{3}]) = [\frac{2(5) - (-3)}{5(5) + 6(-3)}] = [\frac{13}{7}]$$
Matrix of a Linear Transformation

Given a LT $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$, we can write

$T(\bar{x}) = A\bar{x}$, where $A = [T(e_1) \ T(e_2) \ ... \ T(e_n)]$ where

$\bar{e}_j = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$ is the $j^{th}$ entry. 
$\bar{e}_j$ is the $j^{th}$ unit vector.

In the above example, $A = \begin{bmatrix} \bar{g}_1 \\ \bar{g}_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.
Midterm on Thursday, Oct 3, during lecture, in Todd 125.

Practice midterm and study guide are posted on the course web page.

The matrix of an LT

**Theorem 10**

T: \( \mathbb{R}^n \to \mathbb{R}^m \) is a linear transformation. Then

\[
T(\bar{x}) = A\bar{x}, \quad \text{where} \quad A \in \mathbb{R}^{m \times n}
\]

\[
A = \begin{bmatrix} T(\bar{e}_1) & T(\bar{e}_2) & \ldots & T(\bar{e}_n) \end{bmatrix}, \quad \text{where}
\]

\[
\bar{e}_j = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \ldots \]
\]

\[\text{\(j^{th}\) position is the \(j^{th}\) unit vector.}\]

**Proof idea:** Any vector \( \bar{x} \in \mathbb{R}^n \) can be written as a unique linear combination of the unit vectors \( \bar{e}_j \), \( j=1, \ldots, n \).

\[
\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \ldots + x_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \quad \text{or} \quad x_1 \bar{e}_1 + x_2 \bar{e}_2 + \ldots + x_n \bar{e}_n.
\]
For an LT $T$, we have $T(c\vec{u} + d\vec{v}) = cT(\vec{u}) + dT(\vec{v})$. Extending to $n$ vectors (in place of 2), we get that $T(c_1\vec{u}_1 + c_2\vec{u}_2 + \ldots + c_n\vec{u}_n) = c_1T(\vec{u}_1) + c_2T(\vec{u}_2) + \ldots + c_nT(\vec{u}_n)$. Hence,

$$T\left(x_1\vec{e}_1 + x_2\vec{e}_2 + \ldots + x_n\vec{e}_n\right) = x_1T(\vec{e}_1) + x_2T(\vec{e}_2) + \ldots + x_nT(\vec{e}_n).$$

$$= \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & \ldots & T(\vec{e}_n) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \begin{array}{c} A \\ \vec{x} \end{array}$$

We illustrate this result by specifying the matrix of several LTs in 2D that have geometric descriptions.

**Geometric Linear Transformations in 2D**

1. Rotation by an angle $\varphi$ (counter clockwise)

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$T(\vec{e}_1) = \begin{bmatrix} \cos \varphi \\ \sin \varphi \end{bmatrix}$$

$$T(\vec{e}_2) = \begin{bmatrix} -\sin \varphi \\ \cos \varphi \end{bmatrix}$$

$$A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix} = \begin{bmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{bmatrix}.$$
e.g., \( \varphi = 45^\circ \), \( \cos \varphi = \sin \varphi = \frac{1}{\sqrt{2}} \).

\[
A = \begin{bmatrix}
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{bmatrix}
\]

Check out the "Mangle the Coup" link posted on the course web page:
http://www.math.wsu.edu/faculty/hudelson/transform.html

2. **Reflections**

Prob 10, pg 78

\[
T(\vec{e}_1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

\[
T(\vec{e}_2) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}
\]

\[
A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}
\]

3. **Shears**

Prob 4, pg 78

\[
T(\vec{e}_1) = \vec{e}_1, \quad T(\vec{e}_2) = \vec{e}_2 + 2\vec{e}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}
\]

\[
A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}
\]
4. Projections

E.g., project vectors onto the horizontal axis.

\[ T(\overline{e}_1) = \overline{e}_1 \]
\[ T(\overline{e}_2) = \overline{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

\[ \overline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad T(\overline{x}) = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}. \]

The results we have already seen on existence and uniqueness of solutions to \( Ax = \overline{b} \) could be used to answer similar questions in the context of linear transformations. We first define certain types of L'Ts corresponding to these concepts.

Existence and uniqueness questions for L'Ts

Onto and one-to-one transformations

**Def** \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is onto if each \( \overline{b} \) in \( \mathbb{R}^m \) is the image of at least one \( \overline{x} \) in \( \mathbb{R}^n \).

\[ T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is one-to-one if each \( \overline{b} \) in \( \mathbb{R}^m \) is the image of at most one \( \overline{x} \) in \( \mathbb{R}^n \).

\( \Rightarrow \) some \( \overline{b} \) could have no \( \overline{x} \) getting mapped to it, in this case.
$\mathbb{R}^n \quad \mathbb{T} \quad \text{T is not onto} \quad \mathbb{R}^m$

$\mathbb{R}^n \quad \mathbb{T} \quad \text{T is onto} \quad \mathbb{R}^m$

$\mathbb{R}^n \quad \mathbb{T} \quad \text{T is not 1-to-1} \quad \mathbb{R}^m$

$\mathbb{T} \quad \text{T is one-to-one} \quad \mathbb{R}^n \quad \mathbb{R}^m$

range $= \text{codomain}$

two different $\bar{x}$ ($\bar{x}_1$ and $\bar{x}_2$) are mapped to the same $\bar{b}$.

does not have any $\bar{x}$ mapped to it
\[ T: \mathbb{R}^n \rightarrow \mathbb{R}^m \quad T(\bar{x}) = A\bar{x} \]

is onto if \( A \) has a pivot in every row.

is one-to-one if \( A \) has a pivot in every column.

Recall that if \( A \) has a pivot in every row, \( A\bar{x} = \bar{b} \) is consistent for every \( \bar{b} \in \mathbb{R}^m \). Similarly, if \( A \) has a pivot in every column, then there cannot exist any free variables, and hence \( A\bar{x} = \bar{b} \) has a unique solution, or is inconsistent.
22. Let \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) be a linear transformation with \( T(x, y) = (2x - y, 2x + y, 3x) \), find \( x \) such that \( T(x) = (0, 1, -4) \).

Recall: \( \mathbb{R}^n \rightarrow \mathbb{R}^m \) is an LT, then \( T(x) = Ax \), where \( A \in \mathbb{R}^{m \times n} \) with \( A = \begin{bmatrix} T(e_1) & T(e_2) & \cdots & T(e_n) \end{bmatrix} \).

\[ T(x) = Ax \]

The A matrix such that

\[ T(1, 0) = \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 3 & 0 \end{bmatrix} \]

\[ T(0, 1) = \begin{bmatrix} 0 & 2 \\ 1 & 2 \\ 0 & 3 \end{bmatrix} \]

\[ \bar{e} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \]

Find \( x \) s.t. \( T(x) = (0, 1, -4) \), where \( x = \begin{bmatrix} 1 \\ -4 \end{bmatrix} \).
\[
\begin{bmatrix}
2 & -1 & 0 \\
-3 & 1 & -1 \\
2 & -3 & -4 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & -\frac{1}{2} & 0 \\
-3 & 1 & -1 \\
2 & -3 & -4 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & -\frac{1}{2} & 0 \\
0 & -\frac{1}{2} & -1 \\
0 & -2 & -4 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & -\frac{1}{2} & 0 \\
0 & 1 & 2 \\
0 & 0 & 0 \\
\end{bmatrix}
\]

\[\overline{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}\] is the unique vector such that 
\[T(\overline{x}) = \overline{b}\] here.
(as there is a pivot in every column)

**Onto and 1-to-1 transformations**

\[T : \mathbb{R}^n \rightarrow \mathbb{R}^m\] \[T(\overline{x}) = A\overline{x}, \ A \in \mathbb{R}^{m \times n}\]

Recall

\[T\] is **onto** if every \(\overline{b} \in \mathbb{R}^m\) has at least one \(\overline{x} \in \mathbb{R}^n\) such that \(T(\overline{x}) = \overline{b}\).

\[T\] is **1-to-1** if every \(\overline{b} \in \mathbb{R}^m\) has at most one \(\overline{x} \in \mathbb{R}^n\) such that \(T(\overline{x}) = \overline{b}\).

**Theorem 12**

1. \(T\) maps \(\mathbb{R}^n\) onto \(\mathbb{R}^m\) if and only if columns of \(A\) span \(\mathbb{R}^m\), i.e., \(A\) has a pivot in every row.
   \[\Rightarrow \text{"if and only if"}\]

2. \(T\) is one-to-one if and only if the columns of \(A\) are LI, i.e., \(A\) has a pivot in every column.
Probs 29, 30, pg 79

Describe all possible echelon forms.

29. \( T : \mathbb{R}^3 \to \mathbb{R}^4 \) is 1-to-1

We want all echelon forms of a \( 4 \times 3 \) matrix with a pivot in every column.

\[
\begin{bmatrix}
\bullet & \star & \star \\
0 & \bullet & \star \\
0 & 0 & \bullet \\
0 & 0 & 0
\end{bmatrix}
\]

is the only possible echelon form.

30. \( T : \mathbb{R}^4 \to \mathbb{R}^3 \) is onto.

We are looking for echelon forms of a \( 3 \times 4 \) matrix with a pivot in every row.

\[
\begin{bmatrix}
\bullet & \star & \star & \star \\
0 & \bullet & \star & \star \\
0 & 0 & \bullet & \star \\
0 & 0 & 0 & \bullet
\end{bmatrix}, \quad \begin{bmatrix}
\bullet & \star & \star & \star \\
0 & 0 & \bullet & \star \\
0 & 0 & 0 & \bullet
\end{bmatrix}, \quad \begin{bmatrix}
\bullet & \star & \star & \star \\
0 & 0 & 0 & \bullet \\
0 & 0 & 0 & \bullet
\end{bmatrix}, \quad \begin{bmatrix}
0 & \bullet & \star & \star \\
0 & 0 & \bullet & \star \\
0 & 0 & 0 & \bullet
\end{bmatrix}
\]

are the possible echelon forms.

Reminder

\( \bullet \to \) any nonzero number

\( \star \to \) any number (zero or nonzero)

from Section 1.2
24. a. If $A$ is a $4 \times 3$ matrix, then the transformation $x \mapsto Ax$ maps $\mathbb{R}^3$ onto $\mathbb{R}^4$.

b. Every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a matrix transformation.

c. The columns of the standard matrix for a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ are the images of the columns of the $n \times n$ identity matrix under $T$.

d. A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if each vector in $\mathbb{R}^n$ maps onto a unique vector in $\mathbb{R}^m$.

e. The standard matrix of a horizontal shear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ has the form $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$, where $a$ and $d$ are $\pm 1$.

(a) False. A $4 \times 3$ matrix cannot have a pivot in every row.

(b) True. $T : \mathbb{R}^n \to \mathbb{R}^m$ is an LT means $T(x) = Ax$ where $A = [T(e_1) \ T(e_2) \ldots T(e_n)]$. $e_j$ is the $j$th unit vector.

c. True. The $n \times n$ identity matrix is $\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$, which is $[\bar{e}_1 \ \bar{e}_2 \ \cdots \ \bar{e}_n]$.

(d) False. The definition given is satisfied by any transformation, i.e., by any function (or map). For a 1-to-1 mapping, we need that for every $b \in \mathbb{R}^m$, there must exist at most one $x \in \mathbb{R}^n$ that gets mapped to $b$. 

(c) False. Horizontal shear.

\[ T(\vec{e}_1) = \vec{e}_1 \]
\[ T(\vec{e}_2) = \vec{e}_2 + c \vec{e}_1 = \begin{bmatrix} 1 \\ c \end{bmatrix} \]

if \( c > 0 \), we shear to the right.
\( c < 0 \), we shear to the left.

\[ A = [T(\vec{e}_1) \ T(\vec{e}_2)] = \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} \quad c \in \mathbb{R} \quad (c = 0 \text{ creates no change at all}). \]
Review: Practice midterm

Prob 3: \( \bar{u} = \begin{bmatrix} 3 \\ 2 \\ 0.5 \end{bmatrix}, \quad \bar{v} = \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix}, \quad \bar{w} = \begin{bmatrix} 7 \\ 2 \\ -5 \end{bmatrix}, \quad 3\bar{u} - \bar{v} = 2\bar{w}. \)

\[ A = \begin{bmatrix} \bar{v} & \bar{u} & \bar{w} \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0.5 \\ 5 & 3 & 2 \\ 7 & -1 & -5 \end{bmatrix} \]

\( A\bar{x} = \bar{0} \) is the same as \( \bar{v}x_1 + \bar{u}x_2 + \bar{w}x_3 = \bar{0}, \)

where \( \bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \).  \( 3\bar{u} - \bar{v} = 2\bar{w} \) gives \( \begin{bmatrix} -\bar{v} + 3\bar{u} - 2\bar{w} = \bar{0} \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} \)

\( \exists 0, \quad x_1 = -1, \quad x_2 = 3, \quad x_3 = -2 \) is a nontrivial solution.

or \( \bar{x} = \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix} \) is a nontrivial solution.

The idea here is to rewrite the given relation so that it corresponds to the equation in question. As such, we could directly read off a nontrivial solution.
4. \( \mathbf{b} = \begin{bmatrix} 8 \\ -3 \\ 1 \end{bmatrix} \), \( \mathbf{A} \in \mathbb{R}^{3 \times 3} \) such that \( \mathbf{b} \notin \text{span} \{ \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \} \), where \( \mathbf{A} = [\mathbf{a}_1 \mathbf{a}_2 \mathbf{a}_3] \).

Start with \( \mathbf{A} = \begin{bmatrix} 8 & 8 & 8 \\ -3 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix} \), as the last row of this matrix does not have a pivot, while the 3rd entry in \( \mathbf{b} \) is nonzero (\(-1\)).

\[
\begin{align*}
\mathbf{A} & \xrightarrow{R_3 \leftrightarrow R_2} \begin{bmatrix} 8 & 8 & 8 \\ -3 & -3 & -3 \\ -3 & -3 & -3 \end{bmatrix} = \mathbf{A} \\
\text{Or, start with } \mathbf{A} &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \text{ and then } \xrightarrow{R_2 \leftrightarrow R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\end{align*}
\]

you could also, for example, use \( \mathbf{A} = \begin{bmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{bmatrix} \). Just demonstrate that \( \mathbf{A}\mathbf{x} = \mathbf{b} \) is indeed inconsistent.

\[
\begin{align*}
\begin{bmatrix} 5 & 5 & 5 & 8 \\ 5 & 5 & 5 & -3 \\ 5 & 5 & 5 & 1 \end{bmatrix} & \xrightarrow{R_3 - R_1} \begin{bmatrix} 5 & 5 & 5 & 8 \\ 5 & 5 & 5 & -3 \\ 0 & 0 & 0 & -4 \end{bmatrix} \\
\text{inconsistent system!}
\end{align*}
\]
5. \( \overrightarrow{v_1} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}, \overrightarrow{v_2} = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}, \overrightarrow{v_3} = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}, \overrightarrow{v_4} = \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix}. \)

(a) Does \( \{ \overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}, \overrightarrow{v_4} \} \) span \( \mathbb{R}^3 \)?

\[
A = \begin{bmatrix} \overrightarrow{v_1} & \overrightarrow{v_2} & \overrightarrow{v_3} & \overrightarrow{v_4} \end{bmatrix} = \begin{bmatrix} 3 & 6 & 5 & 5 \\ 1 & 2 & -2 & 0 \\ 4 & 1 & 1 & 2 \end{bmatrix}
\]

\[
R_1 \rightarrow R_2 \\
R_2 - 3R_1 \rightarrow R_2 \\
R_3 - 4R_1 \\
\]

\[
\begin{bmatrix} 1 & 2 & -2 & 0 \\ 0 & 0 & 1 & 5 \\ 0 & -7 & 9 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -2 & 0 \\ 0 & 1 & -7 & 9 \\ 0 & 0 & 11 & 5 \end{bmatrix}
\]

pivot in every row. So columns span \( \mathbb{R}^3 \).

(b) Does \( \{ \overrightarrow{v_1}, \overrightarrow{v_2} \} \) span \( \mathbb{R}^3 \)?

No! We need three pivots, but \( A = [\overrightarrow{v_1} \overrightarrow{v_2}] \) can have at most two pivots.

Similarly, \( \{ \overrightarrow{v_1}, \overrightarrow{v_2} \} \) does not span \( \mathbb{R}^2 \), as vectors in \( \mathbb{R}^2 \) have 2 entries each, and cannot be written as linear combinations of \( \overrightarrow{v_1} \) and \( \overrightarrow{v_2} \), which have 3 entries each.

For the same reason, \( \{ \overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}, \overrightarrow{v_4} \} \) cannot span \( \mathbb{R}^4 \).
6. \( T(\bar{e}_2) = \bar{e}_2 + 2\bar{e}_1 \)
   and then reflect on vertical axis
   \[
   T(\bar{e}_1) = -\bar{e}_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}
   \]
   \[
   T(\bar{e}_2) = \bar{e}_2 - 2\bar{e}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}
   \]
   
   \[
   A = \begin{bmatrix} T(\bar{e}_1) & T(\bar{e}_2) \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix}
   \]

8. T/F
   (a) T. See solutions. The reduced echelon form is unique.
   (b) T. There is a pivot in each of the two columns so there are no free variables.
   (c) F. \( A = [0,0] \) \( T(\bar{x}) = A\bar{x} \) is both 1-to-1 and onto.
   (d) \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is an LT, then \( T(\bar{x}) = A\bar{x} \) where

\(\begin{array}{ccc}
\text{domain} & \rightarrow & \text{codomain} \\
A & \in & \mathbb{R}^{m \times n} \\
\end{array}\)

F. \( \mathbb{R}^m \) is the codomain.
\[ \begin{bmatrix} A | b \end{bmatrix} = \begin{bmatrix} 1 & 3 & k \\ 1 & -h & 2 \end{bmatrix} \xRightarrow{R_2 - R_1} \begin{bmatrix} 1 & 3 & k \\ 0 & k - 3 & 2 - k \end{bmatrix} \]

(a) inconsistent if \(-h - 3 = 0\) and \(2 - k \neq 0\), i.e., \(h = -3, \ k \neq 2\).

(b) unique solution if \(-h - 3 \neq 0\), i.e., \(h \neq -3\).

(c) infinitely many solutions when \(-h - 3 = 0\) and \(2 - k = 0\), i.e., \(h = -3, \ k = 2\).
Matrix Operations (Section 2.1)

An $m \times n$ matrix is typically represented as follows.

\[
\begin{bmatrix}
  a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
  a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\
  \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn}
\end{bmatrix}
\]

Row $i$  \[\rightarrow\] \[A_{ij}\]  another notation for matrix $A$

Column representation:

\[A = \begin{bmatrix}
  \vec{a}_1 \\
  \vec{a}_2 \\
  \vdots \\
  \vec{a}_j \\
  \vdots \\
  \vec{a}_n
\end{bmatrix}\]

Each column is an $m$-vector.

Matrix addition

Let $A$ and $B$ be two matrices.

$C' = A + B$ is defined only when both $A$ & $B$ are of the same size, say, $m \times n$. The sum $C'$ is also an $m \times n$ matrix.
Then \[ C_{ij} = A_{ij} + B_{ij} \]

entry in Row-\( i \) & Column-\( j \) of matrix \( C \)

**Scalar multiplication**

Let \( r \) be a scalar. If \( C = rA \), then \( C_{ij} = rA_{ij} \) for all \( i \) and \( j \).

(i.e., multiply each entry by \( r \)).

**Prob 2, pg 116**

\[
A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & 3 \end{bmatrix}
\]

\[
A + 2B = ? \quad 2B = 2\begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & 3 \end{bmatrix} = \begin{bmatrix} 2 \times 7 & 2 \times -5 & 2 \times 1 \\ 2 \times 1 & 2 \times -4 & 2 \times -3 \end{bmatrix} = \begin{bmatrix} 14 & -10 & 2 \\ 2 & -8 & -6 \end{bmatrix}
\]

\[
A + 2B = \begin{bmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix} + \begin{bmatrix} 14 & -10 & 2 \\ 2 & -8 & -6 \end{bmatrix} = \begin{bmatrix} 16 & -10 & 1 \\ 6 & -13 & -4 \end{bmatrix}
\]

**Properties of matrix addition and scalar multiplication**

\( A, B, C \) are \( m \times n \) matrices, \( r, s \) are scalars.

1. \( A + B = B + A \)
2. \( (A+B) + C = A + (B+C) \)
3. \( A + O = O + A = A \)
4. \( r(A+B) = rA + rB \)
5. \( (r+s)A = rA + sA \)
6. \( r(sA) = (rs)A \)

\[ \text{zero matrix}: \text{m} \times \text{n} \text{ matrix of all zeroes} \]
Matrix Products

We have seen $A\vec{x} = [\vec{a}_1 \vec{a}_2 \ldots \vec{a}_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \vec{a}_1 x_1 + \vec{a}_2 x_2 + \ldots + \vec{a}_n x_n$.

Let $A$ be $m \times n$ and $B$ be $n \times p$ matrix.

$C = AB \ ? \quad B = [b_1 \ b_2 \ldots \ b_p]$

We know how to find $A\vec{b}_j$ for $j = 1, 2, \ldots, p$. Then,

$C = [A\vec{b}_1 \ A\vec{b}_2 \ldots \ A\vec{b}_p]$, i.e., find each matrix-vector product $A\vec{b}_j$, stack these products, which are all $m$-vectors, together as columns to get $AB$.

The product $AB$ is defined only when the # columns in $A$ is equal to the # rows in $B$.

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$A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{bmatrix}$, $C = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

$2 \times 3 \quad 2 \times 3 \quad 2 \times 2$

The products $CA$ and $CB$ are defined, but $AC$ (or $BC$) is **not** defined.

In general, $AB \neq BA$. In fact, $BA$ (or $AB$) may not even be defined!


$\begin{bmatrix} 9 & -13 & -5 \\ -13 & 6 & -5 \end{bmatrix}$. 
Row-column definition of $AB$ 

$A_{mxn} \cdot B_{nxp}$

$\begin{bmatrix}
  a_{i1} & a_{i2} & \cdots & a_{in} \\
  \vdots & & & \vdots \\
  b_{i1} & b_{i2} & \cdots & b_{in} \\
\end{bmatrix}$

$\begin{bmatrix}
  c_{ij} \\
  \vdots \\
  b_{nj} \\
\end{bmatrix}$

The product $C$ will be an $m \times p$ matrix.

$C_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$

Properties of matrix multiplication

Assume all matrix products written here are defined.

1. $A(BC) = (AB)C$
2. $A(B+C) = AB + AC$
3. $(B+C)A = BA + CA$
4. $r(AB) = (rA)B = A(rB)$
5. $I_m A = A I_n = A$, where $I_m$ is the $m \times m$ identity matrix

$A_{mxn}$

$I_m = \begin{bmatrix}
  1 & 0 & \cdots & 0 \\
  0 & 1 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & 1 \\
\end{bmatrix}_{m \times m}$
10. Let \( A = \begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix} \), \( B = \begin{bmatrix} -1 & 1 \\ 3 & 4 \end{bmatrix} \), and \( C = \begin{bmatrix} -3 & -5 \\ 2 & 1 \end{bmatrix} \). Verify that \( AB = AC \) and yet \( B \neq C \).

\[
AB = \begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 3 \times 1 + (-6) \times 3 & 3 \times 1 + (-6) \times 4 \\ (-1) \times 1 + 2 \times 3 & (-1) \times 1 + 2 \times 4 \end{bmatrix} = \begin{bmatrix} -21 & -21 \\ 7 & 7 \end{bmatrix}.
\]

\[
AC = \begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -3 & -5 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 \times (-3) + (-6) \times 2 & 3 \times (-5) + (-6) \times 1 \\ (-1) \times (-3) + 2 \times 2 & (-1) \times (-5) + 2 \times 1 \end{bmatrix} = \begin{bmatrix} -21 & -21 \\ 7 & 7 \end{bmatrix}.
\]

Hence \( AB = AC \) here, even though \( B \neq C \).

---

**Transpose of a matrix**

Interchange rows and columns of matrix \( A \) to get \( A^T \).

\[
A = \begin{bmatrix} 2 & 1 & 0 \\ 4 & -2 & 6 \end{bmatrix} \quad A^T = \begin{bmatrix} 2 & 1 \\ 4 & -2 \\ 0 & 6 \end{bmatrix}
\]

If \( A \) is \( m \times n \), then \( A^T \) is \( n \times m \).
Properties of matrix transpose

a. \((A^T)^T = A\).

b. \((A+B)^T = A^T + B^T\)

c. \((rA)^T = rA^T\) → transpose of a product = product of transposes in reverse order.

d. \((AB)^T = B^T A^T\)

e.g., \(A_{2 \times 3}, B_{3 \times 5}\). So \(AB\) is \(2 \times 5\), and \((AB)^T\) is \(5 \times 2\).

\(A^T\) is \(3 \times 2\), \(B^T\) is \(5 \times 3\). Here \(A^T B^T\) is not defined.

But \(B^T A^T\) is \(5 \times 2\).

Probl. 27, pg. 117

\(\vec{u} = \begin{bmatrix} -2 \\ 3 \\ -4 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}\)

\(\vec{u}^T \vec{v} = \begin{bmatrix} -2 & 3 & -4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -2a + 3b - 4c \end{bmatrix}\)

Scalar product of two vectors.

\(\vec{u}^T \vec{v} = \begin{bmatrix} -2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -2a & -2b & -2c \end{bmatrix}\)

\(\begin{bmatrix} 3 \\ -4 \end{bmatrix} \begin{bmatrix} 3 & 3 & 3 \\ -4 & -4 & -4 \end{bmatrix}\)
\[(\mathbf{u} \mathbf{v}^T)^T = (\mathbf{v}^T)^T \mathbf{u}^T = \mathbf{v} \mathbf{u}^T \Rightarrow \text{also a 3x3 matrix}\]
\[(\mathbf{u}^T \mathbf{v})^T = \mathbf{v}^T (\mathbf{u}^T)^T = \mathbf{v}^T \mathbf{u} \Rightarrow \text{same scalar as } \mathbf{u}^T \mathbf{v} (1x1)\]

(Similar to) Problem 17, pg 110

\[A = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}, \quad AB = \begin{bmatrix} -1 & 2 & -1 \\ 6 & -9 & 3 \end{bmatrix}. \quad \text{Find } B?\]

\[2 \times 2 \quad 2 \times 3 \quad \text{AB is 2x3}\]

B should be 2x3, so that AB is indeed 2x3.

Let \[B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}\]

\[AB = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} = \begin{bmatrix} -1 & 2 & -1 \\ 6 & -9 & 3 \end{bmatrix}\]

System 1:

\[\begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \end{bmatrix}\]

\[\begin{bmatrix} 1 & -2 & -1 \\ -2 & 5 & 6 \end{bmatrix} \xrightarrow{R_2+2R_1} \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{R_1+2R_2} \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 4 \end{bmatrix}\]

\[b_{11} = 7, \quad b_{21} = 4\]

System 2:

\[\begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} b_{12} \\ b_{22} \end{bmatrix} = \begin{bmatrix} 2 \\ -9 \end{bmatrix}\]

\[b_{12} = -8, \quad b_{22} = -5\]

System 3:

\[\begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} b_{13} \\ b_{23} \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}\]

\[b_{13} = 1, \quad b_{23} = 1\]

\[B = \begin{bmatrix} 7 & -8 & 1 \\ 4 & -5 & 1 \end{bmatrix}\]
Midterm: Scores for midterm will be curved

Offer:
* If you score 90+/100 in the final, midterm will be replaced by final
* If you score 85-89.9/100, the percentages will be reset as
  midterm - 15%  final - 50%

Inverse of a matrix (Section 2.2)

For nonzero numbers, we have the concept of multiplicative inverse, e.g.,

\[(5)^{-1} = \frac{1}{5},\]

with the property that \[5 \cdot (5)^{-1} = 5 \cdot \frac{1}{5} = 1.\]

We define the analogous concept for matrices.
Def  If \( AB=I \) and \( BA=I \), then \( B \) is called the inverse of \( A \), and \( A \) the inverse of \( B \). We denote this fact by \((A)^{-1}=B\), \(B^{-1}=A\).

\( I \) is the identity matrix. For both \( AB \) and \( BA \) to be defined, \( A,B \) must be square matrices, i.e., \( n \times n \) matrices.

Def  If the inverse of \( A \) exists, then \( A \) is invertible.

e.g., \( A = \begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix} \), \( B = \begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix} \), then

\[
AB = \begin{bmatrix} 2\times4+1\times7 & 2\times(-1)+1\times2 \\ 7\times4+4\times7 & 7\times(-1)+4\times2 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}.
\]
So \( B = A^{-1} \) and \( A = B^{-1} \).

\[
BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\]

In general, if \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \), then \( A^{-1} \) exists if \( ad-bc \neq 0 \), and in this case

\[
A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.
\]

Swap diagonal entries, and change sign of off-diagonal entries.
The quantity $ad - bc$ is called the determinant of the matrix $A$, denoted as $\det(A)$.

A $2 \times 2$ matrix $A$ is invertible if and only if $\det(A) \neq 0$. This extends to $n \times n$ matrices in general.

Check! $AA^{-1} = I$ for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\det(A) = ad - bc \neq 0$

\[
\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} ad + b(c) & -ab + ba \\ cd + d(c) & c(-b) + da \end{bmatrix}
\]

\[
= \frac{1}{ad-bc} \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\]

E.g., $B = \begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix}$  $\det(B) = 4 \cdot 2 - (-1) \cdot (-7) = 1 \neq 0$, so $B^{-1}$ exists.

\[
B^{-1} = \frac{1}{\det(B)} \begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix} = \frac{1}{1} \begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix} (= A)
\]
Why study inverses?

If $x$ is a scalar, and we have the equation $5x = 3$, we could solve for $x$ by multiplying the equation by the inverse of 5:

$$\frac{1}{5}(5x = 3) \Rightarrow \frac{1}{5} \cdot 5x = \frac{1}{5} \cdot 3 \quad \text{i.e.,} \quad x = \frac{3}{5}.$$  

We can extend this result to matrices as follows.

For $A\bar{x} = \bar{b}$ with $A \in \mathbb{R}^{n \times n}$, if $A^{-1}$ exists, then the system has a unique solution for every $\bar{b} \in \mathbb{R}^n$ given as $\bar{x} = A^{-1}\bar{b}$.

**Prob 6 pg 109**

Solve the system using inverses.

$$7x_1 + 3x_2 = -9$$
$$-6x_1 - 3x_2 = 4$$

$A\bar{x} = \bar{b}$ with $A = \begin{bmatrix} 7 & 3 \\ -6 & -3 \end{bmatrix}$, $\bar{b} = \begin{bmatrix} -9 \\ 4 \end{bmatrix}$

$$\det A = 7 \cdot (-3) - (-6) \cdot 3 = -3. \quad \text{So } A^{-1} \text{ exists}.$$  

$$A^{-1} = \frac{1}{-3} \begin{bmatrix} -3 & -3 \\ 6 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -\frac{7}{3} \end{bmatrix}.$$  

$$\bar{x} = A^{-1}\bar{b} = \begin{bmatrix} 1 & 1 \\ -2 & -\frac{7}{3} \end{bmatrix} \begin{bmatrix} -9 \\ 4 \end{bmatrix} = \begin{bmatrix} 5x \cdot -9 + 1 \cdot 4 \\ -2x \cdot 9 + -\frac{7}{3} \cdot 4 \end{bmatrix} = \begin{bmatrix} -5 \\ 24 \end{bmatrix}.$$
Properties of matrix inverses

1. \((A^{-1})^{-1} = A\)
2. \((AB)^{-1} = B^{-1}A^{-1}\)
3. \((A^T)^{-1} = (A^{-1})^T\)

inverse of product = product of inverses in the reverse order

\[ \text{if } AB = I, \text{ then } B = A^{-1} \]
\[ (AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I \]
\[ (B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I \]

How to invert \(n \times n\) matrices in general?

We need to find \(B = A^T\) such that

\[ AB = I \]

We know how to solve \(A\overline{x} = \overline{b}\).

Collection of \(n\) systems all with same \(A\) matrix.

We form the big augmented matrix \([A|I]\), and reduce it to reduced row echelon form.
We saw:

If \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \), then \( A^{-1} \) exists when \( \text{det}(A) = ad - bc \neq 0 \), and in that case \( A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \).

**Inverting \( n \times n \) matrices**

We want to find \( B \) such that \( AB = I \).

**Theorem** \( A \) is invertible if and only if it is row equivalent to \( I \), the identity matrix. In other words, we can transform \( A \) to \( I \) using a sequence of EROs.

The same sequence of EROs will convert \( I \) to \( A^{-1} \).

So \( \begin{bmatrix} A & I \end{bmatrix} \xrightarrow{\text{EROs to reduced echelon form}} \begin{bmatrix} I & A^{-1} \end{bmatrix} \)

If we do not get \( I \) in place of \( A \), \( A \) is not invertible.

If we do not get \( I \) in place of \( A \), it means \( A \) is not row equivalent to \( I \). Hence \( A \) is not invertible.
\[ A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix} \] 

Find \( A^{-1} \) if it exists.

\[ \begin{bmatrix} A & \mathbf{I} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{bmatrix} \]

Perform row operations:

1. \( R_2 + 3R_1 \)
2. \( R_3 - 2R_1 \)
3. \( R_2 + 3R_3 \)
4. \( R_3 \times \frac{1}{2} \)

\[ \begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 7 & 3 & 1 \end{bmatrix} \]

So, \( A \) is invertible, and \( A^{-1} = \begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{bmatrix} \).

Check: \( AA^{-1} = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix} \begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

\[ -3x_1 + x_1 + 4x_\frac{1}{2} = 0 \]

\[ x_1 \times 8 + 0 \times 10 + (-2) \times \frac{7}{2} = 8 - 7 = 1 \]

Of course, you need not check the result in each case. We just did it here for the sake of completeness.
We now look at a few "proof"-type problems, where we will use the many properties of matrix multiplication and inverses that we have discussed so far.

Recall: If there exists B such that \( AB = BA = I \), then \( B = A^{-1} \), \( A = B^{-1} \), \( A \) and \( B \) are both invertible.

**Proof, pg 110**

14. Suppose \((B - C)D = 0\), where \(B\) and \(C\) are \(m \times n\) matrices and \(D\) is invertible. Show that \(B = C\).

\[
(B - C)D = 0 \quad B, C \in \mathbb{R}^{m \times n}, \quad D \text{ is invertible.}
\]

Notice that \(D\) is \(n \times n\), for the product \((B - C)D\) to be defined for a square matrix \(D\).

\[
\begin{align*}
(B - C)D &= 0 \quad \text{multiply on the right by } D^{-1} \\
(B - C)D^{-1} &= 0 \quad \text{as } D^{-1} \text{ exists} \\
(B - C)I &= 0 \\
\therefore \quad B - C &= 0, \quad \text{hence } B = C. \quad \text{(as } AT = A \text{ for any matrix } A) 
\end{align*}
\]
20. Suppose $A$, $B$, and $X$ are $n \times n$ matrices with $A$, $X$, and $A - AX$ invertible, and suppose

$$(A - AX)^{-1} = X^{-1} B$$  \hspace{1cm} (3)$$

a. Explain why $B$ is invertible.
b. Solve equation (3) for $X$. If a matrix needs to be inverted, explain why that matrix is invertible.

(a) We try to find $B^{-1}$. By explicitly giving the expression for $B^{-1}$, we can demonstrate that $B$ is invertible.

\[
X (A - AX)^{-1} = X^{-1} B \\
X (A - AX)^{-1} = X X^{-1} B \\
= I B = B \\
\]

\[
\text{multiply on the left by } X \\
\text{(multiplication is associative)} \\
(XX^{-1} = I) \\
\]

So, $B^{-1} = [X(A-AX)^{-1}]^{-1}$

\[
= [(A-AX)^{-1}]^{-1} X^{-1} \\
= (A-AX) X^{-1} \\
\text{(as } (AB)^{-1} = B^{-1}A^{-1}) \\
\text{(as } (A^{-1})^{-1} = A) \\
\]

Since $B^{-1}$ exists, $B$ is invertible.
(b) To solve \((A-AX)^{-1} = X^{-1}B\) for \(X\), we "isolate" \(X\).

Taking inverse on both sides,

\[
[\begin{pmatrix} A-AX \end{pmatrix}^{-1}]^{-1} = (X^{-1}B)^{-1}
\]


\[
\Rightarrow A-AX = B^{-1}(X^{-1})^{-1} \quad \text{(as } (A^{-1})^\top = A \text{ and } (AB)^{-1} = B^{-1}A^{-1})
\]


\[
\Rightarrow A = AX + B^{-1}X = (A + B^{-1})X \quad \text{(as } A(B+c) = AB + AC)\]

Hence we have \((A + B^{-1})X = A\)

Notice that since \(A\) is invertible, we must have that

\((A + B^{-1})\) is invertible \((\text{since } A^{-1} = X^{-1}[A+B^{-1}]^{-1})\).

\[
\Rightarrow [\begin{pmatrix} A+B^{-1} \end{pmatrix}^{-1}] \begin{cases} (A+B^{-1})X = A \end{cases} \quad (\text{As } A(B+C) = AB + AC)\]

\[
\Rightarrow (A+B^{-1})^{-1}(A+B^{-1})X = (A+B^{-1})^{-1}A
\]

\[
\Rightarrow X = (A+B^{-1})^{-1}A
\]
Properties of invertible matrices (Section 2.3)

We now connect the various properties of systems of equations $A\vec{x} = \vec{b}$, linear transformations, and inverses of matrices. It turns out that many of the properties are equivalent.

Invertible matrix theorem (IMT)

For an $n \times n$ matrix $A$, the following statements are equivalent.

(a) $A$ is invertible.
(b) $A$ is row equivalent to $I_n$.
(c) $A$ has $n$ pivot positions.
(d) $A\vec{x} = \vec{0}$ has only trivial solution.
(e) Columns of $A$ are LI.
(f) The LT $\vec{x} \mapsto A\vec{x}$ is one-to-one.
(g) Columns of $A$ span $\mathbb{R}^n$.

(\text{more statements to follow in the next lecture...})
The invertible matrix theorem

For an \( n \times n \) matrix \( A \), the following statements are equivalent.

(a) \( A \) is invertible.
(b) \( A \) is row equivalent to \( I_n \).
(c) \( A \) has \( n \) pivot positions.
(d) \( A\hat{x} = \hat{0} \) has only trivial solution.
(e) Columns of \( A \) are LI.
(f) The LT \( \hat{x} \mapsto A\hat{x} \) is one-to-one.
(g) \( A\hat{x} = \hat{b} \) has a unique solution for every \( \hat{b} \in \mathbb{R}^n \).
(h) Columns of \( A \) span \( \mathbb{R}^n \).
(i) The LT \( \hat{x} \mapsto A\hat{x} \) maps \( \mathbb{R}^n \) onto \( \mathbb{R}^n \).
(j) There exists \( C \in \mathbb{R}^{n \times n} \) such that \( CA = I_n \).
(k) There exists \( D \in \mathbb{R}^{n \times n} \) such that \( AD = I_n \).
(l) \( A^T \) is invertible. \( (A^{-1})^T = (A^T)^{-1} \)
Determine if matrix is invertible, using as few operations as possible.

4. \[
\begin{bmatrix}
-5 & 1 & 4 \\
0 & 0 & 0 \\
1 & 4 & 9
\end{bmatrix}
\]

There cannot exist \( n = 3 \) pivots (because of the zero row). So matrix is not invertible.

6. \[
\begin{bmatrix}
1 & -3 & -6 \\
0 & 4 & 3 \\
-3 & 6 & 0
\end{bmatrix}
\xrightarrow{R_3 + 3R_1}
\begin{bmatrix}
1 & -3 & -6 \\
0 & 4 & 3 \\
0 & -3 & -18
\end{bmatrix}
\xrightarrow{R_3 + \frac{3}{4}R_2}
\begin{bmatrix}
1 & -3 & -6 \\
0 & 4 & 3 \\
0 & 0 & -63/4
\end{bmatrix}
\]

\( n = 3 \) pivots. So matrix is invertible.

*We could not guess that the matrix is invertible without some EROs here.*

16. If an \( n \times n \) matrix \( A \) is invertible, then the columns of \( A^T \) are linearly independent. Explain why.

If \( A \) is invertible, then \( A^T \) is also invertible (by IMT).

Invertible matrix theorem

So \( A^T \) has a pivot in every column, as it has \( n \) pivots. So, the columns are LI.

For problems in this section, it is not sufficient to bluntly state the IMT as the reason for your conclusions. You need to specify the details of why the conclusion follows. For the same reason, you should not try to memorize the index letters \((b), (f), \text{ or } (j)\), for instance, in the IMT!
22. If \( n \times n \) matrices \( E \) and \( F \) have the property that \( EF = I \), then \( E \) and \( F \) commute. Explain why.

\[ E \text{ and } F \text{ commute means } EF = FE. \text{ In this case, you want to show } EF = FE = I. \]

Since \( EF = I \), by IMT both \( E \) and \( F \) are invertible (statements (j) and (k) in IMT).

Since \( E \) is invertible, and \( EF = I \), \( F = E^{-1} \).

So \( FE = E^{-1}E = I \)

27. Let \( A \) and \( B \) be \( n \times n \) matrices. Show that if \( AB \) is invertible, so is \( A \). You cannot use Theorem 6(b), because you cannot assume that \( A \) and \( B \) are invertible. [Hint: There is a matrix \( W \) such that \( ABW = I \). Why?]

\( AB \) is invertible. By IMT there exist \( n \times n \) matrices \( C \) and \( D \) such that

\[ CAB = I \quad \text{and} \quad ABD = I. \]

Hence \( (CA)B = I \) and \( A(BD) = I \). So there exist \( n \times n \) matrices \( E = CA \) and \( F = BD \) such that \( EB = I \), \( AF = I \).
So both \( A \) and \( B \) are invertible by the IMT.
31. Suppose $A$ is an $n \times n$ matrix with the property that the equation $Ax = b$ has at least one solution for each $b$ in $\mathbb{R}^n$. Without using Theorems 5 or 8, explain why each equation $Ax = b$ has in fact exactly one solution.

If $A$ is invertible, $A\overline{x} = \overline{b}$ has a unique solution $\overline{x} = A^{-1}\overline{b}$

Of course, the result you want to prove here is implied by the theorems you are supposed to "avoid." The idea is to argue from scratch, rather than just state "follows from IMT," for instance.

Since $A\overline{x} = \overline{b}$ has at least a solution for every $\overline{b} \in \mathbb{R}^n$, columns of $A$ span $\mathbb{R}^n$. Hence $A$ has a pivot in every row. But $A$ is $n \times n$, so $A$ has exactly $n$ pivots, and also a pivot in every column. Hence $A\overline{x} = \overline{b}$ has exactly one solution for every $\overline{b} \in \mathbb{R}^n$. 
14. An \( m \times n \) lower triangular matrix is one whose entries above the main diagonal are 0’s (as in Exercise 3). When is a square lower triangular matrix invertible? Justify your answer.

The \( n \times n \) matrix is invertible when all the \( n \) entries in the diagonal are nonzero. In this case, we can use each of these nonzero entries to zero out the entries below the diagonal in each column.

Hence, we will get \( n \) pivots, which makes the matrix invertible by LMT.

Notice that if a diagonal entry is zero, we will not get \( n \) pivots.
Next week: Tuesday: Prof McDonald
Thursday: No class

Subspaces (Section 2.8)

Motivation: \( \mathbb{R}^2 \) is like an infinite sheet of paper sitting on a flat "table", so to say. Now imagine the same infinite sheet sitting in 3D space. When can we guarantee that the desired properties of \( \mathbb{R}^2 \) are all retained by this sheet now sitting in \( \mathbb{R}^3 \)?

**Def**

\( H \) is a **subspace** of \( \mathbb{R}^n \) if

1. The zero vector is in \( H \);
2. For \( \vec{u}, \vec{v} \) in \( H \), \( \vec{u} + \vec{v} \) is also in \( H \); and
3. For \( \vec{u} \) in \( H \) and scalar \( c \), \( c\vec{u} \) is also in \( H \).

\( H \) is closed under vector addition and scalar multiplication.

**Note:** \( \mathbb{R}^n \) as well as \{0\} are both subspaces of \( \mathbb{R}^n \).

Prob 2, pg 151

Exercises 1–4 display sets in \( \mathbb{R}^2 \). Assume the sets include the bounding lines. In each case, give a specific reason why the set \( H \) is not a subspace of \( \mathbb{R}^2 \). (For instance, find two vectors in \( H \) whose sum is not in \( H \), or find a vector in \( H \) with a scalar multiple that is not in \( H \). Draw a picture.)

1. \( c\vec{u} \) is not in \( H \) for any \( c < 0 \).

So \( H \) is not a subspace.

2. \( \vec{u} + \vec{v} \) is not in \( H \).

So \( H \) is not a subspace.
What are (valid) subspaces of $\mathbb{R}^2$?

$\mathbb{R}^2$, $\{0\}$ are subspaces. So are lines passing through the origin.

$\Rightarrow$ not a subspace, as origin is not included.

**Def**

$\text{Span}(\{\overline{v}_1, \ldots, \overline{v}_p\})$ is a subspace of $\mathbb{R}^m$, when each $\overline{v}_j \in \mathbb{R}^{m}$ (the set of all linear combinations of $\overline{v}_1, \ldots, \overline{v}_p$). We call this subspace the subspace generated by, or spanned by, $\overline{v}_1, \ldots, \overline{v}_p$.

**Prob 5, pg 151**

5. Let $v_1 = \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix}$, $v_2 = \begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix}$, and $w = \begin{bmatrix} -3 \\ -3 \\ 10 \end{bmatrix}$. Determine if $w$ is in the subspace of $\mathbb{R}^3$ generated by $v_1$ and $v_2$.

The question asks "is $\overline{w}$ in $\text{span}(\overline{v}_1, \overline{v}_2)$?"

Reword: With $A = [\overline{v}_1 \overline{v}_2]$, is $A\overline{x} = \overline{w}$ consistent?

Notice that we need not solve for $\overline{x}$ — we just need to determine if the system is consistent or not in order to answer the question.
\[
A = \begin{bmatrix}
1 & -2 \\
3 & -3 \\
-4 & 7 \\
\end{bmatrix} \quad \bar{w} = \begin{bmatrix}
-3 \\
-3 \\
10 \\
\end{bmatrix}
\]

\[
[A | \bar{w}] = \begin{bmatrix}
1 & -2 & -3 \\
3 & -3 & -3 \\
-4 & 7 & 10 \\
\end{bmatrix} \xrightarrow{R_2 \rightarrow 3R_1} \begin{bmatrix}
1 & -2 & -3 \\
0 & 3 & 6 \\
0 & -1 & -2 \\
\end{bmatrix} \xrightarrow{R_2 + \frac{1}{3}R_3} \begin{bmatrix}
1 & -2 & -3 \\
0 & 3 & 6 \\
0 & 0 & 0 \\
\end{bmatrix}
\]

System is consistent. So, \( \bar{w} \in \text{span}(\bar{w}_1, \bar{w}_3) \).

We will now study two subspaces related to a matrix \( A \), and their relationships to various concepts we have already seen—consistency of \( A\bar{x} = \bar{b} \), one-to-one and onto LIs defined by \( A \), etc.

**Column Space and Null Space of \( A \in \mathbb{R}^{m \times n} \)**

**Def** The column space of \( A \) is the set of all linear combinations of the columns of \( A \). We denote it \( \text{Col} \ A \).

\( \text{Col} \ A \) is a subspace of \( \mathbb{R}^m \).

\( \bar{b} \in \mathbb{R}^m \) is in \( \text{Col} \ A \) if \( A\bar{x} = \bar{b} \) is consistent.

The null space of \( A \) is the set of all solutions to \( A\bar{x} = \bar{0} \). We denote it by \( \text{Nul} \ A \).

Since any \( \bar{x} \) that is a solution to \( A\bar{x} = \bar{0} \) is in \( \mathbb{R}^n \),

\( \text{Nul} \ A \) is a subspace of \( \mathbb{R}^n \).

Since \( \text{Col} \ A \) is the set of all linear combinations of the columns of \( A \), it satisfies the definition of a subspace in a straightforward manner.
Let us check the definition for Null $A$ being a subspace now.

1. $A\vec{0} = \vec{0}$ (trivial solution). So $\vec{0} \in \text{Null } A$.

2. For $\vec{x}_1, \vec{x}_2$ such that $A\vec{x}_1 = \vec{0}$ and $A\vec{x}_2 = \vec{0}$, indeed

$$A(\vec{x}_1 + \vec{x}_2) = A\vec{x}_1 + A\vec{x}_2 = \vec{0} + \vec{0} = \vec{0}.$$ 
So $\vec{x}_1 + \vec{x}_2$ is also in Null $A$.

3. For $\vec{x}$ in Null $A$, i.e., $A\vec{x} = \vec{0}$, consider $A(c\vec{x})$.

$$A(c\vec{x}) = c(A\vec{x}) = c\vec{0} = \vec{0}.$$ 
So $c\vec{x} \in \text{Null } A$.

7. Let

$$\vec{v}_1 = \begin{bmatrix} 2 \\ -8 \\ 6 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -3 \\ 8 \\ -7 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} -4 \\ 6 \\ -7 \end{bmatrix}.$$ 

$$\vec{p} = \begin{bmatrix} 6 \\ -10 \\ 11 \end{bmatrix}.$$

and $A = [\vec{v}_1 \, \vec{v}_2 \, \vec{v}_3]$.

\[ [A|\vec{p}] = \begin{bmatrix} 2 & -3 & -4 & 6 \\ -8 & 8 & 6 & -10 \\ 6 & -7 & -7 & 11 \end{bmatrix} \]

a. How many vectors are in $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$?

b. How many vectors are in Col $A$?

c. Is $\vec{p}$ in Col $A$? Why or why not?

(a). Three. $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is just a collection of the three vectors.

(b). Infinitely many. Remember, Col $A$ is the set of all linear combinations of the columns of $A$.

(c) $\vec{p} \in \text{Col } A$ if $A\vec{x} = \vec{p}$ is consistent.
\[
[A\vec{p}] = \begin{bmatrix}
2 & -3 & -4 & 6 \\
8 & 8 & 6 & -10 \\
6 & -7 & -7 & 11
\end{bmatrix}
\rightarrow \begin{bmatrix}
2 & -3 & -4 & 6 \\
0 & -4 & -10 & 14 \\
0 & 2 & 5 & -7
\end{bmatrix}
\rightarrow \begin{bmatrix}
2 & -3 & -4 & 6 \\
0 & 4 & -10 & 14 \\
0 & 0 & 0 & 10
\end{bmatrix}
\]

System is consistent. So \( \vec{p} \in \text{Col } A \).

Again, notice that we need not solve the system \( A\vec{x} = \vec{p} \).

\textbf{Prob 9, Pg 151}

9. With \( A \) and \( \vec{p} \) as in Exercise 7, determine if \( \vec{p} \) is in \( \text{Nul } A \).

\( \vec{p} \in \text{Nul } A \) if \( A\vec{p} = \vec{0} \).

\[
A\vec{p} = \begin{bmatrix}
2 & -3 & -4 \\
8 & 8 & 6 \\
6 & -7 & -7
\end{bmatrix}
\begin{bmatrix}
6 \\
-10 \\
11
\end{bmatrix}
= \begin{bmatrix}
2 \times 6 + (-3) \times (-10) + (-4) \times 11 \\
8 \times 6 + 8 \times (-10) + 6 \times 11 \\
6 \times 6 + (-7) \times (-10) + (-7) \times 11
\end{bmatrix}
= \begin{bmatrix}
-2 \\
-62 \\
29
\end{bmatrix} \neq \vec{0}.
\]

So \( \vec{p} \notin \text{Nul } A \).

"not an element of".

\textbf{Basis for a subspace H}

We saw that a subspace, by definition, has infinitely many vectors. Hence we try to work with a finite subset of these vectors which generates the entire subspace. It also makes sense to study such a finite set that is also minimal, i.e., has the smallest number of vectors. It turns out that such a minimal set is LI.
A linearly independent set in $H$ which spans $H$ is a **basis** for $H$.

Equivalently, a basis is a minimal subset of $H$ that generates $H$.

**Example** The unit vectors $\{\overline{e}_1, \overline{e}_2, \overline{e}_3\} = \{[1,0,0]^T, [0,1,0]^T, [0,0,1]^T\}$ form a basis for $\mathbb{R}^3$.

Notice that the set $\{\overline{e}_1, \overline{e}_2, \overline{e}_3\}$ is LI, and any $\mathbf{x} \in \mathbb{R}^3$ can be written as a unique linear combination of $\overline{e}_1, \overline{e}_2, \text{and } \overline{e}_3$.

In general, $\{\overline{e}_1, \overline{e}_2, \ldots, \overline{e}_n\}$ forms a basis for $\mathbb{R}^n$, where $\overline{e}_j$ is the $j$th unit $n$-vector. This is the standard basis for $\mathbb{R}^n$.

In the next lecture, we will talk about **bases** for $\text{Col} \ A$ and $\text{Nul} \ A$. **plural of basis**
Basis for a subspace $H$

A linearly independent set in $H$ that spans $H$ is a basis of $H$.

Set of all unit vectors, $\{ \vec{e}_1, \ldots, \vec{e}_n \}$ is the standard basis for $\mathbb{R}^n$.

(bases is the plural of basis).

Bases for $\text{Nul} A$ can be found from the parametric vector form of the solutions to $A\vec{x} = \vec{0}$.

The collection of vectors that are scaled by each parameter (or free variable) gives a basis for $\text{Nul} A$.

Basis for $\text{Col} A$ : Pivot columns in $A$ (in the original matrix, and not in the echelon form).

It is important to remember that the columns in any basis for $A$ should be chosen from the original matrix $A$.

Dimension of a subspace $H$ : The number of vectors in any basis of $H$. (denoted by $\text{dim } H$)

Any basis for $H$ has the same number of vectors. This number is its dimension.
Problem

\[ A = \begin{bmatrix} -3 & 9 & -2 & -7 \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix} \]

\[ \text{Find bases for } \text{Col}A \text{ and } \text{Nul}A, \text{ and their dimensions.} \]

\[ \text{pivot columns} \]

\[ \begin{bmatrix} -3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix} \]

is a basis of \( \text{Col}A \).

Notice that the vectors in the basis for \( \text{Col}A \) come from \( A \), and not from its echelon form.

\( \dim \text{Col}A = 2 \), as there are 2 pivot columns.

\[ \begin{bmatrix} 1 & -3 & 6 & 9 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ \rightarrow \begin{bmatrix} 1 & -3 & 6 & 9 \\ 0 & 0 & 0 & \frac{5}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ \begin{bmatrix} 1 & -3 & 6 & 9 \\ 0 & 0 & 0 & \frac{5}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ R_2 \rightarrow \begin{bmatrix} 1 & -3 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & \frac{5}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ x_2, x_4 \text{ are free} \]

\[ \bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} + \begin{bmatrix} \frac{3}{2} \\ 0 \\ -\frac{5}{4} \\ 1 \end{bmatrix} t, s, t \in \mathbb{R}. \]

A basis for \( \text{Nul}A \) is \( \left\{ \begin{bmatrix} 3 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{3}{2} \\ 0 \\ -\frac{5}{4} \\ 1 \end{bmatrix} \right\} \).

\( \dim \text{Nul}A = 2 \), as there are 2 free variables.

\( \dim(\{0\}) = ? \) By following the definition, we get \( \dim(\{0\}) = 0 \).

Since \( \dim H \) is the number of vectors in any basis of \( H \), and \( \{0\} \) has no basis. Notice that \( \{0\} \) is LD, and hence has no basis.
Another Problem

\[
A = \begin{bmatrix}
1 & -2 & 9 & 5 & 4 \\
1 & -1 & 6 & 5 & -3 \\
-2 & 0 & -6 & 1 & -2 \\
4 & 1 & 9 & 1 & -9 \\
\end{bmatrix}
\sim \begin{bmatrix}
1 & -2 & 9 & 5 & 4 \\
0 & 0 & -3 & 0 & -7 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Find bases and dimensions of \( \text{Col} \ A \) and \( \text{Nul} \ A \).

Columns 1, 2, and 4 are pivot columns. So, a basis for \( \text{Col} \ A \) is \[
\left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ \vdots \end{bmatrix} \right\} .
\]

\( \dim(\text{Col} \ A) = 3 \), as there are 3 pivot columns.

\( \dim(\text{Nul} \ A) = 2 \), as there are two free variables.

\[
\begin{bmatrix}
1 & -2 & 9 & 5 & 4 \\
0 & 1 & -3 & 0 & -7 \\
0 & 0 & 0 & 1 & -2 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\stackrel{R_1+2R_2}{\longrightarrow}
\begin{bmatrix}
1 & 0 & 3 & 5 & -10 \\
0 & 1 & -3 & 0 & -7 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\stackrel{R_1-5R_3}{\longrightarrow}
\]

\[
\begin{bmatrix}
x_3 \\
x_5 \end{bmatrix}
\text{ are free}
\]

\[
\tilde{x} = \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
\end{bmatrix} = \begin{bmatrix}
-3 \\
3 \\
1 \\
0 \\
0 \\
\end{bmatrix} x_3 + \begin{bmatrix}
0 \\
7 \\
1 \\
0 \\
2 \\
\end{bmatrix} x_5 , \quad x_3, x_5 \text{ are real numbers.}
\]

A basis for \( \text{Nul} \ A \) is \[
\left\{ \begin{bmatrix} -3 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 7 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\} .
\]
Problem

$A_{3 \times 5}$ has 3 pivot columns.

Is $\text{Col } A = \mathbb{R}^3$? Yes! As there are 3 pivots, there is a pivot in every row. So columns of $A$ span $\mathbb{R}^3$.

Is $\text{Nul } A = \mathbb{R}^2$? No! Every solution to $Ax = 0$ sits in $\mathbb{R}^5$.

But, $\dim (\text{Nul } A) = 2$ here.

---

**Rank of a matrix $A$ ($A$ is $m \times n$)**

**Def** The rank of an $m \times n$ matrix $A$, $\text{rank}(A)$ or $\text{rank} A$, is the dimension of $\text{Col } A$. So

$$\text{rank}(A) = \# \text{ pivot columns in } A.$$

**Rank Theorem:**

$$\text{rank}(A) + \dim (\text{Nul } A) = n$$

$n = \# \text{ columns in } A$.

$\# \text{ pivot columns}$

$\# \text{ free variables (or, nonpivot columns)}$

$\text{total } \# \text{ columns}$

---

**Problem**

What is $\text{rank}(A)$ when $A$ is $4 \times 5$ and $\text{Nul } A$ is 3-dimensional?

$$\text{rank}(A) + \dim (\text{Nul } A) = 5.$$ So $\text{rank}(A) = 5 - 3 = 2.$
Problem

Create a $3 \times 4$ matrix $A$ with $\dim(\text{Null } A) = 2$ and $\dim(\text{Col } A) = 2$.

So, $A$ has 2 pivot columns and 2 free variables.

$$A = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ will work.

Basis Theorem: Let $H$ be a $p$-dimensional subspace of $\mathbb{R}^n$.

Then any LI set of exactly $p$ elements (vectors) in $H$ is a basis for $H$. Also, any set of $p$ elements in $H$ that spans $H$ is a basis.

E.g., any 3 LI vectors in $\mathbb{R}^2$ will be a basis for $\mathbb{R}^2$.

We have 3 qualifications, or properties, that a (potential) basis $B$ of $H$ has to satisfy.

1. $B$ spans $H$.
2. $B$ is LI.
3. \# vectors in $B = \dim H$.

If $B$ satisfies any two of these three properties, it is automatically a basis, i.e., the third property is satisfied in this case.
Invertible Matrix Theorem (IMT) \( A_{n \times n} \)

Recall, (a) \( A \) is invertible.

We add equivalent statements related to \( \text{Col } A, \text{Nul } A, \) and their dimensions now.

1. Columns of \( A \) form a basis for \( \mathbb{R}^n \).
2. \( \text{Col } A = \mathbb{R}^n \).
3. \( \dim \text{Col } A = n \).
4. \( \text{rank } A = n \).
5. \( \text{Nul } A = \{ \mathbf{0} \} \).
6. \( \dim \text{Nul } A = 0 \).
Recall (results on bases of subspaces, dimension, rank, etc.)

Basis of a subspace $H$ is an LI set of vectors in $H$ that spans $H$.

dimension of $H$, denoted by $\dim H$ or $\dim(H)$, is the # vectors in any basis of $H$.

$\dim(\{0\}) = 0$ as $\{0\}$ is not LI.

$\text{rank}(A) = \dim(\text{Col}A) = \# \text{ pivot columns}$

$\dim(\text{Nul}A) = \# \text{ free variables}$

A basis for $\text{Col}A$: pivot columns of $A$

A basis for $\text{Nul}A$: vectors in parametric vector form of solutions to $A\vec{x} = \vec{0}$.

**Rank theorem**

For $A \in \mathbb{R}^{m \times n}$,

$$\text{rank}(A) + \dim(\text{Nul}A) = n$$

# pivot columns # free variables total # columns
Invertible matrix theorem (continued...)

(a) \( A \in \mathbb{R}^{n \times n} \) is invertible.
(b) Columns of \( A \) form a basis for \( \mathbb{R}^n \).
(c) \( \text{Col} \ A = \mathbb{R}^n \).
(d) \( \dim \text{Col} \ A = n \).
(e) \( \text{rank} \ A = n \).
(f) \( \text{Nul} \ A = \{0\} \).
(g) \( \dim (\text{Nul} \ A) = 0 \).

We could use any basis to represent a subspace. In any given basis, we can talk about the "coordinates" of any vector in the subspace.

Coordinates

Let \( B = \{\vec{b}_1, \ldots, \vec{b}_p\} \) be a basis for a subspace \( H \). Each \( \vec{x} \in H \) can be written as \( \vec{x} = c_1\vec{b}_1 + c_2\vec{b}_2 + \ldots + c_p\vec{b}_p \) for scalars \( c_1, c_2, \ldots, c_p \), which are unique to \( \vec{x} \). These scalars are called the coordinates of \( \vec{x} \) relative to the basis \( B \). Stacking these scalars into a vector

\[
[\vec{x}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{bmatrix}
\]

is the \( B \)-coordinate of \( \vec{x} \).

When \( B \) is the standard basis, i.e., \( \{\vec{e}_1, \ldots, \vec{e}_n\} \), the \( B \)-coordinate of any \( \vec{x} \) consists of its own entries. (So, the \( B \)-coordinate of \( \vec{x} \) is \( \vec{x} \) itself here)
In Exercises 1 and 2, find the vector \( \mathbf{x} \) determined by the given coordinate vector \([\mathbf{x}]_B\) and the given basis \(B\). Illustrate your answer with a figure, as in the solution of Practice Problem 2.

1. \( B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}, \quad [\mathbf{x}]_B = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \)

\[ \overline{X} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}. \]

\[ \text{We just have to evaluate the linear combination given} \]

In Exercises 3–6, the vector \( \mathbf{x} \) is in a subspace \( H \) with a basis \( B = \{ \mathbf{b}_1, \mathbf{b}_2 \} \). Find the \( B \)-coordinate vector of \( \mathbf{x} \).

5. \( \mathbf{b}_1 = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -2 \\ -7 \\ 5 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 2 \\ 9 \\ -7 \end{bmatrix} \)

\( B = \{ \mathbf{b}_1, \mathbf{b}_2 \} \). Find \([\mathbf{x}]_B\). So, find \( c_1, c_2 \) such that

\[ \overline{X} = c_1 \mathbf{b}_1 + c_2 \mathbf{b}_2 \]

\[
\begin{bmatrix}
1 & -2 \\
4 & -7 \\
-3 & 9
\end{bmatrix}
\begin{bmatrix}
2 \\
3 \\
7
\end{bmatrix}
\]

\[ \begin{bmatrix}
1 & -2 \\
0 & 1 \\
0 & -1
\end{bmatrix}
\begin{bmatrix}
2 \\
1 \\
-1
\end{bmatrix}
\]

\[ \begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & 4
\end{bmatrix}
\]

\[ \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

\[ c_1 = 4, \quad c_2 = 1 \]

or \([\mathbf{x}]_B = \begin{bmatrix} 4 \\ 1 \end{bmatrix}\)

**Why study coordinates?**

One often works with a nonstandard basis for a subspace. Hence we want to study how any vector is expressed in such a basis. For instance, we could start with the standard basis for an image, and do a bunch of geometric transformations, e.g., rotate 90° CCW. After that step, we could just work with the nonstandard basis for further analyses.
In Exercises 17 and 18, mark each statement True or False. Justify each answer. Here \( A \) is an \( m \times n \) matrix.

17. a. If \( B = \{v_1, \ldots, v_p\} \) is a basis for a subspace \( H \) and if \( x = c_1v_1 + \cdots + c_pv_p \), then \( c_1, \ldots, c_p \) are the coordinates of \( x \) relative to the basis \( B \).

b. Each line in \( \mathbb{R}^n \) is a one-dimensional subspace of \( \mathbb{R}^n \).

c. The dimension of \( \text{Col} \ A \) is the number of pivot columns in \( A \).

d. The dimensions of \( \text{Col} \ A \) and \( \text{Nul} \ A \) add up to the number of columns in \( A \).

e. If a set of \( p \) vectors spans a \( p \)-dimensional subspace \( H \) of \( \mathbb{R}^n \), then these vectors form a basis for \( H \).

(a) True. Definition of coordinates.
(b) False. Only if it goes through the origin.
(c) True. Definition.
(d) True. Rank theorem.
(e) True. A set of \( p \) vectors that spans a \( p \)-dimensional subspace will be LI. Hence it's a basis.

23. If possible, construct a \( 3 \times 5 \) matrix \( A \) such that \( \dim \text{Nul} \ A = 3 \) and \( \dim \text{Col} \ A = 2 \).

\[ n=5, \quad \dim \text{Col} \ A + \dim \text{Nul} \ A = 2 + 3 = n \checkmark \]

(\( \text{So, rank theorem holds} \))

\[ A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \] Works. (has 2 pivot columns).
Determinants (Section 3.1)

Recall, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, when
determinant of $A$, $\det A = ad-bc \neq 0$.

In general, for $A \in \mathbb{R}^{n \times n}$, $A$ is invertible if and only if $\det A \neq 0$.

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & a & b \\ 5 & c & d \end{bmatrix}$. When is $A$ invertible?

(what condition should $a, b, c, d$ satisfy in order to make $A$ invertible?)

\[
\begin{bmatrix} 1 & 2 & 3 \\ 4 & a & b \\ 5 & c & d \end{bmatrix}
\rightarrow
\begin{bmatrix} 1 & 2 & 3 \\ 0 & a-8 & b-12 \\ 0 & c-10 & d-15 \end{bmatrix}
\rightarrow
\begin{bmatrix} 2 \end{bmatrix}
\]

\[
\begin{bmatrix} 1 & 2 & 3 \\ 0 & a-8 & b-12 \\ 0 & c-10 & (d-15) - \left( \frac{c-10}{a-8} \right)(b-12) \end{bmatrix}
\]

As we need three pivots

\[
(a-8)(d-15) - (c-10)(b-12) \neq 0
\]

\[
(ad-15a-8d+120) - (bc-12c-10b+120) \neq 0
\]

\[
(ad-bc) - (8d-10b) + (12c-15a) \neq 0
\]
i.e., we need \( 1(ad-bc) - 2(4d-5b) + 3(4c-5a) \neq 0 \) expanding along Row 1

\[
\det \begin{pmatrix}
1 & 2 & 3 \\
4 & a & b \\
5 & c & d
\end{pmatrix}
= 1 \cdot \det \begin{pmatrix}
a & b \\
c & d
\end{pmatrix} - 2 \det \begin{pmatrix}
a & b \\
4 & 5
\end{pmatrix} + 3 \det \begin{pmatrix}
a & b \\
5 & c
\end{pmatrix}
\]

We will see that one could expand along any row or any column to compute \( \det A \). More on the details in the next lecture.
%% commands from the MATLAB session in Lecture 24 on Thursday,  
%% November 7, 2013.

%% To illustrate row reduction, we used Problem 5 from Page 157 of the  
%% book, which was solved in Lecture 23. In this problem, 3-vectors  
%% $b_1, b_2,$ and $x$ are given, and we are asked to the B-coordinates of $x$  
%% in the basis $B = \{b_1, b_2\}$.

%% You should check out some of the MATLAB tutorials listed in the  
%% Computer Project description.

%% We can use the % sign to add comments - MATLAB ignores anything  
%% written in a line following a % sign. Extra comments are added in  
%% between the MATLAB commands here to illustrate.

>> b1 = [1
   4
  -3];
b1 =
  1
  4
 -3

% A' (prime) transposes a matrix or a vector when added to its  
% end. Also, if you do not want MATLAB to display the output of a  
% command, end the same with a ; (semi-colon).

>> b2 = [-2 -7 5];

>> x = [2 9 -7];

>> AugMtx = [b1 b2 x]

AugMtx =
  1   -2    2
  4   -7    9
 -3    5   -7

% Error messages in MATLAB - usually point out where the source of  
% error is. Or, at least tell you from where things go wrong.

>> AugMtx = [b1 b2 x]

??? Error using ==> horzcat CAT  
arguments dimensions are not consistent.

% In the above command, $x$' is 1 x 3, while b1 and b2 are both  
% 3 x 1. Hence the dimensions do not match.

>> AugMtx = [b1 b2 x];

>> rref(AugMtx)
ans =
  1   0   4
  0  1   1
  0   0   0
You need to be aware of at least the basic commands related to matrix/vector operations in MATLAB.

Several computations have functions in built (or, implemented) already in MATLAB. In particular, rref (reduced row echelon form), det (determinant), rank (rank), inv (inverse), are quite useful.

We will revisit the actual problems described in the project once we introduce eigenvalues and eigenvectors.
Determinant of $A \in \mathbb{R}^{n \times n}$ by expanding along Row 1

As illustrated in the previous lecture, we could compute the determinant of any square matrix by expanding along its Row 1.

In general, for $A \in \mathbb{R}^{n \times n}$ with

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13} - \cdots + (-1)^{n+1} a_{1n} \det A_{1n},$$

where $A_{ij}$ is the $(n-1) \times (n-1)$ matrix obtained by removing Row 1 and Column $j$ of $A$.

Pg 167, Prob 2. Compute the determinant by expanding along Row 1.

\[
\begin{vmatrix} 0 & 5 & 1 \\ 4 & -3 & 0 \\ 2 & 4 & 1 \end{vmatrix}
\]

\[
= 0 \begin{vmatrix} 4 & 0 \\ 1 & 2 \end{vmatrix} - 5 \begin{vmatrix} 4 & 0 \\ 1 & 2 \end{vmatrix} + 1 \begin{vmatrix} 4 & -3 \\ 2 & 4 \end{vmatrix}
\]

\[
= 0 (4 \cdot 1 - 4 \cdot 0) - 5 (4 \cdot 1 - 2 \cdot 0) + 1 (4 \cdot 4 - 2 \cdot 3)
\]

\[
= 0 - 20 + 22 = 2.
\]
In fact we can expand along any row or any column to evaluate the determinant.

The result is given as Theorem 1 in the book.

Define $C_{ij} = (-1)^{i+j} \det A_{ij}$ to remove row $i$ column $j$ from matrix $A$.

The $(i,j)$th cofactor of a matrix is the determinant of the submatrix obtained by removing Row $i$ and Column $j$ from the original matrix, multiplied by the appropriate sign that depends on $i+j$, i.e., by $(-1)^{i+j}$.

Expanding along column $j$:

$$\det A = a_{ij}C_{ij} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj}.$$  

Expanding along Row $i$:

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in}.$$  

Notice that the alternating $\pm$ signs are included in the cofactor values.
Compute the determinants in Exercises 9–14 by cofactor expansions. At each step, choose a row or column that involves the least amount of computation.

10. We look for a row or a column with lots of zeros, and expand along that row/column. We repeat this idea for the 3x3 determinant in the next step.

\[
\begin{vmatrix}
3 & 2 & 2 \\
1 & -2 & 5 \\
0 & 0 & 3 \\
\end{vmatrix}
\]

\[
= 3 \cdot (-1) \begin{vmatrix}
-2 & 2 \\
-6 & 5 \\
0 & 4 \\
\end{vmatrix}
= -3 \left( (-2) \cdot (-1)^{2+3} \begin{vmatrix}
2 & 5 \\
5 & 4 \\
\end{vmatrix} + (-6) \cdot (-1)^{3+2} \begin{vmatrix}
1 & 2 \\
5 & 4 \\
\end{vmatrix} \right)
= -3 \left( 2 \left( 8 - 25 \right) + (-6) \left( 4 - 10 \right) \right)
= -3 \left( -34 + 36 \right) = -6.
\]
Determinants using cofactor expansion

Compute the determinants in Exercises 9–14 by cofactor expansions. At each step, choose a row or column that involves the least amount of computation.

\[ \begin{vmatrix} 3 & 2 \\ 4 & 7 \end{vmatrix} = 2 \cdot (-1)^{2+3} \begin{vmatrix} 3 & -5 \\ 7 & 8 \end{vmatrix} = -2 \cdot 3 \cdot (-1)^{2+4} \begin{vmatrix} 3 & -5 \\ 5 & -3 \end{vmatrix} \]

Look for row/column with lots of zeros to expand along!

\[ = -6 \left( 4 \cdot (-1)^{4+1} \begin{vmatrix} 1 & -3 \\ -2 & 3 \end{vmatrix} + 5 \cdot (-1)^{2+4} \begin{vmatrix} 1 & -5 \\ 1 & 2 \end{vmatrix} \right) \]

\[ = -6 \left( 4 \cdot 1 - 5 \cdot 1 \right) = 6. \]

12. Lower triangular matrix—all entries above the main diagonal are zero.

\[ = 4 \cdot (-1)^{4+1} \begin{vmatrix} 0 \end{vmatrix} = 0 \]

\[ = 4 \cdot (-1)^{2+4} \begin{vmatrix} 0 \end{vmatrix} = 0 \]

\[ = -4 \cdot (3 \times 3) = 36 \]

Definition: If all entries above main diagonal are zero, then we have a lower triangular matrix. If all entries below main diagonal are zero, we have an upper triangular matrix.

Theorem 2: If \( A \) is a triangular matrix, then \( \det A = \text{product of the entries in the main diagonal of } A. \)
Properties of determinants (Section 3.2)

22. \[ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow \begin{bmatrix} a & b \\ k & d \end{bmatrix} \]
\[ \det \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = ad - bc \]
\[ \det \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = (a+k) \cdot d - (b+k) \cdot c = ad - bc \]

The ERO does not change the determinant here!

20. \[ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow \begin{bmatrix} a & b \\ kc & kd \end{bmatrix} \]
\[ \det \left( \begin{bmatrix} a & b \\ kc & kd \end{bmatrix} \right) = akd - bkc \]
\[ \det \left( \begin{bmatrix} a & b \\ kc & kd \end{bmatrix} \right) = k \cdot (ad - bc) \]

Determinant is scaled by k as well.

Similarly,
\[ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow \begin{bmatrix} c & d \\ a & b \end{bmatrix} \]
\[ \det \left( \begin{bmatrix} c & d \\ a & b \end{bmatrix} \right) = cb - ad \]
\[ \det \left( \begin{bmatrix} c & d \\ a & b \end{bmatrix} \right) = -(ad - bc) \]

Determinant changes sign.

**Theorem 3**: \( A, B \in \mathbb{R}^{n \times n} \)

1. \( A \xrightarrow{R_i + kR_j} B \) \[ \det B = \det A \]
   - Replacement EROs do not change determinant

2. \( A \xrightarrow{R_i \leftrightarrow R_j} B \) \[ \det B = -\det A \]
   - Swap changes sign of determinant

3. \( A \xrightarrow{kR_i} B \) \[ \det B = k \cdot \det A \]
   - Scaling scales determinant by same number

(pg 168) In Exercises 19–24, explore the effect of an elementary row operation on the determinant of a matrix. In each case, state the row operation and describe how it affects the determinant.

(from 3.1)
So we could use EROs carefully to evaluate determinants! To avoid scaling EROs.

Find the determinants in Exercises 5–10 by row reduction to echelon form.

6. \[
\begin{vmatrix}
1 & 5 & -3 \\
3 & -3 & 3 \\
2 & 13 & -7
\end{vmatrix}
\]

\[
\frac{R_2 - 3R_1}{R_3 - 2R_1} = \begin{vmatrix} 1 & 5 & -3 \\ 0 & -18 & 12 \\ 0 & 3 & -1 \end{vmatrix} \rightarrow \begin{vmatrix} 1 & 5 & -3 \\ 0 & -18 & 12 \\ 0 & 0 & 1 \end{vmatrix} = 1 \cdot -18 \cdot 1 = -18.
\]

Find the determinants in Exercises 15–20, where \( (pg 175) \)

\[
\begin{vmatrix}
a & b & c \\
d & e & f \\
g & h & i
\end{vmatrix} = 7.
\]

15. \[
\begin{vmatrix}
a & b & c \\
d & e & f \\
g & h & i
\end{vmatrix} = 5 \begin{vmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{vmatrix}
\]
as the scaling \( 5R_3 \) gives the new matrix

\[
= 5 \cdot 7 = 35.
\]

Let \( A \in \mathbb{R}^{n \times n} \) be row reduced to \( U \) using \( r \) swaps and any number of replacement EROs (and no scaling EROs). Then

\[
\det A = \begin{cases} (-1)^r (\text{product of pivot entries in } U) & \text{if } A \text{ is invertible} \\ 0 & \text{if } A \text{ is not invertible} \end{cases}
\]

Notice that we do not need scaling EROs to convert \( A \) to echelon form. We might need them to go to reduced echelon form. But echelon form is sufficient here!
In Exercises 24–26, use determinants to decide if the set of vectors is linearly independent.

\[ \begin{bmatrix} -7 & -8 & 7 \\ -4 & 5 & 0 \\ -6 & 7 & -5 \end{bmatrix} \]

\[ A = \begin{bmatrix} 7 & -8 & 7 \\ -4 & 5 & 0 \\ -6 & 7 & -5 \end{bmatrix} \]

\[ R_2 + \frac{4}{7} R_1 \rightarrow \begin{bmatrix} 7 & -8 & 7 \\ 0 & \frac{3}{7} & 4 \\ 0 & \frac{1}{7} & 1 \end{bmatrix} \]

\[ R_3 \leftarrow R_2 \rightarrow \begin{bmatrix} 7 & -8 & 7 \\ 0 & \frac{1}{7} & 1 \\ 0 & 0 & 1 \end{bmatrix} \]

\[ \text{det } A = -\left( 7 \times \frac{1}{7} \times 1 \right) = -1 \neq 0 \]

So the vectors are LI.

**Theorem 4**

A is invertible if and only if det \( A \neq 0 \).

**Theorem 5**

\( \text{det}(A^T) = \text{det } A \).

**Theorem 6**

\( A, B \in \mathbb{R}^{n \times n} \). \( \text{det}(AB) = \text{det } A \cdot \text{det } B \)

Determinant of a product of two matrices is the product of the individual determinants.

**Warning!** \( \text{det } (A+B) \neq \text{det } A + \text{det } B \) in general.
29. Compute \( \det B^5 \), where \( B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} \).

\[
\det(B^5) = \det(B \cdot B \cdot B \cdot B \cdot B) = (\det B)^5.
\]

\[
\det B = 1 \cdot \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = -3 + 1 = -2.
\]

\[
\det(B^5) = (-2)^5 = -32.
\]
Properties of determinants: \[ \det(AB) = \det(A) \cdot \det(B) \]
\[ \det(A^T) = \det(A) \]

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31. Show that if \( A \) is invertible, then \( \det A^{-1} = \frac{1}{\det A} \).

We have \( AA^{-1} = I \)

Taking determinants on both sides,

\[ \det(AA^{-1}) = \det I \]

\[ = 1 \]

\[ \rightarrow \text{product of } n \text{ copies of } 1 \text{ on the diagonal} \]

So \( \det(A) \cdot \det(A^{-1}) = 1 \)

Hence, when \( A \) is invertible, \( \det(A) \neq 0 \), and so

\[ \det(A^{-1}) = \frac{1}{\det A} \]

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36. Suppose that \( A \) is a square matrix such that \( \det A^4 = 0 \).

Explain why \( A \) cannot be invertible.

\[ \det A^4 = (\det A)^4 \]

follows from \( \det(AB) = \det A \cdot \det B \)

Hence if \( \det A^4 = 0 \), \( (\det A)^4 = 0 \), i.e., \( \det A = 0 \).

Hence \( A \) is not invertible.
Let $A$ and $B$ be $4 \times 4$ matrices, with $\det A = -1$ and $\det B = 2$. Compute:

a. $\det AB$

b. $\det B^5$

c. $\det 2A$

d. $\det A^T A$

e. $\det B^{-1} AB$

\begin{align*}
\text{(a)} & \quad \det AB = \det A \cdot \det B = (-1) \cdot 2 = -2. \\
\text{(b)} & \quad \det B^5 = (\det B)^5 = (2)^5 = 32. \\
\text{(c)} & \quad \det (2A) = (2)^4 \cdot \det A = 16 \cdot (-1) = -16 \\
\end{align*}

\text{Every row of $A$ is scaled by 2, and there are 4 rows. For $A \in \mathbb{R}^{n \times n}$, $\det (cA) = c^n \det(A)$. $n$-rows getting scaled by $c$ each.}

\begin{align*}
\text{(d)} & \quad \det (A^T A) = \det (A^T) \cdot \det A = \det (A) \cdot \det (A) = (-1)^2 = 1. \\
\text{(e)} & \quad \det (B^{-1} AB) = \det (B^{-1}) \cdot \det (A) \cdot \det (B) \\
& \quad = \frac{1}{\det B} \cdot \det (A) \cdot \det (B) \quad \text{as $\det (B) = 2 \neq 0$} \\
& \quad = -1. \\
\end{align*}

Also, $\det (B^{-1} AB) = \det A = -1$

Similarly, $\det (A'tBA') = \det (A') \cdot \det (B) \cdot \det (A')$

$$= \frac{1}{\det A} \cdot \det B \cdot \frac{1}{\det A} = -1 \cdot 2 \cdot -1 = 2.$$
Eigenvalues and eigenvectors (Chapter 5)

Motivation

Given $A \in \mathbb{R}^{n \times n}$, can we say something more about the images of the LT $\tilde{x} \mapsto A\tilde{x}$, apart from the basis for $\text{Col}(A)$?

In particular, are there vectors $\tilde{x}$ whose images under the LT “look very much like” $\tilde{x}$?

More precisely, the images are just scaled versions of $\tilde{x}$.

The zero vector always fits this criterion, but we are interested in non-trivial vectors.

**Def**

$\tilde{x} \in \mathbb{R}^n$ is an eigenvector of $A \in \mathbb{R}^{n \times n}$ if $\tilde{x}$ is nonzero, and for some scalar $\lambda$, we have $A\tilde{x} = \lambda\tilde{x}$.

In this case, $\lambda$ is an eigenvalue of $A$, and $\tilde{x}$ is the eigenvector corresponding to $\lambda$.

At least one entry is $\neq 0$.

**Some 2x2 examples**

(a) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ Matrix of the geometric LT that reflects points across the $y=x$ line (flips $x$ and $y$-coordinates)
Need to find \( \bar{x} \) and \( \lambda \) such that \( A\bar{x} = \lambda \bar{x} \), and \( \bar{x} \neq \bar{0} \).

\[
A\bar{x} = \lambda \bar{x} \\
\Rightarrow A\bar{x} - \lambda I \bar{x} = \bar{0} \\
\Rightarrow (A - \lambda I) \bar{x} = \bar{0}
\]

which is a \( 2 \times 2 \) matrix with two unknowns - \( \bar{x} \) and \( \lambda \).

We want \( A - \lambda I \) to be not invertible, as we are looking for non-trivial solutions to \( (A - \lambda I) \bar{x} = \bar{0} \).

So \( \det(A - \lambda I) = 0 \) \( \Rightarrow \) only one unknown (\( \lambda \)).

\[
A - \lambda I = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 - \lambda & 1 \\ 1 & 0 - \lambda \end{bmatrix} = \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix}
\]

\[
\det(A - \lambda I) = (-\lambda)^2 - 1 = \lambda^2 - 1 = 0 \text{ when } \lambda = \pm 1.
\]

Hence, there are two eigenvalues to \( A \), \( \lambda_1 = 1 \), \( \lambda_2 = -1 \).

To find an eigenvector corresponding to \( \lambda_1 = 1 \), we find a non-trivial solution to \( (A - \lambda_1 I) \bar{x} = \bar{0} \).

\[
A - \lambda_1 I = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \xrightarrow{R_2 + R_1} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \xrightarrow{-R_1} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}
\]

\( x_2 \text{ free. } x_1 - x_2 = 0 \), i.e., \( x_1 = x_2 \)

\[
\Rightarrow \bar{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} s, \text{ } s \in \mathbb{R}
\]

So \( \bar{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) is an eigenvector of \( A \) corresponding to the eigenvalue \( \lambda = 1 \).
(b) \[ A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}, \quad A - \lambda I = \begin{bmatrix} 3 - \lambda & 1 \\ 1 & 3 - \lambda \end{bmatrix} \]

\[ \det(A - \lambda I) = 0 \implies (3 - \lambda)^2 - 1 = 0 \]
\[ \implies \lambda^2 - 6\lambda + 8 = 0 \]
\[ (\lambda - 2)(\lambda - 4) = 0 \]

\[ \det A = 3 \times 3 - 1 \times 1 = 8 \]  
\[ \text{det } A = 3 \times 3 - 1 \times 1 = \text{ad - bc} \]

There are two eigenvalues, \( \lambda = 2, \lambda = 4 \).

In general, for \( A \in \mathbb{R}^{2 \times 2} \), we have

\[ \det(A - \lambda I) = \lambda^2 - (\text{trace}(A)) + \det A. \]

**Def** \( \text{trace}(A) = \text{sum of diagonal entries}, \) when \( A \in \mathbb{R}^{n \times n} \).

We can find an eigenvector corresponding to the eigenvalue \( \lambda = 4 \), for instance, just as in the previous example.

With \( \lambda = 4 \), \( A - \lambda I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \)

\[ \begin{align*}
\text{R}_2 + \text{R}_1 & \implies \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \\
\text{R}_1 \times (-1) & \implies \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}
\end{align*} \]

\[ \bar{x} = \begin{bmatrix} 1 \end{bmatrix} \text{s, s} \in \mathbb{R}. \]

Thus, \( \bar{x} = \begin{bmatrix} 1 \end{bmatrix} \) is an eigenvector of \( A \) corresponding to the eigenvalue \( \lambda = 4 \).
(c) \( A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \) rotates vectors 90° CCW about the origin

\[ \det(A - \lambda I) = 0 \quad \begin{bmatrix} A - \lambda I \end{bmatrix} = \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} \]

\[ \Rightarrow (-\lambda)^2 + 1 = 0 \quad \text{or} \quad \lambda^2 + 1 = 0. \]

No real eigenvalues \( \ddagger! \)

\[ \text{Result} \quad \text{If } A \text{ is symmetric, i.e., } A_{ij} = A_{ji}, \text{ or } A^T = A, \]
\[ \text{then } A \text{ has only real eigenvalues.} \]

\[ \text{If } A \text{ is anti-symmetric, i.e., } A_{ij} = -A_{ji}, \]
\[ A \text{ has only non-real eigenvalues.} \]

In Math 220, we will typically concern ourselves with real eigenvalues.
Recall that if $\overline{x} \neq 0$ such that $A\overline{x} = \lambda \overline{x}$ for some $\lambda$, then $\overline{x}$ is an eigenvector of $A$ corresponding to the eigenvalue $\lambda$. $\lambda = 0$ is okay.

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5. Is $\overline{x} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ an eigenvector of $A = \begin{bmatrix} -4 & 3 & 3 \\ 2 & -3 & -2 \\ -1 & 0 & -2 \end{bmatrix}$? If so, find the eigenvalue.

Just check if $A\overline{x} = \lambda \overline{x}$ for some $\lambda$.

$$A\overline{x} = \begin{bmatrix} -4 & 3 & 3 \\ 2 & -3 & -2 \\ -1 & 0 & -2 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -15 \\ 10 \\ -5 \end{bmatrix} = -5 \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = -5\overline{x}.$$  

So $\overline{x}$ is an eigenvector of $A$ corresponding to the eigenvalue $\lambda = -5$.  

7. Is \( \lambda = 4 \) an eigenvalue of \[
\begin{bmatrix}
3 & 0 & -1 \\
2 & 3 & 1 \\
-3 & 4 & 5
\end{bmatrix}
\]? If so, find one corresponding eigenvector.

\( \lambda = 4 \) is an eigenvalue if \((A - \lambda I)x = 0\) has nontrivial solutions.

\[
A - \lambda I = \begin{bmatrix}
3 - \lambda & 0 & -1 \\
2 & 3 - \lambda & 1 \\
-3 & 4 & 5 - \lambda
\end{bmatrix} = \begin{bmatrix}
-1 & 0 & -1 \\
2 & -1 & 1 \\
-3 & 4 & 1
\end{bmatrix}
\]

\[-R_1, R_2 - 2R_1, 3R_1 + R_1 \]

\[
\begin{bmatrix}
1 & 0 & 1 \\
0 & -1 & 1 \\
0 & 4 & 4
\end{bmatrix}
\]

\[-R_2 \]

So \( \lambda = 4 \) is an eigenvalue. And \( \bar{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, s \epsilon \mathbb{R} \) is a corresponding eigenvector.

**Def.** The *eigenspace* of \( A \) corresponding to the eigenvalue \( \lambda \) is the null space of \( A - \lambda I \).

Notice that the eigenspace is indeed a subspace, i.e., it contains the zero vector. Hence the eigenspace of \( A \) corresponding to the eigenvalue \( \lambda \) consists of the zero vector along with all corresponding eigenvectors.
In Exercises 9–16, find a basis for the eigenspace corresponding
to each listed eigenvalue.

14. \( A = \begin{bmatrix} 4 & 0 & -1 \\ 3 & 0 & 3 \\ 2 & -2 & 5 \end{bmatrix}, \lambda = 3 \)

\[
A - \lambda I = \begin{bmatrix} 4-\lambda & 0 & -1 \\ 3 & 0-\lambda & 3 \\ 2 & -2 & 5-\lambda \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 3 & -3 & 3 \\ 2 & -2 & 2 \end{bmatrix} \xrightarrow{R_2 - 3R_1} \begin{bmatrix} 1 & 0 & -1 \\ 0 & -3 & 6 \\ 0 & -2 & 4 \end{bmatrix} \xrightarrow{\frac{1}{3}R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_3 + 2R_2}
\]

Hence a basis for the eigenspace is \( \{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \} \).

We now consider certain special cases where we could
guess some eigenvalues and eigenvectors more directly.

**Eigenvalues of triangular matrices**

Recall that if \( A \) is a triangular matrix, then \( \text{det} \ A \) is
the product of the diagonal entries.

9b. \( A = \begin{bmatrix} a & * & * \\ 0 & b & * \\ 0 & 0 & c \end{bmatrix} \), \( \text{det} \ A = abc \).

Similarly, \( \text{det} (A - \lambda I) = \text{det} \left( \begin{bmatrix} a-\lambda & * & * \\ 0 & b-\lambda & * \\ 0 & 0 & c-\lambda \end{bmatrix} \right) = (a-\lambda)(b-\lambda)(c-\lambda) \).

To find eigenvalues, we solve \( \text{det} (A - \lambda I) = 0 \), i.e.,
\( (a-\lambda)(b-\lambda)(c-\lambda) = 0 \).

Hence \( \lambda = a, b, c \) are the eigenvalues of \( A \).
Theorem: The eigenvalues of a triangular matrix are the entries on its diagonal.

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Find the eigenvalues of the matrices in Exercises 17 and 18.

18. \[
\begin{bmatrix}
5 & 0 & 0 \\
0 & 0 & 0 \\
-1 & 0 & 3 \\
\end{bmatrix}
\]

The matrix is lower triangular. Hence \( \lambda = 5, 0, 3 \) are its eigenvalues.

Other "easy" cases

20. Without calculation, find one eigenvalue and two linearly independent eigenvectors of
\( A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \).

Notice that all rows add up to the same number 6. Adding the entries in all rows simultaneously \( = A \bar{x} \) for \( \bar{x} = [1] \).

Check \( \bar{x} = [1] \):
\( A \bar{x} = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 6 \end{bmatrix} = 6 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \).

If \( A = [a_1, a_2, a_3] \),
\( A \bar{x} = a_1 + a_2 + a_3 \).

So, \( \lambda = 6 \) is an eigenvalue, and \( \bar{x} = [1] \) is an eigenvector. But we want two LI eigenvectors!

What about \( \bar{u} = [1, 0] \)?
\( A \bar{u} = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \neq \lambda \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \)

for any \( \lambda \).

\( \downarrow \)

Does not work \( \Theta \)!
Columns of \( A \) are not LI. So \( A \overline{x} = \overline{0} \) has nontrivial solutions. Hence \( (A - \lambda I) \overline{x} = \overline{0} \) has nontrivial solutions when \( \lambda = 0 \).

\[
\sqrt{\overline{a_1} - \overline{a}_2} = \overline{0} \quad \sqrt{\overline{a}_2 - \overline{a}_2} = \overline{0}
\]

In fact, \( \overline{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) and \( \overline{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \) are both nontrivial solutions to \( A \overline{x} = \overline{0} \), and hence for \( (A - \lambda I) \overline{x} = \overline{0} \) for \( \lambda = 0 \). So, \( \overline{u}_1, \overline{u}_2 \) are LI eigenvectors corresponding to the eigenvalue \( \lambda = 0 \).

Notice that \( \overline{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) and \( \overline{u}_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \) are indeed LI. But they correspond to two different eigenvalues, \( \lambda = 6 \) and \( \lambda = 0 \), respectively.

On the other hand, \( \overline{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) and \( \overline{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \) are two LI eigenvectors corresponding to the same eigenvalue \( \lambda = 0 \).

Similarly, \( \overline{u}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \) is another eigenvector corresponding to the eigenvalue \( \lambda = 0 \). In fact, the set \( \{\overline{u}_1, \overline{u}_2, \overline{u}_3\} \) is LI.
30. Consider an $n \times n$ matrix $A$ with the property that the column sums all equal the same number $s$. Show that $s$ is an eigenvalue of $A$. [Hint: Use Exercises 27 and 29.]

27. Show that $\lambda$ is an eigenvalue of $A$ if and only if $\lambda$ is an eigenvalue of $A^T$. [Hint: Find out how $A - \lambda I$ and $A^T - \lambda I$ are related.]

29. Consider an $n \times n$ matrix $A$ with the property that the row sums all equal the same number $s$. Show that $s$ is an eigenvalue of $A$. [Hint: Find an eigenvector.]

First, we show that the eigenvalues of $A$ and $A^T$ are the same (Prob 27).

$$\det (A-\lambda I) = 0 \text{ if } \lambda \text{ is an eigenvalue of } A$$

So, $$\det [(A-\lambda I)^T] = \det (A-\lambda I) = 0 \quad \text{as } \det A^T = \det A$$

hence, $$\det (A^T-\lambda I^T) = \det (A^T-\lambda I) = 0 \quad \text{as } (A+B)^T = A^T+B^T, \quad \text{and } I^T = I$$

So $\lambda$ is an eigenvalue of $A^T$.

Then we prove the row sum eigenvalue property (Prob 29). Notice that finding the sum of all rows of $A$ is equivalent to multiplying it by the vector of ones.

With $A = [\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n]$ and $\bar{u} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$, we have

$$A \bar{u} = \begin{bmatrix} \bar{a}_1 & \bar{a}_2 & \cdots & \bar{a}_n \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \bar{a}_1 + \bar{a}_2 + \cdots + \bar{a}_n = \begin{bmatrix} s \\ s \\ \vdots \\ s \end{bmatrix} \quad \text{here.}$$

all rows add up to $s$. 

\[
Hence we have \( A\vec{u} = \begin{bmatrix} 8 \\ \vdots \\ 8 \end{bmatrix} = 8 \begin{bmatrix} 1 \\ \vdots \end{bmatrix} = 8\vec{u}. \)

In other words, \( \lambda = 8 \) is an eigenvalue of \( A \), and \( \vec{u} = \begin{bmatrix} 1 \\ \vdots \end{bmatrix} \) is a corresponding eigenvector.

Now, notice that if the columns of \( A \) all add up to 8, the rows of \( A^T \) all add up to 8. Hence we can combine the two results above (problems 27 and 29) to conclude that \( \lambda = 8 \) is an eigenvalue of \( A \).
Homework on Chapter 6 - due before the final will count for 10 pts in the final.

Final Exam: Tue, Dec 10, 7-9 pm in Heald G3 (auditorium) will count for 90 pts in the final.

The invertible matrix theorem (IMT)

1. \( A \in \mathbb{R}^{n \times n} \) is invertible

2. zero is not an eigenvalue of \( A \).

3. \( \det A \neq 0 \).

4. If \( \lambda \) is an eigenvalue, then \( A \bar{x} = \lambda \bar{x} \) for some \( \bar{x} \neq 0 \). Hence if \( \lambda = 0 \) is an eigenvalue, \( A \bar{x} = 0 \) has a nontrivial solution. So \( A \) is not invertible.

The Characteristic Equation

\[ \det(A - \lambda I) = 0 \] is the characteristic equation of \( A \) (in unknown \( \lambda \)).

The polynomial given by \( \det(A - \lambda I) \) is called the characteristic polynomial of \( A \).
Find the characteristic polynomial and the real eigenvalues of the matrices in Exercises 1–8.

\[
\text{det}(A - \lambda I) = \begin{vmatrix} 8-\lambda & 2 \\ 3 & 3-\lambda \end{vmatrix} = (8-\lambda)(3-\lambda) - 2 \times 3
\]
\[
= \lambda^2 - 11\lambda + 24 - 6
\]
\[
= \lambda^2 - 11\lambda + 18 \quad \text{characteristic polynomial}
\]

\[
\lambda^2 - 11\lambda + 18 = 0 \quad \text{is the characteristic equation}
\]

\[
\Rightarrow (\lambda - 2)(\lambda - 9) = 0 \quad \text{So, } \lambda = 2, 9 \text{ are the eigenvalues. (eigenvalues are solutions or roots of the characteristic equation)}
\]

Exercises 9–14 require techniques from Section 3.1. Find the characteristic polynomial of each matrix, using either a cofactor expansion or the special formula for $3 \times 3$ determinants described prior to Exercises 15–18 in Section 3.1. [Note: Finding the characteristic polynomial of a $3 \times 3$ matrix is not easy to do with just row operations, because the variable $\lambda$ is involved.]

\[
\text{det}(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 & 1 \\ 0 & 5-\lambda & 0 \\ -2 & 0 & 7-\lambda \end{vmatrix} \quad \text{expand!}
\]

\[
= (-1)^{2+2} (5-\lambda) \begin{vmatrix} 3-\lambda & 1 \\ -2 & 7-\lambda \end{vmatrix} = (5-\lambda) \left[ (3-\lambda)(7-\lambda) - 1 \times (-2) \right]
\]

\[
= (5-\lambda) \left[ \lambda^2 - 10\lambda + 21 + 2 \right] = 5\lambda^2 - 50\lambda + 115 - \lambda^3 + 10\lambda^2 - 23\lambda
\]

\[
= -\lambda^3 + 15\lambda^2 - 73\lambda + 115 \quad \text{characteristic polynomial}
\]
**Def**

(Algebraic) multiplicity of an eigenvalue \( \lambda \) is the number of times it appears as a root of the characteristic equation.

For the matrices in Exercises 15–17, list the real eigenvalues, repeated according to their multiplicities.

17. \[
\begin{bmatrix}
3 & 0 & 0 & 0 & 0 \\
-5 & 1 & 0 & 0 & 0 \\
3 & 8 & 0 & 0 & 0 \\
0 & -7 & 2 & 1 & 0 \\
-4 & 1 & 9 & -2 & 3
\end{bmatrix}
\]

The matrix is (lower) triangular, hence the eigenvalues are the entries in the diagonal.

The eigenvalues are \(3, 3, 1, 1, 0\).

Alternatively, we can express each eigenvalue along with its multiplicity in braces, i.e., \(3(2), 1(2), 0(1)\).

\(\lambda = 0\) appears as an eigenvalue once.

**Q:** How many eigenvalues can \(A \in \mathbb{R}^{n \times n}\) have?

Recall that eigenvalues are roots of the characteristic equation, which is \(\text{det}(A-\lambda I)=0\). In \(A-\lambda I\), the \(n\) entries along the diagonal have \(\lambda\) in them. As such, \(\text{det}(A-\lambda I)\) is a polynomial of degree at most \(n\). Hence \(A\) has at most \(n\) eigenvalues. But they need not all be distinct.
It can be shown that the algebraic multiplicity of an eigenvalue $\lambda$ is always greater than or equal to the dimension of the eigenspace corresponding to $\lambda$. Find $h$ in the matrix $A$ below such that the eigenspace for $\lambda = 4$ is two-dimensional:

$$A = \begin{bmatrix} 4 & 2 & 3 & 3 \\ 0 & 2 & h & 3 \\ 0 & 0 & 4 & 14 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Rewrite: Find $h$ such that $\dim(\text{Nul}(A-\lambda I)) = 2$, i.e., $A-\lambda I$ has 2 free variables.

$$A-\lambda I = \begin{bmatrix} 0 & 2 & 3 & 3 \\ 0 & -2 & h & 3 \\ 0 & 0 & 0 & 14 \\ 0 & 0 & 0 & -2 \end{bmatrix} \xrightarrow{R_2+R_1} \begin{bmatrix} 0 & 2 & 3 & 3 \\ 0 & 0 & h+3 & 6 \\ 0 & 0 & 0 & 14 \\ 0 & 0 & 0 & 0 \end{bmatrix} = 0$$

So $h = -3$ makes the eigenspace corresponding to the eigenvalue $\lambda = 4$ 2-dimensional.

**Similar Matrices**

**Def:** Given $A, B \in \mathbb{R}^{n \times n}$, $A$ is similar to $B$ if there is an invertible matrix $P \in \mathbb{R}^{n \times n}$ such that $P^{-1}AP = B$.

In this case, $PBP^{-1} = A$, so $B$ is similar to $A$. So we just say that $A$ and $B$ are similar, and write $A \sim B$.

Using $P^{-1}$ as the invertible matrix, $(P^{-1})^tBP^{-1} = A$. 

We have already seen that if $A \sim B$, then
\[
\det A = \det B, \quad \text{as} \quad A = P^t BP \quad \text{where} \quad P \text{ is invertible.}
\]
So \( \det A = \det(P^t) \cdot \det B \cdot \det(P) \quad \text{as} \quad \det AB = \det A \cdot \det B
\]
\[
= \frac{1}{\det(P)} \cdot \det B \cdot \det(P). \quad \text{as} \quad \det(A^{-1}) = \frac{1}{\det A}
\]
when \( \det A \neq 0 \)

**Theorem**

If $A \sim B$, then they have the same characteristic polynomial, and hence the same eigenvalues.

Given $B = P^{-1}AP$

\[
B - \lambda I = P^{-1}AP - \lambda P^{-1}P = P^{-1}(AP - \lambda P) = P^{-1}(A - \lambda I)P.
\]

\[
\begin{array}{c}
B - \lambda I \\
\sim \\
B \quad I
\end{array}
\]

as \( A(B+C) = AB + AC \)

Hence $(A - \lambda I) \sim (B - \lambda I)$.

So \( \det(B - \lambda I) = \det(A - \lambda I) \quad (\text{as shown above}) \)
ER0s and eigenvalues

We have seen previously that a replacement ER0 does not change the determinant. Thus, if \( A \xrightarrow{R_i \leftrightarrow R_j} B \), then \( \det B = \det A \).

How does ER0s affect eigenvalues?

In Exercises 21 and 22, \( A \) and \( B \) are \( n \times n \) matrices. Mark each statement True or False. Justify each answer.

22. d. A row replacement operation on \( A \) does not change the eigenvalues.

False!

Consider the following example.

Let \( A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \). \( \det (A - \lambda I) = \begin{bmatrix} 1 - \lambda & 0 \\ 1 & 1 - \lambda \end{bmatrix} = (1 - \lambda)^2 = \lambda^2 - 2\lambda + 1 \)

The eigenvalue is \( 1 \) with multiplicity 2.

Let \( A \xrightarrow{R_i \leftrightarrow R_2} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = A' \). \( \det (A' - \lambda I) = \begin{vmatrix} 2 - \lambda & 1 \\ 1 & 1 - \lambda \end{vmatrix} = (2 - \lambda)(1 - \lambda) - 1 \cdot 1 \)

\[ = \lambda^2 - 3\lambda + 2 - 1 \]
\[ = \lambda^2 - 3\lambda + 1 \]

The eigenvalues are different!

\[ \lambda = \frac{3 \pm \sqrt{5}}{2} \]

are the two eigenvalues of \( A' \).

\( \alpha \pm \sqrt{b^2 - 4ac} \)

are the solutions to \( ax^2 + bx + c = 0 \).
If the real eigenvalue with the largest absolute value is < 1 (in absolute value), then the population will become extinct eventually.
Inner product, length (norm), orthogonality

Basis for a subspace:
Consider $\mathbb{R}^2$, for example. Let
\[
\overrightarrow{v_1} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad \overrightarrow{v_2} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}.
\]
Then
\[B_1 = \{ \overrightarrow{v_1}, \overrightarrow{v_2} \}\]
is a basis.

But \[B_e = \{ e_1, e_2 \} = \{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \}\]
is also a basis for $\mathbb{R}^2$.

In fact, it is the standard basis.

Notice that $\overrightarrow{e_1}$ and $\overrightarrow{e_2}$ are perpendicular to each other, and are also of unit length. "most LI"

$\overrightarrow{v_1}$ and $\overrightarrow{v_2}$ are not collinear, and hence are LI, and so
\[B_1 = \{ \overrightarrow{v_1}, \overrightarrow{v_2} \}\]
is indeed a basis for $\mathbb{R}^2$. But $\overrightarrow{e_1}$ and $\overrightarrow{e_2}$ are the "most LI", as they are the farthest away from being collinear.

We can extend these results to any subspace — given any basis for a subspace, we can find a basis that has vectors of unit length, and are "perpendicular" to each other.
**Inner product (or scalar product or dot product)**

\[ \vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}. \]

Then \[ \vec{u}^\top \vec{v} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n. \]

\[ \vec{u} \cdot \vec{v} = \vec{u}^\top \vec{v} = \vec{v}^\top \vec{u} = u_1 v_1 + \cdots + u_n v_n. \]

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Compute the quantities in Exercises 1–8 using the vectors

\[ \vec{w} = \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix} \]

2. \( \vec{w} \cdot \vec{w}, \vec{x} \cdot \vec{w}, \text{ and } \frac{\vec{x} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \)

\[ \begin{align*}
\vec{w} \cdot \vec{w} &= \begin{bmatrix} 3 & -1 & -5 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix} = 3 \cdot 3 + (-1)(-1) + (-5)(-5) \\
&= 9 + 1 + 25 = 35 \\
\vec{x} \cdot \vec{w} &= 6 \cdot 3 + (-2)(-1) + 3(-5) \\
&= 18 + 2 - 15 = 5 \\
\frac{\vec{x} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} &= \frac{5}{35} = \frac{1}{7}.
\end{align*} \]

**Length of a vector (also called norm)**

\[ \| \vec{v} \| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}. \quad (\text{Notice that } \vec{v} \cdot \vec{v} = \| \vec{v} \|^2) \]

E.g., \( \vec{v}_2 = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \) \[ \| \vec{v}_2 \| = \sqrt{4^2 + 3^2} = \sqrt{16 + 9} = \sqrt{25} = 5. \]

\[ \text{(hypotenuse)}^2 = \text{(base)}^2 + \text{(altitude)}^2 \]

Thus, we are extending the Pythagorean theorem to higher dimensions here to define the length of a vector.
Distance between two vectors \( \vec{u}, \vec{v} \)

\[
dist(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\| \quad \text{(length of the difference)}
\]

\[
dist(\begin{bmatrix} 3 \\ 2 \\ \frac{4}{3} \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ \frac{1}{3} \end{bmatrix}) = \|\begin{bmatrix} 3-4 \\ 2-(-3) \\ \frac{4}{3}-\frac{1}{3} \end{bmatrix}\| = \sqrt{(-1)^2 + 5^2} = \sqrt{26}
\]

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14. Find the distance between \( \vec{u} = \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix} \) and \( \vec{z} = \begin{bmatrix} -4 \\ -1 \\ 8 \end{bmatrix} \).

\[
dist(\vec{u}, \vec{z}) = \|\vec{u} - \vec{z}\| = \|\begin{bmatrix} 0-(-4) \\ -5-(-1) \\ 2-8 \end{bmatrix}\| = \|\begin{bmatrix} 4 \\ -4 \\ -6 \end{bmatrix}\| = \sqrt{4^2 + (-4)^2 + (-6)^2} = \sqrt{68}
\]

Unit vector in the direction of \( \vec{v} \)

\[
\vec{\hat{v}} = \frac{\vec{v}}{\|\vec{v}\|} \quad \text{is a unit vector, i.e., vector of length 1, along } \vec{v}.
\]

\[
\vec{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \|\vec{v}_1\| = \sqrt{3^2 + 2^2} = \sqrt{13}
\]
\[
\vec{v}_2 = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \quad \|\vec{v}_2\| = 5
\]

\[
\vec{\hat{v}}_1 = \frac{1}{\sqrt{13}} \vec{v}_1 = \frac{1}{\sqrt{13}} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \end{bmatrix}
\]
\[
\vec{\hat{v}}_2 = \frac{1}{5} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{4}{5} \\ \frac{3}{5} \end{bmatrix}
\]
$\overrightarrow{u}$ and $\overrightarrow{v}$ (two vectors in $\mathbb{R}^n$) are orthogonal if $\overrightarrow{u} \cdot \overrightarrow{v} = 0$. We denote $\overrightarrow{u} \perp \overrightarrow{v}$.

E.g., $\overrightarrow{u} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$, $\overrightarrow{v} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$, $\overrightarrow{u} \cdot \overrightarrow{v} = 3 \cdot 0 + 0 \cdot 4 = 0$

$\overrightarrow{v}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $\overrightarrow{v}_3 = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$

$\overrightarrow{v}_1 \cdot \overrightarrow{v}_3 = 3 \cdot 2 + 2 \cdot (-3) = 0$. 

Orthogonal set

A set $\{\overrightarrow{u}_1, \overrightarrow{u}_2, \ldots, \overrightarrow{u}_p\}$ in $\mathbb{R}^n$ is an orthogonal set if $\overrightarrow{u}_i \cdot \overrightarrow{u}_j = 0$ for all $i \neq j$, i.e., $\overrightarrow{u}_i \perp \overrightarrow{u}_j$ for $i \neq j$.

$\left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is an orthogonal set.

An orthogonal basis is a basis which is an orthogonal set. An orthonormal basis is an orthogonal basis where the vectors each have unit length.
\[
\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\]
\] is an orthogonal basis for \(\mathbb{R}^3\).

And \[
\left\{ \frac{1}{2} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \frac{1}{3} \begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}
\] is an orthonormal basis for \(\mathbb{R}^3\).

Orthogonal projection

\[\text{The length of the "shadow of } \vec{w} \text{ along } \vec{u} \text{" is the length of } \vec{w} \text{ along } \vec{u}.\]

The orthogonal projection of \(\vec{w}\) on to \(\vec{u}\) is the vector \(\vec{y} = \alpha \vec{u}\) where \(\alpha = \frac{\vec{w} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\).

Consider \(\vec{w}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}\) and \(\vec{w}_2 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}\) again. The orthogonal projection of \(\vec{w}_2\) on to \(\vec{w}_1\) is the vector

\[\vec{y} = \frac{\vec{w}_2 \cdot \vec{w}_1}{\vec{w}_1 \cdot \vec{w}_1} \vec{w}_1\]
We get

\[
\overline{y} = \left( \frac{\left[ \frac{4}{3} \cdot \frac{3}{2} \right]}{\left[ \frac{3}{2} \cdot \frac{3}{2} \right]} \right) \left[ \frac{3}{2} \right]
\]

\[
= \left( \frac{4 \cdot 3 + 3 \cdot 2}{3^2 + 2^2} \right) \left[ \frac{3}{2} \right]
\]

\[
= \frac{18}{13} \left[ \frac{3}{2} \right] = \left[ \frac{54}{13} \right] = \left[ \frac{36}{13} \right]
\]
14. Let \( \mathbf{y} = \begin{bmatrix} 2 \\ 6 \end{bmatrix} \) and \( \mathbf{u} = \begin{bmatrix} 7 \\ 1 \end{bmatrix} \). Write \( \mathbf{y} \) as the sum of a vector in \( \text{Span} \{ \mathbf{u} \} \) and a vector orthogonal to \( \mathbf{u} \).

Write \( \overline{\mathbf{y}} = \overline{\mathbf{y}}_\mathbf{u} + \overline{\mathbf{v}} \) where

\[ \overline{\mathbf{y}}_\mathbf{u} = \alpha \mathbf{u} \quad \text{and} \quad \overline{\mathbf{y}}_\mathbf{u} \perp \overline{\mathbf{v}} \]

Can find the orthogonal projection of \( \overline{\mathbf{y}} \) on to \( \mathbf{u} \) to get \( \overline{\mathbf{y}}_\mathbf{u} \).

\[
\alpha = \frac{\overline{\mathbf{y}} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} = \frac{\begin{bmatrix} 2 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 1 \end{bmatrix}}{(7)^2 + (1)^2} = \frac{2 \times 7 + 6 \times 1}{49 + 1} = \frac{20}{50} = \frac{2}{5}
\]

\[
\begin{bmatrix} 7 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 1 \end{bmatrix} \overline{\mathbf{y}}_\mathbf{u} = \frac{2}{5} \mathbf{u} = \frac{2}{5} \begin{bmatrix} 7 \\ 1 \end{bmatrix}.
\]

\[
\overline{\mathbf{v}} = \overline{\mathbf{y}} - \overline{\mathbf{y}}_\mathbf{u} = \begin{bmatrix} 2 \\ 6 \end{bmatrix} - \frac{2}{5} \begin{bmatrix} 7 \\ 1 \end{bmatrix} = \frac{5}{5} \begin{bmatrix} 2 \\ 6 \end{bmatrix} - \frac{2}{5} \begin{bmatrix} 7 \\ 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} -4 \\ 28 \end{bmatrix} = \frac{4}{5} \begin{bmatrix} -1 \\ 7 \end{bmatrix}.
\]

So, \( \overline{\mathbf{y}} = \begin{bmatrix} 2 \\ 6 \end{bmatrix} = \frac{2}{5} \begin{bmatrix} 7 \\ 1 \end{bmatrix} + \frac{4}{5} \begin{bmatrix} -1 \\ 7 \end{bmatrix} \)

are orthogonal!

as \( \begin{bmatrix} 7 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 7 \end{bmatrix} = -7 + 7 = 0 \)
Properties of scalar products

\[ \vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w} \]
\[ \vec{u} \cdot (c \vec{v}) = (c \vec{u}) \cdot \vec{v} = c (\vec{u} \cdot \vec{v}) \]

Review of final

5. Justify your answer for each of the following.

(a) If \( A \) has more columns than rows, can the columns of \( A \) be linearly independent?

\[ \text{No.} \]
\[ \text{All columns cannot be pivot columns.} \]
\[ \text{or There will be free variables.} \]

(b) If \( A \) is a 5 \( \times \) 5 matrix and the rank of \( A \) is 5, is \( \det(A) = 0? \)

\[ \text{No. A has five 5 pivot columns and no free variables.} \]
\[ \text{So A is invertible. So } \det(A) \neq 0. \]

(c) Do six linearly independent vectors in \( \mathbb{R}^9 \) span a subspace of dimension six?

\[ \text{dimension} = \# \text{ vectors in any basis} \]
\[ \text{The six LI vectors form a basis for the} \]
\[ \text{span of these vectors. Hence the answer is YES.} \]
\[ \text{But they do NOT span } \mathbb{R}^6 \text{, as each vector is in } \mathbb{R}^9 \text{ to start with.} \]
(d) If $A$, $B$, and $C$ are $n \times n$ matrices and $AB = AC$, must $B = C$?

No. $B = C$ only if $A$ is invertible.

\[ A^{-1}(AB = AC) \]
\[ (A^{-1}A)B = (A^{-1}A)C \quad \text{or} \quad B = C \]

\[ I \quad I \]

7. Suppose that $A$ is a matrix with $\text{rank}(A) = 3$, $\text{dim Nul}(A) = 2$, and such that the row reduced echelon form of $A$ has one row of zeros. How many rows does $A$ have? How many columns does $A$ have?

The rank theorem says $\text{rank}(A) + \text{dim Nul}(A) = n$ when $A$ is $m \times n$.

Here $n = 3 + 2 = 5$.

Since $\text{rank}(A) = 3$, there are three pivots. So there should be 3 nonzero rows in $\text{rref}(A)$. Since there is one zero row in $\text{rref}(A)$, $A$ has $3 + 1 = 4$ rows.

9. Let $x \in \mathbb{R}^n$ be an eigenvector of both the $n \times n$ matrices $A$ and $B$. Show that $x$ is an eigenvector of the matrix $AB$.

Let $\lambda, \mu$ be the eigenvalues of $A$ and $B$ corresponding to the eigenvector $x$. So

\[ A\bar{x} = \lambda \bar{x} \quad \text{So} \quad AB \bar{x} = A(B \bar{x}) = A(\mu \bar{x}) = \mu A\bar{x} \]
\[ B\bar{x} = \mu \bar{x} \quad = \mu (\lambda \bar{x}) = \lambda \mu \bar{x} \]

So $\bar{x}$ is an eigenvector of $AB$ for the eigenvalue $\lambda \mu$. 

Practice final

7. (7) Construct a nonzero $3 \times 3$ matrix $A$ with rank 2, and a vector $b$ that is not in Nul $A$.

\[
\begin{bmatrix}
1 & 2 & 3 \\
0 & 1 & 2 \\
0 & 0 & 0
\end{bmatrix}
\text{ has rank 2.}
\]

\[
\begin{bmatrix}
1 & 2 & 3 \\
0 & 1 & 2 \\
0 & 0 & 0
\end{bmatrix}
\xrightarrow{R_2+R_1}
\begin{bmatrix}
1 & 2 & 3 \\
1 & 3 & 5 \\
1 & 2 & 3
\end{bmatrix}
\]

\[
A = \begin{bmatrix}
1 & 2 & 3 \\
1 & 3 & 5 \\
1 & 2 & 3
\end{bmatrix}
\text{ works.}
\]

\[
\bar{b} = \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}
\text{ is not in Nul } A, \text{ as}
\]

\[
A\bar{b} = \begin{bmatrix}
1 & 2 & 3 \\
1 & 3 & 5 \\
1 & 2 & 3
\end{bmatrix} \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix} = \begin{bmatrix}
1
\end{bmatrix} \neq \bar{b}.
\]

\[
A = \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1
\end{bmatrix}
\text{ also works. Or, you could directly write down } A = \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1
\end{bmatrix}, \text{ for instance. There are two pivots in the matrix here, and all entries are nonzero. The same } \bar{b} \text{ given above works in both cases here as well, since } A\bar{b} \neq 0 \text{ in both cases.}
\]
(10) Let $A + B$ and $C$ be $n \times n$ invertible matrices. Solve the following equation for $X$. Justify each step in your solution.

\[ C^{-1} (XB + XA)C = C^T. \]

\[
C^{-1} (X(B+A))C = C^T \quad \text{as} \quad A(B+C) = AB + AC
\]

\[ C^{-1} (X(B+A))C C^{-1} = C C^T C^{-1} \quad \text{as} \quad C \text{ is invertible} \]

\[ I (X(B+A)) I = C C^T C^{-1} \quad \text{as} \quad CC^T = I \]

\[ X (B+A) (B+A)^{-1} = C C^T C^{-1} (B+A)^{-1} \quad A+B = B+A \]

\[ I \]

\[ X = C C^T C^{-1} (A+B)^{-1}. \]

Notice that $C^T C^{-1} \neq I$! We had seen earlier that $(C^{-1})^T = (C^T)^{-1}$. But here we have $C^T C^{-1}$, which cannot be simplified further.