2.4 Polynomial and Rational Functions

Polynomial Functions

Given a linear function $f(x) = mx + b$, we can add a square term, and get a quadratic function $g(x) = ax^2 + f(x) = ax^2 + mx + b$. We can continue adding terms of higher degrees, e.g. we can add a cube term and get $h(x) = cx^3 + g(x) = cx^3 + ax^2 + mx + b$, and so on. $f(x)$, $g(x)$, and $h(x)$ are all special cases of a polynomial function.

**Definition (Polynomial Function)**

A polynomial function is a function that can be written in the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

for $n$ a nonnegative integer, called the degree of the polynomial. The coefficients $a_n, a_{n-1}, \ldots, a_1, a_0$ are real numbers with $a_n \neq 0$.

Note that although $a_n \neq 0$, the remaining coefficients $a_{n-1}, a_{n-2}, \ldots, a_1, a_0$ can very well be 0.

**Domain of Polynomial Function**

The domain of a polynomial function is $\mathbb{R}$, the set of all real numbers.

The domain of $f(x) = x^n$ is $\mathbb{R}$ regardless the value of $n$ (any nonnegative integer), and so is the domain of $g(x) = ax^n$, where $a$ is some real number. Clearly, if you add, say $k$, such functions with different degrees ($n$) the domain of the resulting function will still be $\mathbb{R}$. 

Consider a function \( f(x) = (x - 1)(x - 2)(x - 3) \). It could be rewritten as

\[
\begin{align*}
  f(x) &= (x - 1)(x - 2)(x - 3) \\
  &= (x - 1)(x^2 - 2x - 3x + 6) \\
  &= (x - 1)(x^2 - 5x + 6) \\
  &= x^3 - 5x^2 + 6x - x^2 + 5x - 6 \\
  &= x^3 - 6x^2 + 11x - 6.
\end{align*}
\]

So, \( f(x) \) is a polynomial function of degree 3.

**Question:** How many \( y \) intercepts does \( f(x) \) have?

**Answer:** Only one, \( y = f(0) = -6 \). Any function can have at most one \( y \) intercept, otherwise it will not pass the vertical line test.

*y Intercept of a Polynomial Function*

If \( f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 \) is a polynomial function, it has exactly one \( y \) intercept \( y = a_0 \).

**Question:** How many \( x \) intercepts does \( f(x) \) have?

**Answer:** \( f(x) \) has 3 intercepts. \( 0 = (x - 1)(x - 2)(x - 3) \implies x = 1 \) or \( x = 2 \) or \( x = 3 \).

*x Intercept of a Polynomial Function*

A polynomial of degree \( n \) can have, at most, \( n \) linear factors. Therefore, the graph of a polynomial function of positive degree \( n \) can intersect the \( x \) axis at most \( n \) times. The \( x \) intercepts of \( f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 \) could be found by solving \( a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 = 0 \).
Consider a function $h(x) = (x^2 + 1)(x - 2)(x - 3)$.

\[
h(x) = (x^2 + 1)(x - 2)(x - 3) = (x^2 + 1)(x^2 - 2x - 3x + 6) = (x^2 + 1)(x^2 - 5x + 6) = x^4 - 5x^3 + 6x^2 + x^2 - 5x + 6 = x^4 - 5x^3 + 7x^2 - 5x + 6.
\]

$h(x)$ is a polynomial function of degree 4, but has just 2 $x$ intercepts, because the equation $0 = (x^2 + 1)(x - 2)(x - 3)$ has just 2 roots (zeros), which are $x = 2$ and $x = 3$. 

3
Note that $f(x) = x^3 - 6x^2 + 11x - 6$ has degree 3, which is an odd number. It starts negative, ends positive, and crosses the $x$ axis odd number of times.

$h(x) = x^4 - 5x^3 + 7x^2 - 5x + 6$ has degree 4, which is an even number. It starts positive, ends positive, and cross the $x$ axis even number of times.

Consider $m(x) = -f(x) = -(x^3 - 6x^2 + 11x - 6) = -x^3 + 6x^2 - 11x + 6$, and $n(x) = -g(x) = -(x^4 - 5x^3 + 7x^2 - 5x + 6) = -x^4 + 5x^3 - 7x^2 + 5x - 6$. 


Definition (Leading Coefficient)
Given a polynomial function \( f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \), the coefficient \( a_n \) of the highest-degree term is called the leading coefficient of a polynomial function \( f(x) \).

Graph of a Polynomial Function
Given a polynomial function \( f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \):

(a) if \( a_n > 0 \) and \( n \) is odd, then the graph of \( f(x) \) starts negative, ends positive, and crosses the \( x \) axis odd number of times but at least once;

(b) if \( a_n < 0 \) and \( n \) is odd, then the graph of \( f(x) \) starts positive, ends negative, and crosses the \( x \) axis odd number of times but at least once;

(c) if \( a_n > 0 \) and \( n \) is even, then the graph of \( f(x) \) starts positive, ends positive, and crosses the \( x \) axis even number of times or does not cross it at all;

(d) if \( a_n < 0 \) and \( n \) is even, then the graph of \( f(x) \) starts negative, ends negative, and crosses the \( x \) axis even number of times or does not cross it at all.

Note: (c) is a reflection in the \( x \) axis of (a), and (d) is a reflection in the \( x \) axis of (b).

Also note that a polynomial function always either increases or decreases without bound as \( x \) goes to either negative or positive infinity.
Continuity and "Smoothness" of Polynomial Function

Consider $f(x) = \frac{2|x|}{x}$. $f(x)$ has a discontinuous break at $x = 0$. 
Consider \( g(x) = |x| - 2 \). \( g(x) \) is continuous, but not smooth due to a sharp corner at \((0, -2)\).

Consider \( h(x) = \frac{2}{x - 1} \). \( h(x) \) has a discontinuous break at \( x = 1 \).

**Graph of a Polynomial Function**

The graph of a polynomial function is **continuous**, with no holes or breaks. That is, the graph can be drawn without removing a pen from the paper. Also, the graph of a polynomial is "**smooth**", i.e. has no sharp corners.
Rational Functions

Just as rational numbers are defined in terms of quotients of integers, rational functions are defined in terms of quotients of polynomials.

**Definition (Rational Function)**

A **rational function** is any function that can be written in the form

\[ f(x) = \frac{n(x)}{d(x)}, \quad d(x) \neq 0 \]

where \( n(x) \) and \( d(x) \) are polynomials.

For example,

\[ f(x) = \frac{1}{x}, \quad g(x) = \frac{x - 2}{x^2 - x - 6}, \quad h(x) = \frac{x^{13} - 8}{x^5} \]

\[ p(x) = x^4 - 5x^3 + 7x^2, \quad q(x) = 123, \quad r(x) = 0 \]

are all rational functions.

If \( n(x) \) and \( d(x) \) are polynomials, then they both have domain \( \mathbb{R} \). However,

**Domain of a Rational Function**

If \( f(x) = \frac{n(x)}{d(x)} \) is a rational function, then its domain is the set of all real numbers such that \( d(x) \neq 0 \).
Example 1
Find the domain of \( f(x) = \frac{x^2 + 1}{x^2 - 7x + 10} \)
Vertical and Horizontal Asymptotes

Recall that a polynomial function is always continuous and "smooth". It is also true that if \( x \) increases or decreases without bound, then function also increases or decreases without bound. However, this may not be true for a rational function. Also, a rational function may not have a \( y \) intercept.

Consider a rational function \( f(x) = \frac{x-3}{x-2} \). Its domain \((-\infty, 2] \cup [2, \infty)\), or all real numbers except for \( x = 2 \),

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
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<tbody>
<tr>
<td>1.5</td>
<td>( \frac{1.5-3}{1.5-2} = -1.5 )</td>
</tr>
<tr>
<td>1.75</td>
<td>( \frac{1.75-3}{1.75-2} = -1.25 )</td>
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<tr>
<td>1.9</td>
<td>( \frac{1.9-3}{1.9-2} = -1.1 )</td>
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<tr>
<td>1.95</td>
<td>( \frac{1.95-3}{1.95-2} = -1.05 )</td>
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<td>1.999</td>
<td>( \frac{1.999-3}{1.999-2} = -1.001 )</td>
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<td>1.9999</td>
<td>( \frac{1.9999-3}{1.9999-2} = -1.0001 )</td>
</tr>
<tr>
<td>1.99999</td>
<td>( \frac{1.99999-3}{1.99999-2} = -1.00001 )</td>
</tr>
<tr>
<td>2</td>
<td>undefined</td>
</tr>
<tr>
<td>2.00001</td>
<td>( \frac{2.00001-3}{2.00001-2} = -0.999999 )</td>
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<tr>
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<td>2.001</td>
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<td>2.01</td>
<td>( \frac{2.01-3}{2.01-2} = -0.999999 )</td>
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<tr>
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<td>( \frac{2.05-3}{2.05-2} = -0.999999 )</td>
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<tr>
<td>2.1</td>
<td>( \frac{2.1-3}{2.1-2} = -0.999999 )</td>
</tr>
<tr>
<td>2.25</td>
<td>( \frac{2.25-3}{2.25-2} = -0.999999 )</td>
</tr>
</tbody>
</table>
The graph of \( f(x) \) gets closer to the line \( x = 2 \) as \( x \) gets closer to 2. Line \( x = 2 \) is a vertical asymptote for \( f(x) \).

The graph of \( f(x) \) gets closer to the line \( y = 1 \) as \( x \) increases or decreases without bound. The line \( y = 1 \) is a horizontal asymptote for \( f(x) \).
Definition (Vertical Asymptote)
A vertical line \( x = a \) is called a vertical asymptote for a function \( f(x) \) if the graph of \( f(x) \) gets closer to the line \( x = a \) as \( x \) gets closer to \( a \).

Note: the number of vertical asymptotes of a rational function \( f(x) = \frac{n(x)}{d(x)} \) is at most equal to the degree of \( d(x) \).

Definition (Horizontal Asymptote)
A horizontal line \( y = b \) is called a horizontal asymptote for a function \( f(x) \) if the graph of \( f(x) \) gets closer to the line \( y = b \) as \( x \) gets increases or decreases without bound.

Note: a rational function has at most one horizontal asymptote. Moreover, the graph of a rational function approaches the horizontal asymptote (when one exists) both as \( x \) increases and decreases without bound.

\[
f(x) = \frac{8}{x^2 - 4} = \frac{8}{(x-2)(x+2)}
\]

\[
f(x) = x + \frac{1}{x} = \frac{x^2 + 1}{x}
\]
Example 2
Given the rational function \( f(x) = \frac{3x+3}{x^2-9}, \)

(a) Find the domain.

(b) Find the \( x \) and \( y \) intercepts.

(c) Find the equations of all vertical asymptotes.

(d) If there is a horizontal asymptote, find its equation.

(e) Using the information from parts (a)-(d) and additional points as necessary, sketch a graph of \( f \) for \(-10 \leq x \leq 10\).
Consider the rational function \( g(x) = \frac{3x^2 - 3x - 36}{x^3 - 4x^2 - 9x + 36} \).
Example 3

Find the vertical and horizontal asymptotes of the rational function

\[ f(x) = \frac{x^3 - 4x}{x^2 + 5x} . \]
Applications

Example 4 (Employee Training)
A company that manufactures computers has established that, on the average, a new employee can assemble \( N(t) \) components per day after \( t \) days of on-the-job training, as given by

\[
N(t) = \frac{25t + 5}{t + 5}, \quad t \geq 0
\]

Sketch a graph of \( N \), \( 0 \leq t \leq 100 \), including any vertical or horizontal asymptotes. What does \( N(t) \) approach as \( t \) increases without bound?