Micro motions of a swimmer in the 3-D incompressible fluid governed by the nonstationary Stokes equation

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Abstract

In this paper we consider a model describing the self-propelled motion of a “small” abstract swimmer in the 3-D incompressible fluid, governed by the nonstationary Stokes equation. Typically this fluid is associated with the low Reynolds numbers when inertia is viewed as negligible. It is assumed that the swimmer’s body consists of finitely many substantially connected parts, identified with the fluid they occupy, linked to each other by rotational and elastic Hooke’s forces [21]-[22]. Our goal is to derive a formula for asymptotically small motions of this swimmer in the 3-D incompressible fluid. These results can be useful for biological and engineering applications dealing with the study and design of propulsion systems in fluids. They can also be instrumental to study controllability properties of this swimming model.

Key words: Swimming models, hydrodynamics, hybrid pde/integro-differential ode systems, nonstationary 3-D Stokes equation.

1. Introduction. In [22] we introduced the following hybrid model for a “small” swimmer in the 3-D incompressible fluid, governed by the nonstationary Stokes equation (see Figs. 1-2):

\[
\begin{align*}
\frac{\partial y}{\partial t} &= \nu \Delta y + F(z, v) - \nabla p \quad \text{in } Q_T = \Omega \times (0, T), \quad y = (y_1, y_2, y_3) \\
\text{div } y &= 0 \quad \text{in } Q_T, \quad y = 0 \quad \text{in } \Sigma_T = \partial \Omega \times (0, T), \quad y |_{t=0} = y_0 \quad \text{in } \Omega,
\end{align*}
\]

\[
\frac{dz_i}{dt} = \frac{1}{\text{mes } \{S_0\}} \int_{S_i(z_i(t))} y(x, t) dx, \quad z_i(0) = z_{i0}, \quad i = 1, \ldots, n, \quad n > 2,
\]

where for \( t \in [0, T] \):

\[
z(t) = (z_1(t), \ldots, z_n(t)), \quad z_i(t) \in \mathbb{R}^3, \quad i = 1, \ldots, n, \quad v(t) = (v_1(t), \ldots, v_{n-2}(t)) \in \mathbb{R}^{n-2},
\]

\[
F(z, v) = \sum_{i=2}^{n} |v_{i-1}(t)|k_{i-1} \left( \frac{\|z_i(t) - z_{i-1}(t)\|_{\mathbb{R}^3} - l_{i-1}}{\|z_i(t) - z_{i-1}(t)\|_{\mathbb{R}^3}} \right) (z_i(t) - z_{i-1}(t))
\]

\[
+ \xi_i(x, t)k_{i-1} \left( \frac{\|z_i(t) - z_{i-1}(t)\|_{\mathbb{R}^3} - l_{i-1}}{\|z_i(t) - z_{i-1}(t)\|_{\mathbb{R}^3}} \right) (z_{i-1}(t) - z_i(t))
\]

\[
+ \sum_{i=2}^{n-1} v_{i-1}(t) \left[ \xi_{i-1}(x, t)A_i(z_{i-1}(t) - z_i(t)) - \xi_{i+1}(x, t) \left( \|z_{i-1}(t) - z_i(t)\|_{\mathbb{R}^3}^2 B_i(z_{i+1}(t) - z_i(t)) \right) \right]
\]

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\[ -\sum_{i=2}^{n-1} \xi_i(x, t)v_{i-1}(t) \left[ A_i(z_{i-1}(t) - z_i(t)) - \frac{\|z_{i-1}(t) - z_i(t)\|_{R^3}^2}{\|z_{i+1}(t) - z_i(t)\|_{R^3}^2} B_i(z_{i+1}(t) - z_i(t)) \right]. \tag{1.3} \]

Equation (1.1) is for the nonstationary 3-D Stokes equation for the fluid excited by the force term \( F(z, v) \), which in turn is generated by the internal forces of the swimmer placed inside the fluid. The position of swimmer in space is described by system of integro-differential odes (1.2) for the centers of mass \( z_i(t) \)'s of the respective identical separate parts of its body. The swimmer’s center of mass respectively will be the point

\[ z(t) = \frac{1}{n} \sum_{i=1}^{n} z_i(t) \in \Omega. \]

We assume that the swimming motion occurs in a bounded domain \( \Omega \subset R^3 \) with boundary \( \partial \Omega \) of class \( C^2 \), \( y(x, t) \) and \( p(x, t) \) are respectively the velocity and the pressure of the fluid at point \( x = (x_1, x_2, x_3) \in \Omega \) at time \( t \), while \( \nu \) is a kinematic viscosity constant.

\[ \text{Figure 1: A 4-piece swimmer shown with all elastic forces and rotation forces about } z_2(t) \text{ active only, when it is not in a fluid. (compare to Fig.2)} \]
Let us now elaborate on the terms in (1.1)-(1.3).

**Swimmer:** In the general case the swimmer is modeled as a collection of \( n \) bounded sets \( S_i(z_i(t)), i = 1, \ldots, n \) of non-zero measure (such as balls, parallelepipeds, etc.), identified with the fluid within the space which they occupy. These sets are assumed to be open bounded connected sets symmetric relative to the points \( z_i(t) \)’s which are their centers of mass. The sets \( S_i(z(t)) \)’s are viewed as the given sets \( S_i(0) \)’s (“0” stands of the origin) that have been shifted to the respective positions \( z_i(t) \)’s without changing their orientation in space. Respectively,

\[
\xi_i(x, t) = \begin{cases} 
1, & \text{if } x \in S_i(z_i(t)), \\
0, & \text{if } x \in \Omega \setminus S(z_i(t)), 
\end{cases} \quad i = 1, \ldots, n. \tag{1.4}
\]

We assume that each \( S_i(0) \) lies in a “small” neighborhood of the origin of given radius \( r > 0 \), while \( S_i(a) \) denotes the set \( S_i(0) \) shifted to point \( a \). Throughout this paper we assume that the swimmer consists of identical parallelepipeds of dimensions \( p \times q \times l \):

\[
S_0 = \{ x \mid -p/2 < x_1 < p/2, -q/2 < x_2 < q/2, -l/2 < x_3 < l/2 \}, \quad 0 < l \leq q \leq p. \tag{1.5}
\]

The spatial orientation of these rectangles as sets \( S_i(z_i(t)), i = 1, \ldots, n \) in (1.1)-(1.3) can be different.

**Forces:** We assume that parallelepipeds \( S_i(z_i(t)) \)’s are subsequently linked by forces described by the term \( F(z, v) \). No “actual” physical links between sets \( S_i(z_i(t)) \) are assumed (i.e., they are assumed to be negligible in terms of affect on the resisting surrounding fluid). The forces in (1.3) are *internal*, relative to the swimmer – their sum is zero. We assume that a force applied to a set \( S(z(t)) \) acts evenly over all its points, and, as such, it creates a transformed external force on the incompressible fluid surrounding \( S(z(t)) \) (see Figs. 1-2, and the respective results in Theorems 2.2-2.5 and A.2-A.5).

The structural integrity of the swimming object is preserved by the elastic forces acting according to Hooke’s Law. They act along the lines connecting the respective adjacent centers \( z_i(t) \)’s when the distances between any two adjacent points \( z_{i-1}(t) \) and \( z_i(t), i = 2, \ldots, n \) deviate from the respective given values \( l_{i-1} > 0, i = 2, \ldots, n \) as described in the first sum in (1.3). The constant parameters \( k_i > 0, i = 1, \ldots, n-1 \) characterize the rigidity of the links \( z_{i-1}(t)z_{i}(t), i = 2, \ldots, n \). The matching pairs of these forces between \( z_{i-1}(t) \) and \( z_i(t) \), and between \( z_i(t) \) and \( z_{i+1}(t) \) are shown on Fig. 1.

The 2nd sum in (1.3) describes the rotation forces about any of the points \( z_i(t), i = 2, \ldots, n-1 \) which make the adjacent points to rotate about it perpendicular to the lines connecting the respective \( z_i(t) \)’s. To satisfy the 3rd Newton’s Law, these forces lie in the same plane along with the matching counter-force given in the 3rd sum in (1.3). Respectively,

\[
A_i = A_i(z_{i-1}(t), z_i(t), z_{i+1}(t)) : R^3 \to \text{span} \{ z_{i-1}(t) - z_i(t), z_{i+1}(t) - z_i(t) \},
\]

\[
B_i = B_i(z_{i-1}(t), z_i(t), z_{i+1}(t)) : R^3 \to \text{span} \{ z_{i-1}(t) - z_i(t), z_{i+1}(t) - z_i(t) \}
\]

denote nonlinear mappings, defined at each moment of time by three vectors \( z_{i-1}(t), z_i(t), z_{i+1}(t) \), such that for \( i = 2, \ldots, n-1 \):

* \( (z_{i-1}(t) - z_i(t))^T[A_i(z_{i-1}(t) - z_i(t))] = 0, \quad (z_{i+1}(t) - z_i(t))^T[B_i(z_{i+1}(t) - z_i(t))] = 0; \)

* \( \| A_i(z_{i-1}(t) - z_i(t)) \|_{R^3} = \| z_{i-1}(t) - z_i(t) \|_{R^3}, \)

\( \| B_i(z_{i+1}(t) - z_i(t)) \|_{R^3} = \| z_{i+1}(t) - z_i(t) \|_{R^3}; \)
and the directions of vectors \( (A_i(z_{i-1}(t) - z_i(t))) \) and \( (B_i(z_{i+1}(t) - z_i(t))) \) are such that they correspond to either folding or unfolding motion of lines \( z_{i-1}(t)z_i(t) \) and \( z_i(t)z_{i+1}(t) \) relative to the point \( z_i(t) \).

A matching pair of rotation forces, generated by \( z_2(t) \) for the adjacent points, is shown on Fig. 1.

The magnitudes and directions of the rotation forces are determined by the given coefficients \( v_i(t), \ i = 1, \ldots, n - 2 \). The choice of fractional coefficients at terms \( A_i(z_{i+1}(t) - z_i(t)) \) in (1.3) ensures that the momentum of swimmer’s internal forces is conserved at any \( t \in (0, T) \) (see the respective 2-D calculations in [19]).

Swimmer’s motion. Dynamics of points \( z_i(t)\xi_i(x,t), i = 1, \ldots, n \) are determined by the average motion of the fluid within their respective supports \( S_i(z_i(t)) \)'s as described in (1.2).

Note that, when the adjacent points in the swimmer’s body share the same position in space, the forcing term \( F \) in (1.3) and hence model (1.1)-(1.3) become undefined. While such situation mathematically seems possible, it does not have to happen. First of all, one can address the issue of local existence of solutions to (1.1)-(1.3) on some “small” time-interval \( (0, T) \), assuming that initially model (1.1)-(1.3) is well-defined in the above sense. This was the primary subject of [22] (see the appendix). Then the question of global existence can be viewed as the issue of suitable selection of coefficients \( v_i \)'s with the purpose to ensure that the aforementioned ill-posed situation is avoided.

Comments on the choice of fluid equation. In model (1.1)-(1.3) we chose the fluid governed by the nonstationary Stokes equation which, along with its stationary version, is a typical choice of fluid for micro-swimmers (the case of Low Reynolds numbers). The empiric reasoning behind this is that, due to the small size of swimmer, the inertia terms in the Navier-Stokes equation, containing the 1-st order derivatives in \( t \) and \( x \), can be omitted, provided that the frequency parameter of the swimmer at hand is a quantity of order unity. However, it was noted that a micro-swimmer (e.g., a nano-size robot) may use a rather high frequency of motion, which may justify at least in some cases the need for the term \( y_i \) in Stokes model equations. In general, it seems reasonable to suggest that the presence of this term (in a number of cases) can provide a better approximation of the Navier-Stokes equation than the lack of it. We also point out that in [6], [7], [32] the full-size Navier-Stokes equation is used for micro-swimmers. It also seems that the methods we use for the nonstationary Stokes equation (as opposed to stationary Stokes equation), may serve as a natural step toward the swimming models, based on the Navier-Stokes equation.

Remarks on bibliography in the field. It seems that the first quantitative research in the area of swimming phenomenon was aimed at the biomechanics of specific biological species: Gray [10](1932), Gray and Hancock [11] (1951), Taylor [37] (1951), [38] (1952), Wu [42] (1971), Lighthill [26] (1975), and others. These efforts resulted in the derivation of a number of mathematical models (linked to the size of Reynolds number) for swimming motion in the whole \( R^2 \)- or \( R^3 \)-spaces with the swimmer to be used as the reference frame, see, e.g., Childress [3] (1981) and the references therein. Such approach however requires some modification if one wants to track the actual position of swimmer in a fluid.

A different modeling approach was proposed by Peskin in the computational mathematical biology (see Peskin [31] (1975), Fauci and Peskin [6] (1988), Fauci [7] (1993), Peskin and McQueen [32] (1994) and the references therein), where a swimmer is modeled as an immaterial immersed boundary identified with the fluid, further discretized for computational purposes on some grid. In this case a fluid equation is to be complemented by a coupled infinite dimensional differential equation for the aforementioned “immersed boundary”.

In this paper we intend to deal with the swimming phenomenon in the framework of nonstationary PDE’s along the immersed body approach summarized in Khapalov [20] (2005-2010).
Namely, in [18] (2005), inspired by the ideas of the above-cited Peskin’s method, we introduced a 2-D model for “small” flexible swimmers assuming that their bodies are identified with the fluid occupying their shapes. This approach views such a swimmer as the already discretized aforementioned immersed boundary supported on the respective grid cells, see, e.g., Fig. 1 below. Our model offered two novel features: (1) it was set in a bounded domain with (2) governing equations to be a fluid equation coupled with a system of ODE’s describing the spatial position of swimmer within the space domain. We established the wellposedness of this model up to the contact of a swimmer at hand either with the boundary of space domain or with itself. The need of such type of models was motivated by the intention to investigate controllability properties of swimming phenomenon (see [20]).

It should be noted that, the classical mathematical issue of wellposedness of a swimming model as a system of PDE’s for the first time was apparently addressed by Galdi [8] (1999) for a model of swimming micromotions in $R^3$ (with the swimmer serving as the reference frame).

Another available approach to modeling of swimming motion (apparently, initiated by the work Shapere and Wilczek [35] (1989)) exploits the idea that the swimmer’s body shape transformations during the actual swimming process can be viewed as a set-valued map in time. The respective models describe swimmer’s position via such maps, see [13] (1981), [34] (2008), [4] (2011) and the references therein. Some models treat these maps as a priori prescribed, in which case the crux of the problem is to identify which maps are admissible, i.e., compatible with the principle of self-propulsion of swimming locomotion. In the case when the aforementioned motion map is not a priori prescribed (i.e., it will be defined at each moment of time by swimmer’s internal forces and the interaction of its body with the resisting surrounding medium), the model will have to include extra equations, see, e.g., [41] in the framework of the immersed boundary method and the references therein.

More recently, a number of significant efforts, both theoretical and experimental, were made to study models of possible bio-mimetic mechanical devises which employ the change of their geometry, inflicted by internal forces, as the means for self-propulsion, see, e.g., S. Hirose [16] (1993), Mason and Burdick [27] (2000); McIsaac and Ostrowski [28] (2000); Martinez and J. Cortes [29] (2001); Trinhafyllou et al. [40] (2000); Morgansen et al. [30] (2001); Fakuda et al. [5] (2002); Guo et al. [12] (2002); Hawthorne et al. [14] (2004), and the references therein. It was also recognized that sophistication and complexity of design of bio-mimetic robots give rise to control-theoretic methods, see, e.g., Koiller et al. [23] (1996); McIsaac and Ostrowski [28] (2000); Martinez and Cortes [29] (2001); Trinhafyllou et al. [40] (2000); San Martin et al. [33](2007), Alouges et al. [1] (2008), Sigalotti and Vivalda [36] (2009), and the references therein. It should be noted however that the above-cited results deal with control problems in the framework of ODE’s only.

A number of attempts were made along these lines to introduce various reduction techniques to convert swimming model equations into systems of ODE’s, namely, by making use of applicable analytical considerations, empiric observations and experimental data, see, e.g., Becker et al [2] (2003); Kanso et al. [17] (2005); San Martin et al. [33] (2007); Alouges et al. [1] (2008), and the references therein.

2. Formulation of main results. In this paper our goal is to study the asymptotically small (or as we call it – “micro”) motions of model (1.1)-(1.3) in the case when the swimmer’s body consists of identical rectangles (1.5). More precisely, we consider the motions of the swimming object during some “small” time-interval under the following assumptions.

**Assumption 2.1.** Let $t_0 > 0$ be given and for $t \in [0, t_0]$ the multiplicative controls $v_i, i = 1, \ldots, n - 1$ remain constant, and the spatial orientations of rectangles $S_i(x_i(t))$’s do not change.

**Assumption 2.2.** For the given $r \geq \frac{1}{2} \sqrt{n^2 + q^2 + l^2}$, defining the size of rectangles $S_i(0)$ in
(1.4)-(1.5), assume that
\[ l_{i-1} > 2r, \ i = 2, \ldots, n; \ \overline{S_i}(z_i(0)) \subset \Omega, \ \|z_{i,0} - z_{j,0}\|_{R^3} > 2r, \ i, j = 1, \ldots, n, \ i \neq j; \quad (2.1) \]

Note next that in the case of rectangular-shaped sets \( S_i(0) \)'s we have:
\[
\int_{(S_i(0) \cup S_i(h)) \setminus (S_i(0) \cap S_i(h))} dx = \int_{\Omega} |\xi_i(x) - \xi_i(x - h)| dx \leq Cpq \|h\|_{R^3} \quad \forall h \in B_{h_0}(0) \quad (2.2)
\]
for some positive constants \( h_0 \), where \( \xi_i(x) \) is the characteristic function of \( S_i(0) \) and \( B_{h_0}(0) = \{x \parallel x\|_{R^3} < h_0\} \subset R^3 \). Here and below the generic notation \( C > 0 \) is used for possibly different constants.

Conditions (2.1) mean that at time \( t = 0 \), any two sets \( S_i(z_i(0)) \) do not overlap, and that the swimmer lies in \( \Omega \). Property (2.2) is a regularity assumption of Lipschitz type.

**Assumption 2.3.** Assume that within some \( (R^3)^n \) neighborhood \( G(z(0)) \subset (R^3)^n \) of the initial datum in (1.2) the mappings \( A_i \) and \( B_i \) are uniformly Lipschitz for all \( i = 2, \ldots, n-1 \) in the following sense:
\[
\| A_i(a_{i-1}, a_i, a_{i+1}) - A_i(b_{i-1}, b_i, b_{i+1}) \|_{R^3} \\
\leq L \|a_{i-1} - b_{i-1}\|_{R^3} + \|a_i - b_i\|_{R^3} + \|a_{i+1} - b_{i+1}\|_{R^3}, \\
\| B_i(a_{i-1}, a_i, a_{i+1}) - B_i(b_{i-1}, b_i, b_{i+1}) \|_{R^3} \\
\leq L \|a_{i-1} - b_{i-1}\|_{R^3} + \|a_i - b_i\|_{R^3} + \|a_{i+1} - b_{i+1}\|_{R^3},
\]
for any \( a_{i-1}, a_i, a_{i+1}, b_{i-1}, b_i, b_{i+1} \in G(z(0)) \), where \( L > 0 \) is a constant and \( A_i \)'s and \( B_i \)'s are defined as in the introduction by three respective vectors \( a_{i-1}, a_i, a_{i+1} \).

Assumption 2.3 can be satisfied if, e.g., the points \( a_{i-1} = z_{i-1}(0), a_i = z_i(0), a_{i+1} = z_{i+1}(0), i = 2, \ldots, n-1 \) are not on the same line and the mappings \( A_i \) and \( B_i \) are selected in (1.3) by making use of Gramm-Schmidt orthogonalization procedure for vectors \( a_{i-1} - a_i \) and \( a_{i+1} - a_i \). Alternatively, we can define \( A_i \)'s and \( B_i \)'s, making use of the cross-product:
\[
(A_i(z_{i-1}(t) - z_i(t))(t) = e_1(t) \| z_{i-1}(t) - z_i(t) \|_{R^3}, \\
(B_i(z_{i+1}(t) - z_i(t))(t) = e_2(t) \| z_{i+1}(t) - z_i(t) \|_{R^3}, \quad e_i(t) = \frac{s_i(t)}{s_i(t)} \parallel s_i(t) \parallel_{R^3}, \quad i = 1, 2, \\
s_1(t) = (z_{i-1}(t) - z_i(t)) \times [(z_{i-1}(t) - z_i(t)) \times (z_{i+1}(t) - z_i(t))], \\
s_2(t) = [(z_{i+1}(t) - z_i(t))(z_{i+1}(t) - z_i(t))] \times (z_{i+1}(t) - z_i(t)).
\]

**Assumption 2.4.** We assume below that, for the given choice of the initial datum and \( k_i \)'s and \( l_i \)'s in system (1.1)-(1.5), the distances between the respective \( S_i(z_i(t)) \)'s, \( t \in [0, t_*] \) (where \( t_* \) is from Assumption 2.1) and from them to the boundary \( \partial \Omega \) exceed some positive value \( d_* > 0 \), while the values
\[
\frac{\|z_{i-1}(t) - z_i(t)\|^2_{R^3}}{\|z_{i+1}(t) - z_i(t)\|^2_{R^3}}, \quad i = 2, \ldots, n-2
\]
do not exceed \( d^* \).

This assumption holds for sufficient small \( t_* > 0 \), provided that it holds strictly at time \( t = 0 \), due to the regularity result of Theorem A.1 in the appendix.
We assume that Assumptions 2.1-2.4 hold throughout the rest of the paper.

Let \( \hat{J}(\Omega) \) denote the set of infinitely differentiable vector functions with values in \( \mathbb{R}^3 \) which have compact support in \( \Omega \) and are divergence-free, i.e., \( \text{div}\phi = 0 \) in \( \Omega \). Denote by \( J_0(\Omega) \) the closure of this set in the \((L^2(\Omega))^3\)-norm and by \( G(\Omega) \) denote the orthogonal complement of \( J_0(\Omega) \) in \((L^2(\Omega))^3\) (see, e.g., [24], [39]). In \( \hat{J}(\Omega) \) introduce the scalar product

\[
[\phi_1, \phi_2] = \int_\Omega \sum_{j=1}^3 \sum_{i=1}^3 \phi_{1j_i}\phi_{2j_i} \, dx, \quad \phi_1(x) = (\phi_{11}, \phi_{12}, \phi_{13}), \quad \phi_2(x) = (\phi_{21}, \phi_{22}, \phi_{23}).
\]

Denote by \( H(\Omega) \) the Hilbert space which is the completion of \( \hat{J}(\Omega) \) in the norm

\[
\|\phi_1\|_{H(\Omega)} = \left( \int_\Omega \sum_{j=1}^3 \sum_{i=1}^3 \phi_{1j_i}^2 \, dx \right)^{1/2}.
\]

In the appendix we quote Theorem A.1 from [22] providing the wellposedness result for system (1.1)-(1.5) under Assumptions 2.2 and 2.3.

Denote the orthogonal projection operator from the space \((L^2(\Omega))^3\) onto \( J_0(\Omega) \) by \( P \). Here is our first result.

**Lemma 2.1.** Let \( T \geq 0 \) be given. Then we have the following formula for solutions to system (1.1)-(1.3) as \( t \to 0^+ \) (\( t \in (0, T) \)):

\[
\frac{dz_i(t)}{dt} = \frac{1}{pql} \int_{S_i(z_i(0))} y(x,0) dx + \frac{t}{pql} \int_{S_i(z_i(0))} (PF(z,v))(x,0) dx + u(t) + t\nu(t), \quad i = 1, \ldots, n,
\]

where

\[
\|u(t)\|_{R^3} \leq \frac{1}{\sqrt{pql}} \mu(t, y(\cdot,0))
\]

\[
+ \frac{2}{pql} \max_{\{\text{mes } S_i(z_i(0)) \leq S_i(z_i(t)) \leq S_i(z_i(0))\}} \|y(\cdot,0)\|_{L^2(\Omega)^3}
\]

and \( \mu(t, y(\cdot,0)) \to 0, \nu(t) \to 0 \) as \( t \to 0^+ \), \( 0 \leq \mu(t, y(\cdot,0)) \leq \|y(\cdot,0)\|_{L^2(\Omega)^3} \).

More precisely,

\[
\mu(t, y(\cdot,0)) = \left( \sum_{k=1}^{\infty} \left( e^{-\lambda_k t} - 1 \right)^2 \left( \int_\Omega y(\cdot,0)^T \omega_k \, dx \right)^2 \right)^{1/2},
\]

\[
0 \leq \nu(t) \leq Cw_\ast(\cdot) + \frac{C}{\sqrt{pql}} \gamma(t)
\]

\[
+ \frac{C}{\sqrt{pql}} \max_{i=1,\ldots,n} \left( \text{mes}^{1/2} S_i(z_i(t)) \right) w(t).
\]
In (2.6) $C > 0$ is, as usual, a generically denoted positive constant,

\[
w(t) = \max_{j \in J} \| f_j(\cdot) \|_{(C[0,t])^3}, \quad (2.7a)
\]

\[
w_*(t) = \max_{j \in J} \| f_j(\cdot) - f_j(0) \|_{(C[0,t])^3}, \quad (2.7b)
\]

$f_j(t), j \in J$ denote all the coefficients at (respectively repeated) terms $\xi_i(x,t)$'s ($i = 1, \ldots, n$) in (1.3) in the order of appearance (or any other order), and

\[
\gamma(t) = \max_{j \in J} \left( \sum_{k=1}^{\infty} \left[ 1 - e^{-\lambda_k t} \right] \right)^2 \left( \int_{\Omega} f_j^T(0) \xi_j(x,0) \omega_k \, dx \right)^2 \frac{1}{2}. \quad (2.7c)
\]

Lemma 2.1 yields that, if sufficiently large constant (see Assumption 2.1) scalar parameters $v_i$'s are applied, the directions at which the points $z_i(t)$'s will try to move from their current positions are primarily determined by the projections of swimmer’s internal forces on the fluid velocity space at this moment, averaged over their corresponding supports $S_i(z_i(t))$'s. Lemma 2.1 applies to the sets $S_i(z_i)$'s in (1.2) of arbitrary geometry.

In our next results we refine the relations (2.3)-(2.7a-c) to the particular cases when these sets are small parallelepipeds.

Let, for each moment $t$, $\Pi_1 b = \Pi_{11}(t)b, i = 1, \ldots, n$ (we further may omit parameter $t$ in this notation) be the vector projection of a vector $b \in R^3$ on the straight line co-linear to the longest side of the parallelepiped $S_i(z_i(t))$. For example, if $b = (b_1, b_2, b_3)$ and $S_i(z_i(t))$ is oriented parallel to the $x_1 x_2 x_3$ axes, we have $\Pi_1 b = (b_1,0,0)$ at time $t$. Respectively, $\Pi_{12}$ and $\Pi_{13}, i = 1, \ldots, n$ stand for the vector projections on sides of lengths $q$ and $l$.

In [21] we studied the terms forming the expression $PF(z,v)$ in (2.3) in Lemma 2.1 for parallelepipeds of certain asymptotic proportions and small balls. We quote the respective results in the appendix as Theorems A.2-A.5. The following theorems make use of these results.

In the proof below (see (4.12)-(4.13)) we obtained the following parameters which we use to formulate Theorem 2.2:

\[
\beta_*(t) = \frac{C_0 \alpha t}{\sqrt{pqt(1 - C_0 n \alpha_*(t))}} \| g(\cdot,0) \|_{(L^2(\Omega))^3} + \frac{C_0 \alpha t^2 \max_{i=1,\ldots,n-1} |k_i|}{1 - n \alpha_*(t)},
\]

provided that

\[
\alpha_*(t) = C_0 \beta^2 \max\left\{ \frac{1}{n}, \frac{1}{n} \right\},
\]

where $c_0$ is some constant satisfying (4.11). We also set:

\[
\alpha = \max_{i=1,\ldots,n-2} |v_i| + \max_{i=1,\ldots,n-1} k_i.
\]

**Theorem 2.2:** Swimming micro motions for model (1.1)-(1.5) for parallelepipeds whose 3rd dimension is substantially smaller than the other two. Let $0 < \ell < q \leq p$. Then, provided that $t > 0$ is sufficiently small (for how small - see the proof below and the above notations) for the parallelepipeds described in Theorem A.2:

\[
z_i(t) = z_i(0) + \sum_{j=1}^{2} \frac{l^2}{2} \Pi_{ij}(0) [F_i^*(z,v)] + W_i(t) + R_i(t), \quad i = 1, \ldots, n, \quad (2.8)
\]

In (2.7) $C > 0$ is, as usual, a generically denoted positive constant,
where $F_i^v(z, v)$ denotes the 3-D vector coefficient at $\xi_i(x, t)$ in (1.3) and for some constant $C > 0$ and function $\gamma_*(t, z(0))$:

$$\| W_{ii}(t) \|_{R^3} \leq Ct^2 \alpha \max\{\frac{\ell^{1-\varepsilon}}{q}, \ell^{\varepsilon}, p\},$$

$$\| R_{ii}(t) \|_{R^3} \leq Ct^2 \sqrt{rql} \alpha + \frac{Ct^2}{\sqrt{pql}} \gamma_*(t, z(0)) \alpha$$

$$+ \frac{Ct}{\sqrt{pql}} \| y(\cdot, 0) \|_{(L^2(\Omega))^3} + Ct^2 \alpha \beta_* (t) + \frac{C}{\sqrt{l}} t^2 \alpha \beta_*^{1/2} (t), \quad i = 1, \ldots, n,$$

and $\gamma_*(t, z(0)) \to 0^+$ as $t \to 0^+$, provided that for some $\varepsilon \in (0, 1)$ the values of $p, \frac{\ell^{1-\varepsilon}}{q}$ are small enough.

Figure 2: Transformation of swimmer’s forces from Figure 1, acting upon the centers of mass of its body parts, when it is placed inside a fluid.

Respectively, Theorems A.3-A.5, provide the following modifications of Theorem 2.2.
Theorem 2.3: Swimming micro motions for model (1.1)-(1.5) for parallelepipeds which have two equal dimensions that are substantially smaller than the third one. Let $0 < \ell = q \leq p$. Then, provided that $t > 0$ is sufficiently small, for the parallelepipeds described in Theorem A.3:

$$z_i(t) = z_i(0) + \frac{t^2}{2} \Pi_i(0) [F_i^*(z,v)] + \frac{1}{2} \sum_{j=2}^{3} \frac{t^2}{2} \Pi_{ij}(0) [F_i^*(z,v)] + W_{2i}(t) + R_{2i}(t), \ i = 1, \ldots, n,$$  

where

$$\| W_{2i}(t) \|_{R^3} \leq Ct^2 \alpha(t) \max\{q^{1-\epsilon}/p, q^\epsilon, p\},$$  

and $R_{2i}(t)$’s are similar to (2.10), provided that for some $\epsilon \in (0,1)$ the values of $p, \frac{q^{1-\epsilon}}{p} \to 0$ are small enough.

Theorem 2.4: Swimming micro motions for model (1.1)-(1.5) for parallelepipeds which are cubes. Let $p = q = l$. Then, provided that $t > 0$ is sufficiently small, for the parallelepipeds described in Theorem A.4:

$$z_i(t) = z_i(0) + \frac{2}{3} \sum_{j=1}^{3} \frac{t^2}{2} \Pi_{ij}(0) [F_i^*(z,v)] + W_{3i}(t) + R_{3i}(t), \ i = 1, \ldots, n,$$  

where

$$\| W_{3i}(t) \|_{R^3} \leq Ct^2 \alpha(t)p,$$  

and $R_{3i}(t)$’s are similar to (2.10), provided that $p$ is small enough.

Theorem 2.5: Swimming micro motions for model (1.1)-(1.5) consisting of balls. Let the sets $S_i(z_i(t))$’s are balls of radius $r$. Then, provided that $t > 0$ is sufficiently small, for the balls described in Theorem A.5:

$$z_i(t) = z_i(0) + \frac{2}{3} \sum_{j=1}^{3} \frac{t^2}{2} \Pi_{ij}(0) [F_i^*(z,v)] + W_{4i}(t) + R_{4i}(t), \ i = 1, \ldots, n,$$  

where

$$\| W_{4i}(t) \|_{R^3} \leq Ct^2 \alpha(t)r,$$  

and $R_{4i}(t)$’s are similar to (2.10), provided that $r$ is small enough.

Our main results, given by relations (2.8)-(2.16), can be viewed as the formulas for the “linearized” motion of the swimmer near its initial position. They imply that, under certain conditions (namely, when the remainders $W_{ji}$’s and $R_{ji}$’s are “relatively small”), the asymptotically small motions of the swimmer after the instant $t$ are defined by the respective 3-D vector projections $\Pi_{ij}(t)$’s of the components in the forcing term (1.3) at this instant. These formulas can be used to approximate the actual trajectory of a swimmer in a fluid.

Remark 2.1. Theorems 2.2-2.5 are formulated for the case when all sets $S_i(z_i(t))$’s forming the body of swimmer are identical (the same parallelepipeds or balls). These theorems, of course, can be applied to the case when swimmer’s body made up of different parts selected from those described in these theorems.

Step 1. As it is noted in Section 2, the unique solution to (1.1)-(1.3) lies in the space \( J_0(\Omega) \) at all times. It admits the following implicit Fourier series representation:

\[
y(x, t) = \sum_{k=1}^{\infty} e^{-\lambda_k t} \left( \int_{\Omega} y_0^T \omega_k(x) \omega_k(x) \right) + \sum_{k=1}^{\infty} \int_0^t e^{-\lambda_k(t-\tau)} \left( \int_{\Omega} (F(z, v))^T \omega_k(x) \right) d\tau \omega_k(x). \tag{3.1}
\]

Here, the 3-D vector functions \( \omega_k, k = 1, \ldots \) and the real numbers \(-\lambda_k, k = 1, \ldots\) denote respectively the orthonormalized in \((L^2(\Omega))^3\) eigenfunctions and eigenvalues of the spectral problem associated with (1.1):

\[
\nu \Delta \omega_k - \nabla p_k = -\lambda \omega_k \quad \text{in} \quad \Omega, \quad \text{div} \omega_k = 0 \quad \text{in} \quad \Omega, \quad \omega_k = 0 \quad \text{in} \quad \partial \Omega.
\]

The functions \( \{\omega_k\}_{k=1}^\infty \) also form a basis in \( J_0(\Omega) \). Therefore, making use of the orthogonal projection operator \( P \) from the space \((L^2(\Omega))^3\) onto \( J_0(\Omega) \), we can rewrite (3.1) as follows:

\[
y(x, t) = \sum_{k=1}^{\infty} e^{-\lambda_k t} \left( \int_{\Omega} y_0^T \omega_k(x) \omega_k(x) \right) + \sum_{k=1}^{\infty} \int_0^t e^{-\lambda_k(t-\tau)} \left( \int_{\Omega} (PF(z, v))^T \omega_k(x) \right) d\tau \omega_k(x).
\]

Combining (3.1) and (1.2) yields:

\[
\frac{dz_i}{dt} = \frac{1}{\text{mes } \{S_0\}} \times \int_{S_i(z_i(t))} \left( \sum_{k=1}^{\infty} e^{-\lambda_k t} \left( \int_{\Omega} y_0^T \omega_k(x) \omega_k(x) \right) + \sum_{k=1}^{\infty} \int_0^t e^{-\lambda_k(t-\tau)} \left( \int_{\Omega} (F(z, v))^T \omega_k(x) \right) d\tau \omega_k(x) \right) \, dx
\]

\[
= \frac{1}{\text{mes } \{S_0\}} \times \int_{S_i(z_i(t))} \left( \sum_{k=1}^{\infty} e^{-\lambda_k t} \left( \int_{\Omega} y_0^T \omega_k(x) \omega_k(x) \right) + \sum_{k=1}^{\infty} \int_0^t e^{-\lambda_k(t-\tau)} \left( \int_{\Omega} (PF(z, v))^T \omega_k(x) \right) d\tau \omega_k(x) \right) \, dx.
\]

Step 2. We will show first that the first term in (3.3) generates the first term in (2.3) and \( u \) in (2.4), namely, that,

\[
\frac{1}{\text{mes } \{S_0\}} \int_{S_i(z_i(0))} \left( \sum_{k=1}^{\infty} e^{-\lambda_k t} \left( \int_{\Omega} y_0^T \omega_k(x) \omega_k(x) \right) \right) \, dx = \frac{1}{\text{mes } \{S_0\}} \int_{S_i(z_i(0))} y_0 \, dx + u(t). \tag{3.4}
\]

Indeed, making use of (3.3),

\[
\left\| \frac{1}{\text{mes } \{S_0\}} \int_{S_i(z_i(t))} \left( \sum_{k=1}^{\infty} e^{-\lambda_k t} \left( \int_{\Omega} y_0^T \omega_k(x) \right) \omega_k(x) \right) \, dx - \frac{1}{\text{mes } \{S_0\}} \int_{S_i(z_i(0))} y_0 \, dx \right\|_{L^3}
\]

\[
\leq \frac{1}{\text{mes } \{S(0)\}} \left\| \int_{S_i(z_i(0)) \cap S_i(z_i(t))} \sum_{k=1}^{\infty} (e^{-\lambda_k t} - 1) \int_{\Omega} y_0^T \omega_k(x) d\omega_k(x) \right\|_{L^3}
\]
where $\mu(t, y_0) \to 0$ as $t \to 0+$, $0 \leq \mu(t, y_0) \leq C \| y_0 \|_{(L^2(\Omega))^3}$ is as in (2.5), and $C > 0$ denotes a (generic) constant.

Step 3.  Note that each of the separate terms in the solution formula (3.1), associated with the sums forming the forcing term $F(z, v)$ in (1.3), admits the following representation:

$$\sum_{k=1}^{\infty} \int_0^t e^{-\lambda_k(t-\tau)} \left( \int_{\Omega} \xi_j(s, \tau) W^T(\tau) \omega_k ds \right) d\tau \omega_k(x) \quad (3.6)$$

for some 3-D function $W(t)$, i.e., for some of the vector coefficients in (1.3) at $\xi_j(s, \tau)$. Making use of (3.3), to show that (2.6) holds, we need to evaluate, e.g., the following expression:

$$\frac{1}{\text{mes} \{S_0\}} \left\| \int_{S_i(z_i(t))} \sum_{k=1}^{\infty} \int_0^t e^{-\lambda_k(t-\tau)} \left( \int_{\Omega} \xi_i(s, \tau) W^T(\tau) \omega_k ds \right) d\tau \omega_k(x) dx \right\|_{R^3}$$

$$- \frac{1}{\text{mes} \{S_0\}} \left\| \int_{S_i(z_i(0))} \sum_{k=1}^{\infty} \left( \int_{\Omega} \xi_i(s, 0) W^T(0) \omega_k ds \right) \omega_k(x) dx \right\|_{R^3}$$

$$= \frac{1}{\text{mes} \{S_0\}} \left\| \int_{S_i(z_i(t))} \sum_{k=1}^{\infty} \int_0^t e^{-\lambda_k(t-\tau)} \left( \int_{\Omega} \xi_i(s, \tau) W^T(\tau) \omega_k ds \right) d\tau \omega_k(x) dx \right\|_{R^3}$$

$$- \frac{1}{\text{mes} \{S_0\}} \left\| \int_{S_i(z_i(0))} \left( P(\xi_i(\cdot, 0) W^T(0)) \right) (x) dx \right\|_{R^3} \leq \frac{1}{\text{mes} \{S_0\}}$$

$$\times \left\| \int_{S_i(z_i(0)) \cap S_i(z_i(t))} \sum_{k=1}^{\infty} \int_0^t e^{-\lambda_k(t-\tau)} \left( \int_{\Omega} [\xi_i(s, \tau) W(\tau) - \xi_i(s, 0) W(0)]^T \omega_k ds \right) d\tau \omega_k(x) dx \right\|_{R^3}$$

$$+ \frac{1}{\text{mes} \{S_0\}} \left\| \int_{S_i(z_i(t)) \cap S_i(z_i(0))} \sum_{k=1}^{\infty} \int_0^t [e^{-\lambda_k(t-\tau)} - 1] \left( \int_{\Omega} \xi_i(s, 0) W^T(0) \omega_k ds \right) d\tau \omega_k(x) dx \right\|_{R^3}$$

$$+ \frac{1}{\text{mes} \{S_0\}} \left\| \int_{S_i(z_i(t)) \cap S_i(z_i(0))} \sum_{k=1}^{\infty} \int_0^t e^{-\lambda_k(t-\tau)} \left( \int_{\Omega} \xi_i(s, \tau) W^T(\tau) \omega_k ds \right) d\tau \omega_k(x) dx \right\|_{R^3}$$

$$+ 12.$$
\[+ \frac{t}{\text{mes} \{S_0\}} \left\| \int \sum_{k=1}^{\infty} \left( \int_{\Omega} \xi_i(s, \tau) W^T(0) \omega_k ds \right) d\tau \omega_k(x) dx \right\|_{R^3}
\]
\[\leq \frac{t^{1/2} \text{mes}^{1/2} \{S_i(z_i(0)) \cap S_i(z_i(t))\}}{\text{mes} \{S_0\}} \left( \| \xi_i(\cdot, \cdot) W(\cdot) - \xi_i(\cdot, 0) W(0) \|_{(L^2(Q_t))^3} \right)
\]
\[+ \frac{1}{\text{mes}^{1/2} \{S_0\}} \left( \sum_{k=1}^{\infty} t^2 \left[ \frac{1 - e^{-\lambda_k t}}{\lambda_k t} - 1 \right]^2 \left( \int_{\Omega} \xi_i(s, 0) W^T(0) \omega_k ds \right) \right)^{1/2}
\]
\[+ \frac{t^{1/2} \text{mes}^{1/2} \{S_i(z_i(t)) \setminus S_i(z_i(0))\}}{\text{mes} \{S_0\}} \| \xi_i(\cdot, \cdot) W(\cdot) \|_{(L^2(Q_t))^3}^3
\]
\[+ t \frac{\text{mes}^{1/2} \{S_i(z_i(0)) \setminus S_i(z_i(t))\}}{\text{mes} \{S_0\}} \| W(\cdot) \|_{(C[0,t])^3}^3\]
\[+ t \frac{\text{mes}^{1/2} \{S_i(z_i(t)) \setminus S_i(z_i(0))\}}{\text{mes}^{1/2} \{S_0\}} \| W(\cdot) \|_{(C[0,t])^3}^3 + t \frac{\text{mes}^{1/2} \{S_i(z_i(0)) \setminus S_i(z_i(t))\}}{\text{mes}^{1/2} \{S_0\}} \| W(\cdot) \|_{R^3}, \quad (3.7)
\]

where \( \gamma(t) \) is from (2.7c), \( \gamma(t) \to 0^+ \) as \( t \to 0^+ \), and we used the estimate
\[\left| \frac{1 - e^{-s}}{s} - 1 \right| < 1, \quad s > 0.\]

This ends the proof of Lemma 2.1.

4. Proof of Theorem 2.2.

Formula for \( z_i(t) - z_i(0) \). The integration of (2.3) over \((0, t), t \in (0, t_*)\) yields:
\[z_i(t) - z_i(0) = \frac{t}{pql} \int_{S_i(z_i(0))} y(x, 0) dx + \frac{t^2}{2pql} \int_{S_i(z_i(0))} (PF(z, v))(x, 0) dx
\]
\[+ \int_0^t u(\tau) d\tau + \int_0^t \tau v(\tau) d\tau, \quad i = 1, \ldots, n. \quad (4.1)
\]

We now invoke the following lemma from [21].

**Lemma 4.1 [21].** Let \( b = (b_1, b_2, b_3) \) be a given 3-D vector. Let \( S_0 \subset \Omega \) be strictly separated from \( \partial \Omega \). Then for any subset \( Q \) of \( \Omega \) of positive measure which lies strictly outside of \( S_0 \) and is strictly separated from \( \partial \Omega \), we have
\[\left\| \frac{1}{\text{mes} \{Q\}} \int_Q (Pb, \xi) dx \right\|_{R^3} \leq \frac{C \|b\|_{L^3} \{ \text{mes}^{1/2} \{\partial \Omega\} + \text{mes}^{1/2} \{S_0\} \}}{\text{mes} \{Q\} \text{mes}^{1/2} \{S_0\}}
\]
where \( C > 0 \) is a (generic) constant, \( \xi (x) \) is the characteristic function of \( S_0 \), and \( d_0 \) is the smallest out of the distances from \( Q \) to \( S_0 \) and from \( Q \) to \( \partial \Omega \).

Let the expressions \( F_i (z, v), i = 1, \ldots, n \) represent the terms in (1.3) supported respectively on \( S_i (z_i (0)) \). Making use of (1.3) and Lemma 4.1, we further obtain, for example for \( z_1 (t) \):

\[
\frac{1}{p q l} \int_{S_1 (z_1 (0))} (P F(z, v))(x, 0) dx = \frac{1}{p q l} \int_{S_1 (z_1 (0))} (P F_1(z, v))(x, 0) dx + C \sqrt{p q l} \zeta (0),
\]

where \( C > 0 \) is a constant depending on \( d_* \) and \( d^* \), and

\[
\zeta (0) = \max_{i=1, \ldots, n-1} \{ k_i \mid z_{i+1} (0) - z_i (0) \parallel_{R^3} - l_i \} + \max_{i=1, \ldots, n-1} \| z_{i+1} (0) - z_i (0) \parallel_{R^3} \max_{i=1, \ldots, n-2} \{ | v_i | \},
\]

and, from (1.3):

\[
(F_1 (z, v))(x, 0) = \xi_1 (x, 0) \left[ k_1 \left\{ \frac{z_2 (0) - z_1 (0)}{R^3} - l_1 \right\} \frac{z_2 (0) - z_1 (0)}{R^3} - z_1 (0) \right] + \xi_1 (x, 0) v_1 A_1 (z_2 (0) - z_1 (0)).
\]

**Evaluation of the 3-rd and the 4-th terms in (4.1).** Making use of (2.4)-(2.5), we obtain:

\[
\left\| \int_0^t u(\tau) d\tau \right\|_{L^2} \leq \frac{t}{\sqrt{p q l}} \| y(\cdot, 0) \|_{(L^2 (\Omega))^3}
\]

\[
+ \frac{2t \max_{t \in [0, t]} \{ \max^{1/2} \{ \text{mes} S_i (z_i (0)) \cap S_i (z_i (\tau)) \}, \text{mes} S_i (z_i (\tau)) \cap S_i (z_i (0)) \} \} \| y(\cdot, 0) \|_{(L^2 (\Omega))^3}
\]

\[
\leq \frac{3t}{\sqrt{p q l}} \| y(\cdot, 0) \|_{(L^2 (\Omega))^3}.
\]

Furthermore, making use of (2.2), (2.6)-(2.7a-c),

\[
\left\| \int_0^t \tau v(\tau) d\tau \right\|_{R^3} \leq \frac{C l^2}{2} \kappa (t),
\]

where

\[
\kappa (t) = w_*(t) + \frac{1}{\sqrt{p q l}} \| \gamma \|_{C [0, t]}
\]

\[
+ \frac{1}{\sqrt{p q l}} \max_{t \in [0, t]} \max_{\tau = 0} \left( \text{mes}^{1/2} \{ S_i (z_i (\tau)) \cap S_i (z_i (0)) \} + \text{mes}^{1/2} \{ S_i (z_i (\tau)) \cap S_i (z_i (0)) \} \right) w(t)
\]

\[
\leq w_*(t) + \frac{1}{\sqrt{p q l}} \| \gamma \|_{C [0, t]} + \frac{1}{\sqrt{t}} \max_{i=1, \ldots, n} \| z_i (\cdot) - z_i (0) \|^{1/2}_{C [0, t]} w(t).
\]

In turn, since in \( f_j (0) \)'s in (2.7c) the parameters \( v_i \)'s and \( k_i \)'s enter as factors, and

\[
\frac{1 - e^{-\lambda t}}{\lambda t} \rightarrow 1 \quad \text{as} \quad t \rightarrow 0,
\]

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we obtain from (2.7c) that
\[
\| \gamma \|_{C[0,t]} \leq \gamma_*(t, z(0)) \left( \max_{i=1,\ldots,n-1} |k_i| + \max_{i=1,\ldots,n-2} |v_i| \right) = \alpha \gamma_*(t, z(0)), \tag{4.7}
\]
where \( \alpha \) was introduced before Theorem 2.2, \( \gamma_*(t, z(0)) \to 0 \) as \( t \to 0 \) for any \( z(0) \) and \( \gamma_*(t, z(0)) \) does not depend on \( k_i \)'s and \( v_i \)'s.

**Evaluation of the 2-nd term in (4.1).** Combining (2.2), (4.1)-(4.7) and Theorem A.2 produces the following result.

**Lemma 4.2.** We have the following estimates for the local motions of points \( z_i, i = 1, \ldots, n \) in model (1.1)-(1.5) as \( \frac{t^{1-\varepsilon}}{q} \to 0 \) as \( p \to 0 \) for some \( \varepsilon \in (0,1) \):

\[
\begin{align*}
\| z_i(t) - z_i(0) - \sum_{j=1}^{\frac{t^2}{2} \Pi_{ij}(0)} [F_i^*(z, v)] \|_{R^3} & \leq C t^2 \alpha \left\{ \max \left\{ \frac{t^{1-\varepsilon}}{q}, t^2, p \right\} + \sqrt{pq} \right\} \zeta(0) + \frac{C t^2}{\sqrt{pq}} \gamma_*(t, z(0)) \alpha \\
& + \frac{C t}{\sqrt{pq}} \| y(\cdot, 0) \|_{(L^2(\Omega))^3} + C t^2 w_*(t) + \frac{C t^2}{\sqrt{t}} \max_{i=1,\ldots,n} \| z_i(\cdot) - z_i(0) \|_{[C(0,t)]^3}^{1/2} w(t),
\end{align*}
\tag{4.8}
\]
where \( C > 0 \) depends on \( d_* \) and \( d_* \) and \( F_i^*(z, v) \) denotes the 3-D vector coefficient at \( \xi_i(x,t) \) in (1.3).

**Evaluation of \( w(t) \) and \( w_*(t) \).** To this end, we need to evaluate the following values:

\[
\gamma_i(0) = \max_{\tau \in [0,t]} \| z_i(\tau) - z_i(0) \|_{R^3}, \quad i = 1, \ldots, n.
\tag{4.9}
\]

It follows from (1.2) and (3.1)-(3.3) that we have (similar to (4.1)-(4.2)) for \( \tau \in [0,t] \):

\[
\begin{align*}
\| z_i(\tau) - z_i(0) \|_{R^3} & \leq \frac{C t}{\sqrt{pq}} \| y(\cdot, 0) \|_{(L^2(\Omega))^3} \\
& + C t^2 \max_{i=1,\ldots,n-1} |k_i| \sum_{i=1}^{n-1} \max_{s \in [0,\tau]} \| z_{i+1}(s) - z_i(s) \|_{R^3} - l_i | \\
& + C t^2 \max_{i=1,\ldots,n-2} |v_i| \sum_{i=1}^{n-1} \| z_i - z_{i+1} \|_{[C(0,\tau)]^3}, \quad i = 1, \ldots, n
\end{align*}
\tag{4.10}
\]
for some (generically denoted) constant.

Next, noting that

\[
\max_{\tau \in [0,t]} | \| z_{i+1}(\tau) - z_i(\tau) \|_{R^3} - l_i | \leq \text{diam}(\Omega), \quad i = 1, \ldots, n - 1,
\]

after taking a maximum over \( \tau \in [0,t] \) in (4.10), we obtain that:

\[
\gamma_i(0) \leq \frac{C d t}{\sqrt{pq}} \| y(\cdot, 0) \|_{(L^2(\Omega))^3} + C d t^2 \max_{i=1,\ldots,n-1} |k_i| + C d t^2 \max_{i=1,\ldots,n-2} |v_i| \sum_{j=1}^{n} \gamma_j(0) \tag{4.11}
\]
for some constant $C_0 > 0$. Summing up (4.11) over $i = 1, \ldots, n$ and resolving the for $\sum_{i=1}^{n} \gamma_i(0)$ yields the following result.

**Lemma 4.3.** Fix $t > 0$ as in Assumption 2.1. We have the following estimate:

$$
\sum_{i=1}^{n} \| z_i(\cdot) - z_i(0) \|_{C(0,t)}^3 \leq \beta_n(t)
$$

$$
= \frac{C_0 n t}{\sqrt{pq(1 - n\alpha_n(t))}} \| y(\cdot, 0) \|_{L^2(\Omega)}^3 + \frac{C_0 n t^2 \max_{i=1,\ldots,n-1} | k_i |}{1 - n\alpha_n(t)},
$$

(4.12)

provided that $\alpha_n(t) = C_0 t^2 \max_{i=1,\ldots,n-1} | v_i | < \frac{1}{n}$ (i.e., for sufficiently small $t \in [0, t_s]$ in Assumption 2.1).

Note that, in view of (2.7a-c) and (1.3),

$$
\| w(t) \|_{R^3} \leq C \left( \max_{i=1,\ldots,n-1} k_i + \max_{i=1,\ldots,n-2} | v_i | \right) = C \alpha
$$

(4.13)

for some (generically denoted) constant. In turn, Lemma 4.2 allows us to evaluate $w_n(t)$ as follows:

$$
\| w_n(t) \|_{R^3} \leq C \beta_n(t) \alpha.
$$

(4.14)

Combining Lemmas 4.2 and 4.3 with (4.13) and (4.14) yields Theorem 2.2.

**Proofs of Theorems 2.3-2.5.** These are proved exactly as Theorem 2.2 except that, instead of Theorem A.2, one should use Theorems A.3-A.5 in the appendix.

6. **Concluding remarks and an example.** In this paper we considered a model describing the self-propelled motion of a small flexible swimmer in the 3-D incompressible fluid, governed by the nonstationary Stokes equation. We obtained an asymptotic formula for its micro motions. This formula asserts that the swimmer’s movements are strongly defined by the geometry of its body, because the latter plays the crucial role in the transformation (both direction- and magnitude-wise) of the original swimmer’s internal forces when it is placed inside an incompressible fluid ([20], [21]). In particular, we derived explicit motion formulas for the case when the swimmer’s body consists of parallelepipeds of three different proportions and/or small balls. Such results can be used to approximate the actual trajectory of swimmer in a fluid. They can be useful in biological and engineering applications dealing with the study and design of propulsion systems in fluids and be an instrumental approach to study their controllability properties ([20]).

**Remark 6.1.** At no extra effort, all the above results can be extended to a more simple forcing term in place of (1.3) . Namely, (1.3) can be replaced by the following term where we assume that we can change the structural forces as will:

$$
F(z, v) = \sum_{i=2}^{n} (\xi_{i-1}(x, t)k_{i-1}(z_i(t) - z_{i-1}(t)) + \xi_i(x, t)k_i(z_i(t) - z_i(t))
$$

$$
+ \sum_{i=2}^{n-1} v_{i-1}(t) \left[ \xi_{i-1}(x, t)A_i(z_{i-1}(t) - z_i(t)) - \xi_{i+1}(x, t) \frac{\| z_{i-1}(t) - z_i(t) \|_{R^3}^2}{\| z_{i+1}(t) - z_i(t) \|_{R^3}^2} B_i(z_{i+1}(t) - z_i(t)) \right]
$$

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\[- \sum_{i=2}^{n-1} \xi_i(x, t) v_{i-1}(t) \left[ A_i(z_{i-1}(t) - z_i(t)) - \frac{\|z_{i-1}(t) - z_i(t)\|^2_{R^3}}{\|z_{i+1}(t) - z_i(t)\|^2_{R^3}} B_i(z_{i+1}(t) - z_i(t)) \right]. \quad (6.1)\]

**Example 6.1.** The following four figures illustrate a possible schematics of how one can apply Theorem 2.2 to subsequently move the points $z_1(t)$ and $z_2(t)$ of swimmer’s body to the left. Here we assume that we can change the orientation of sets $S_i(z_i(t))$’s and turn on and off any sets of respective forces, both rotational and structural. The latter can be achieved in equations (1.1)-(1.2), (6.1) by varying values of the coefficients $v_i(t)$’s and $k_i(t)$’s. (Of course, it is just one of infinitely many possible schematics for locomotion).

![Fig. A1: Rotation forces about $z_2$ only when not in the fluid](image-url)

→ Fig. A2 next:
Fig. A1 → Fig. A2: Resulting rotation forces about $z_2$ when put in the fluid.
Thus: point $z_1$ moves to the left, other points don’t move.
Fig. A3: Structural forces between $z_1$ and $z_2$ only when not in the fluid. → Fig. 4 next:
Fig. A3 → Fig. A4: Resulting structural forces when put in the fluid.
Thus: point $z_2$ moves to the left, other points don’t move.

References


Appendix: On wellposedness of model (1.1)-(1.3) and projections of forces in the 3-D fluid. Here we are quoting several results from [22] and [21] used in this paper.

**Theorem A.1 [22]:** Wellposedness of (1.1)-(1.3). Let \( y_0 \in H(\Omega); T > 0; k_i > 0, i = 1, \ldots, n - 1; v_i \in L^\infty(0,T); i = 1, \ldots, n - 2; \) and \( z_i(0) \in \Omega, i = 1, \ldots, n, \) and let Assumptions 2.2 and 2.3 hold. Then there exists a \( T^* \in (0,T) \) such that system (1.1) - (1.3) admits a unique solution \( \{y,p,z\} \) on \( (0,T^*) \). \( \{y,\nabla p,z\} \in L^2(0,T^*; J_r(\Omega)) \times L^2(0,T^*; G(\Omega)) \times [C([0,T^*]; R^3)]^n. \) Moreover, \( y \in C([0,T^*]; H(\Omega)), y_t, y_{x_j, p_\xi} \in (L^2(Q_T))^3, \) where \( i, j = 1, 2, 3, \) and equations (1.1) and (1.2) are satisfied almost everywhere, while Assumptions 2.2 and 2.3 hold in \([0,T^*] \).

The fact that conditions Assumptions 2.2 and 2.3 hold in \([0,T^*] \) implies that we are able to guarantee that within \([0,T^*] \) no parts of the swimmer’s body will collide, and that it stays strictly inside of \( \Omega. \) These conditions allow us to maintain the mathematical and physical wellposedness of model (1.1)-(1.3). Moreover, this proof in [22] allows further extension of the solutions to (1.1)-(1.3) in time as long as Assumptions 2.2 and 2.3 continue to hold. This depends on the choice of parameters \( v_1(t), \ldots, v_{n-2}(t). \)

In [21] we investigated how the geometric shape of a swimmer affects the forces acting upon it in a 3-D incompressible fluid, such as governed by the non-stationary Stokes or Navier-Stokes equations. We were interested in the following question: *How will the swimmer’s internal forces (i.e., not moving the center of swimmer’s mass when it is not inside a fluid) “transform” their actions when the swimmer is placed into a fluid (thus, possibly, creating its self-propelling motion)?* We considered a number geometric shapes for swimmer’s bodies and below cite some of the obtained results.

In the first result we analyze the aforementioned forces that act upon parallelepipeds whose 3rd dimension is substantially smaller than the other two. The included configurations of parallelepipeds include, e.g., the case when \( p = q \) and with \( \ell \) to be significantly smaller, or the case when \( p >> q >> \ell. \)

**Theorem A.2** Let \( b = (b_1, b_2, b_3) \) be a given 3-D vector and \( 0 < \ell < q \leq p. \) Then
\[
\frac{1}{\text{mes}(S_o)} \int_{S_o} (Pb \xi)(x) dx = (b_1, b_2, 0) + \left[ O(\ell^\varepsilon) + O\left(\frac{\ell^{1-\varepsilon}}{q}\right) + O(p) \right] \|b\|_{R^3},
\]
(A.1)

provided that \( \frac{\ell^{1-\varepsilon}}{q} \to 0 \) as \( p \to 0 \) for some \( \varepsilon \in (0,1). \)

This result is illustrated by the following figures showing the transformation of forces acting upon the center of mass of parallelepiped \( S_o \) when it is not in a fluid and when it placed inside of an incompressible fluid.
The next result deals with parallelepipeds which have two equal dimensions that are substantially smaller than the third one.

**Theorem A.3** Let \( b = (b_1, b_2, b_3) \) be a given 3-D vector, \( 0 < \ell = q \leq p \). Then

\[
\frac{1}{\text{mes}\{S_o\}} \int_{S_o} (Pb\xi)(x)dx = (b_1, \frac{b_2}{2}, \frac{b_3}{2}) + \left[ O(q^{\tau}) + O\left(\frac{q^{1-\tau}}{p}\right) + O(p)\right] \|b\|_{R^3},
\]

provided that for some \( \tau \in (0, 1) \) we have \( \frac{q^{1-\tau}}{p} \to 0 \) as \( p \to 0 \).

This result is illustrated by the following figures:

**Experimental illustration.** In [25] Leal provided a series of photos illustrating the motion of a *slanted-down-to-the-right* thin long particle (i.e., *thin rod-like as in Theorem A.3*, but with rounded edges), which falls down in the fluid due to the gravity. Its motion is perfectly consistent with the expectations due to Theorem A.3 in terms of motion of the center of mass of this particle, i.e., *down to the right*, as it can be seen from the following diagrams:

Gravity force distribution not in fluid. Transformation of forces inside the fluid due to Th. A.3.

The following multiple-image photographs from [25] show exactly the same pattern of motion of the thing long particles as expected from the above diagram in two types of fluid glycerin ("inertia free") and in water (when inertia is present).
Figure 8. Multiple-image photograph of a slender rod-like particle ($r = 66$) sedimenting through 99.5\% glycerine.
The third result deals with cubes.

**Theorem A.4** Let \( b = (b_1, b_2, b_3) \) be a given 3-D vector and \( p = q = l \). Then

\[
\frac{1}{\text{mes}\{S_0\}} \int_{S_0} (Pb\xi)(x)dx = \frac{2}{3}(b_1, b_2, b_3) + O(p) \|b\|_{R^3} \text{ as } p \to 0.
\]

This result is illustrated by the following figures:
The forth result is aimed at small balls.

**Theorem A.5. Forces acting upon small balls in a fluid.** Let $b = (b_1, b_2, b_3)$ be a given 3-D vector. Let $B_r$ be a ball of radius $r$ with center at the origin lying in $\Omega$. Then,

$$\frac{1}{\text{mes}(B_r)} \int_{B_r} (Pb\xi_{B_r})(x) dx = \frac{2}{3} (b_1, b_2, b_3) + O(r) \|b\|_{R^3}$$

as $r \to 0^+$, where $\xi_{B_r}(x)$ is the characteristic function of $B_r$.\[\Box\]