Semimonotone Matrices

Megan Wendler

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A matrix $A \in M_n(\mathbb{R})$ is **semimonotone** if

$$0 \neq x \succeq 0 \text{ where } x \in \mathbb{R}^n \quad \Rightarrow \quad x_k > 0 \text{ and } (Ax)_k \geq 0 \text{ for some } k$$

A matrix $A \in M_n(\mathbb{R})$ is called **strictly semimonotone** if $(Ax)_k \geq 0$ is replaced with $(Ax)_k > 0$ in the above definition.
The Definition of Semimonotone & Strictly Semimonotone

Definition

A matrix $A \in M_n(\mathbb{R})$ is **semimonotone** if

$$0 \neq x \geq 0 \text{ where } x \in \mathbb{R}^n \Rightarrow x_k > 0 \text{ and } (Ax)_k \geq 0 \text{ for some } k$$

Definition

A matrix $A \in M_n(\mathbb{R})$ is called **strictly semimonotone** if $(Ax)_k \geq 0$ is replaced with $(Ax)_k > 0$ in the above definition.
Example

Let \( A = \begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix} \).

Clearly, if \( x_1 = 0 \), then we must have that \( x_2 > 0 \) and we get that \((Ax)_2 = 3x_2 > 0\).

Similarly, if \( x_2 = 0 \), then we must have that \( x_1 > 0 \) and we get that \((Ax)_1 = 2x_1 > 0\).

Now suppose \( x_1, x_2 > 0 \). In this case, \( Ax = \begin{bmatrix} 2x_1 - x_2 - 2x_1 + 3x_2 \end{bmatrix} \).

Suppose \( Ax < 0 \). Then \( x_2 > 2x_1 \). Thus, we must have \(-2x_1 + 3x_2 > -2x_1 + 3(2x_1) = 4x_1 > 0\), a contradiction.

Thus, we have shown that \( A \) is semimonotone. In fact, \( A \) is strictly semimonotone.
Example of a Semimonotone Matrix

Example

Let $A = \begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix}$. Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$ where $0 \leq x \leq 0$.

Clearly, if $x_1 = 0$, then we must have that $x_2 > 0$ and we get that $(Ax)_2 = 3x_2 > 0$.

Similarly, if $x_2 = 0$, then we must have that $x_1 > 0$ and we get that $(Ax)_1 = 2x_1 > 0$.

Now suppose $x_1, x_2 > 0$. In this case, $Ax = \begin{bmatrix} 2x_1 - x_2 - 2x_1 + 3x_2 \\ -2x_2 \end{bmatrix}$.

Suppose $Ax < 0$. Then $x_2 > 2x_1$. Thus, we must have $-2x_1 + 3x_2 > -2x_1 + 3(2x_1) = 4x_1 > 0$, a contradiction.

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- Now suppose \( x_1, x_2 > 0 \). In this case,

\[
Ax = \begin{bmatrix} 2x_1 - x_2 \\ -2x_1 + 3x_2 \end{bmatrix}
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Suppose \( Ax < 0 \). Then \( x_2 > 2x_1 \). Thus, we must have

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Outline

1 Introduction
- The definition of semimonotone & an example
- Some observations and previous results
- Questions

2 Some Results
- What kinds of matrices are semimonotone?
- Properties of semimonotone matrices

3 Conjectures

4 Future Directions
A few simple observations about a semimonotone matrix

Suppose $A \in M_n(\mathbb{R})$ is semimonotone.

- By letting $x = e_k$, we obtain that $a_{kk} \geq 0$, for each $k = 1, 2, \ldots, n$. This means that the diagonal entries of $A$ must be nonnegative.
A few simple observations about a semimonotone matrix

Suppose $A \in M_n(\mathbb{R})$ is semimonotone.

- By letting $x = e_k$, we obtain that $a_{kk} \geq 0$, for each $k = 1, 2, \ldots, n$. This means that the diagonal entries of $A$ must be nonnegative.
- Every principal submatrix $A(\alpha, \alpha)$ must be semimonotone, where $\alpha \subseteq \{1, 2, \ldots, n\}$. (This can be shown by taking any $x$ where $x[\alpha] > 0$ and all the other entries are zero.)
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Proposition

A matrix $A \in M_n(\mathbb{R})$ is semimonotone if and only if

1. Every proper principal submatrix of $A$ is semimonotone, and
2. For every $x > 0$, $(Ax)_k \geq 0$ for some $k$. 

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Before we discuss previous results, we need to first recall some definitions.

**Definition**

A matrix $A \in M_n(\mathbb{R})$ is a $P$-matrix ($P_0$-matrix) if all its principal minors are positive (nonnegative).
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A matrix $A \in M_n(\mathbb{R})$ is *copositive* if $x^T A x \geq 0$ for all $x \geq 0$. 
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**Definition**

A matrix $A \in M_n(\mathbb{R})$ is *semipositive* if there exists an $x \geq 0$ such that $A x > 0$. By continuity of a matrix as a linear map, this is equivalent to saying that there exists an $x > 0$ such that $A x > 0$. The class of semipositive matrices is denoted $S$. 

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**Definition**

A matrix $A \in M_n(\mathbb{R})$ is said to be an $S_0$ matrix if there exists a $0 \neq x \geq 0$ such that $Ax \geq 0$. 
Semimonotone matrices have been studied a little in the past. Below are some useful results obtained by Cottle, Pang, and Stone [1]:

1. Every nonnegative matrix is semimonotone.
2. Every $P_0$-matrix is semimonotone. Every $P$-matrix is strictly semimonotone.
3. All copositive matrices are semimonotone.
4. $A$ is semimonotone if and only if $A$ and all its proper principal submatrices belong to $S_0$.
5. $A$ is strictly semimonotone if and only if $A$ and all its proper principal submatrices are semipositive.
6. $A$ is semimonotone if and only if $A^T$ is semimonotone.
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5. $A$ is strictly semimonotone if and only if $A$ and all its proper principal submatrices are semipositive.
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Besides these few results, however, not much can be currently found about semimonotone matrices.
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   - Properties of semimonotone matrices

3. **Conjectures**

4. **Future Directions**
Some Questions

- What kinds of matrices are semimonotone matrices? When is a matrix that is not a $P_0$ matrix or a copositive matrix a semimonotone matrix?

What are some properties of semimonotone matrices?

What are the possible spectrums of a semimonotone matrix?

Given a matrix $A$, what is the best way to tell if $A$ is a semimonotone matrix?

How does one create a generic semimonotone matrix?
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Proposition

If $A \in M_n(\mathbb{R})$ is diagonally dominant with nonnegative diagonal entries, then $A$ is semimonotone.
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Proof

It is not difficult to show the above result directly. One could also show that if $A$ is diagonally dominant with nonnegative diagonal entries, then $A \in P_0$. Hence $A$ is semimonotone.
Proposition

If \( A \in M_n(\mathbb{R}) \) is skew-symmetric, then \( A \) is semimonotone.
Proposition

If $A \in M_n(\mathbb{R})$ is skew-symmetric, then $A$ is semimonotone.

Proof

It can be easily shown that if $A$ is skew-symmetric, then $x^T Ax = 0$. Hence, $A$ is copositive. The result follows. Alternatively, one can show that if $A$ is skew-symmetric, then it is $P_0$. Hence, $A$ is semimonotone.
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Proof

It can be easily shown that if $A$ is skew-symmetric, then $x^T A x = 0$. Hence, $A$ is copositive. The result follows. Alternatively, one can show that if $A$ is skew-symmetric, then it is $P_0$. Hence, $A$ is semimonotone.

Note neither of these results are interesting since we already knew that all $P_0$ matrices are semimonotone.
Proposition

Suppose $A$ is a $Z$-matrix. Then $A$ is semimonotone if and only if $A$ is an $M$-matrix.

Proposition

Suppose $A$ is a $Z$-matrix. Then $A$ is strictly semimonotone if and only if $A$ is a nonsingular $M$-matrix.
Proposition

Suppose $A \in M_n(\mathbb{R})$ has all proper principal submatrices semimonotone. If $A$ has a row or column of nonnegative entries, then $A$ is semimonotone.
Matrices with a nonnegative row or column whose proper principal submatrices are semimonotone

**Proposition**

Suppose $A \in M_n(\mathbb{R})$ has all proper principal submatrices semimonotone. If $A$ has a row or column of nonnegative entries, then $A$ is semimonotone.

Some matrices which have a row or column of nonnegative entries, and whose principal submatrices are semimonotone, are neither $P_0$ nor copositive. An example might be

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 3 & -1 \\ 6 & -8 & 4 \end{bmatrix}$$
Proposition

Let $A$ be a $2 \times 2$ real matrix with a nonnegative diagonal. Then $A$ is semimonotone if and only if either all entries in $A$ are nonnegative or the determinant of $A$ is nonnegative.
Proposition

Let $A$ be a $2 \times 2$ real matrix with a nonnegative diagonal. Then $A$ is semimonotone if and only if either all entries in $A$ are nonnegative or the determinant of $A$ is nonnegative.

Thus, we see that if $A \in M_2(\mathbb{R})$ with a nonnegative diagonal, then $A$ is not semimonotone if and only if $A = \begin{bmatrix} a & -b \\ -c & d \end{bmatrix}$ where $\det A < 0$. 
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Some basic properties of semimonotone matrices

**Proposition**

Let $A$ be a semimonotone matrix. If $E$ is a nonnegative matrix, then $A + E$ is semimonotone.
Some basic properties of semimonotone matrices

Proposition

Let \( A \) be a semimonotone matrix. If \( E \) is a nonnegative matrix, then \( A + E \) is semimonotone.

Proposition

Let \( P \in M_n(\mathbb{R}) \) be a permutation matrix. Then a matrix \( A \in M_n(\mathbb{R}) \) is semimonotone if and only if \( PAP^T \) is semimonotone.
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Let $P \in M_n(\mathbb{R})$ be a permutation matrix. Then a matrix $A \in M_n(\mathbb{R})$ is semimonotone if and only if $PAP^T$ is semimonotone.

Proposition

Let $A \in M_n(\mathbb{R})$ be a block upper triangular matrix with diagonal blocks $A_1, A_2, \ldots, A_n$ which are semimonotone. Then $A$ is semimonotone.
Multiplying a semimonotone matrix by a diagonal matrix with nonnegative diagonal entries

Proposition

Let $A \in M_n(\mathbb{R})$ and let $D = \text{diag}(d_1, d_2, \ldots, d_n)$ where $d_i \geq 0$. If $A$ is semimonotone, then the matrices $AD$ and $DA$ are semimonotone.
Proposition

Let $A \in M_n(\mathbb{R})$ and let $D = \text{diag}(d_1, d_2, \ldots, d_n)$ where $d_i \geq 0$. If $A$ is semimonotone, then the matrices $AD$ and $DA$ are semimonotone.

Note the converse is not true. Take, for example

$$A = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

$A$ is not semimonotone but both

$$AD = \begin{bmatrix} 0 & -2 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad DA = \begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix}$$

are semimonotone.
What if the diagonal entries of $D$ are all positive?

However, if $D = \text{diag}(d_1, d_2, \ldots, d_n)$ where $d_i > 0$, then we get the following.
What if the diagonal entries of $D$ are all positive?

However, if $D = \text{diag}(d_1, d_2, \ldots, d_n)$ where $d_i > 0$, then we get the following.

**Proposition**

Let $A \in M_n(\mathbb{R})$ and let $D = \text{diag}(d_1, d_2, \ldots, d_n)$ be a diagonal matrix with $d_i > 0$. Then the following statements are equivalent.

(i) $A$ is semimonotone

(ii) $DA$ is semimonotone

(iii) $AD$ is semimonotone
Proposition

Given any real $n \times n$ matrix with nonnegative trace and spectrum $\sigma$, there exists a semimonotone matrix $A$ such that $\sigma(A) = \sigma$. 

Proof (Outline)

Let $\sigma$ be the spectrum of any real $n \times n$ matrix $M$ with nonnegative trace. The characteristic polynomial of $M$ will be in the form

$$p(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$$

where $a_{n-1} = -\text{tr}(M) \leq 0$.

It can be shown that the companion matrix of $p(x)$ given by

$$A = \begin{bmatrix}
0 & 0 & \cdots & 0 \\
-1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 \\
-\alpha_{n-1} & -\alpha_{n-2} & \cdots & -\alpha_0
\end{bmatrix}$$

is semimonotone.
Spectral restrictions of semimonotone matrices

**Proposition**

*Given any real $n \times n$ matrix with nonnegative trace and spectrum $\sigma$, there exists a semimonotone matrix $A$ such that $\sigma(A) = \sigma$.*

**Proof (Outline)**

- Let $\sigma$ be the spectrum of any real $n \times n$ matrix $M$ with nonnegative trace.
- The characteristic polynomial of $M$ will be in the form
  
  $$p(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$$

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- It can be shown that the companion matrix of $p(x)$ given by
  
  $$A = \begin{bmatrix}
  0 & 0 & \cdots & 0 & -a_0 \\
  1 & 0 & \cdots & 0 & -a_1 \\
  0 & 1 & \cdots & 0 & -a_2 \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & \cdots & 1 & -a_{n-1}
  \end{bmatrix}$$

  is semimonotone.
Suppose $A \in M_n(\mathbb{R})$ with all proper principal submatrices semimonotone. Then $A$ is semimonotone if and only if for all invertible diagonal matrices $D \geq 0$ where $D \in M_n(\mathbb{R})$, $A + D$ does not have a positive nullvector.
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Possible characterization of semimonotone matrices

**Conjecture**

Suppose $A \in M_n(\mathbb{R})$ has all proper principal submatrices semimonotone. Then

$$A \text{ is not semimonotone} \iff \text{adj}(A) \geq 0, \text{ with all non-diagonal entries strictly greater than zero, and } \det A < 0.$$  

The backwards direction of this conjecture can easily be proven. However, the forward direction remains unknown.
Since $A \text{adj}(A) = \det(A)I$, the previous conjecture is equivalent to the following one (as long as $A$ is invertible).

**Conjecture**

Suppose $A \in M_n(\mathbb{R})$ has all proper principal submatrices semimonotone. Then

\[
A \text{ is not semimonotone} \iff A^{-1} \leq 0, \text{ with all non-diagonal entries strictly less than zero, and } \det A < 0.
\]
How do almost semimonotone matrices act on vectors with mixed signs?

- It is well-known that a $P$-matrix does not completely reverse the sign of any nonzero vector $x$. For example, if $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and $x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, then $Ax = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ which completely changed the sign of the vector.
How do almost semimonotone matrices act on vectors with mixed signs?

- It is well-known that a $P$-matrix does not completely reverse the sign of any nonzero vector $x$.
- Strictly semimonotone matrices act the same way on positive vectors and negative vectors. However, they don’t necessarily act the same way on vectors containing positive and negative entries.

\[ A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad Ax = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \]

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- For example, if $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and $x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, then

\[
Ax = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\]

which completely changed the sign of the vector.
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- But what if we now take a matrix which is not semimonotone but whose proper principal submatrices are all (strictly) semimonotone? We’ll call this type of matrix *almost (strictly) semimonotone*. 

Every $2 \times 2$ almost semimonotone matrix acts the same way on vectors of mixed sign as $P$-matrices do in that they don’t completely reverse their sign.

Is this true for larger matrices?

**Conjecture**

Suppose $A \in \mathbb{M}_n(\mathbb{R})$ is almost semimonotone. Then for all vectors $x$ of mixed sign, there exists a $k$ such that $x_k(Ax)_k > 0$. 

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- But what if we now take a matrix which is not semimonotone but whose proper principal submatrices are all (strictly) semimonotone? We’ll call this type of matrix *almost (strictly) semimonotone*.

- We’ll also call $x$ a *vector of mixed sign* if $x$ contains both positive and negative entries, but no zero entries.
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Definition

A signature matrix $S$ is a diagonal matrix with each diagonal entry being $\pm 1$. 

A related conjecture to the previous one is the following.

Conjecture

Suppose $A$ is almost semimonotone. Then for any signature matrix $S \neq \pm I$, $SAS$ is semimonotone.

For this to be true we'd also need to prove the following.

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Suppose $A$ is almost semimonotone. Then for any signature matrix $S$, all proper principal submatrices of $SAS$ are semimonotone.
Almost semimonotone and signature similarities

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Theorem

$A \in M_n(\mathbb{R})$ is a $P$-matrix ($P_0$-matrix) if and only if for every signature matrix $S \in M_n(\mathbb{R})$, $SAS$ is an $S$-matrix ($S_0$-matrix).
Signature similarities, $P$-matrices, and strictly semimonotone matrices

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$A \in M_n(\mathbb{R})$ is a $P$-matrix ($P_0$-matrix) if and only if for every signature matrix $S \in M_n(\mathbb{R})$, $SAS$ is an $S$-matrix ($S_0$-matrix).

Since a matrix $A$ is (strictly) semimonotone if and only if $A$ and all its proper principal submatrices are $S_0$-matrices ($S$-matrices) we can get the following result.

**Proposition**

The following are equivalent:

(a) $A$ is a $P_0$-matrix ($P$-matrix).
(b) $SAS$ is an $S_0$-matrix ($S$-matrix) for all signature matrices $S$.
(c) $SAS$ is (strictly) semimonotone for all signature matrices $S$. 
Almost semimonotone implies all proper principal submatrices are $P_0$?

Let us look at the previous conjecture again.

**Conjecture**

Suppose $A$ is almost semimonotone. Then for any signature matrix $S$, all proper principal submatrices of $SAS$ are semimonotone.
Almost semimonotone implies all proper principal submatrices are $P_0$?

Let us look at the previous conjecture again.

Conjecture
Suppose $A$ is almost semimonotone. Then for any signature matrix $S$, all proper principal submatrices of $SAS$ are semimonotone.

This could only be true if all the proper principal submatrices of $A$ were $P_0$-matrices.

Conjecture
Suppose $A$ is almost semimonotone. Then all proper principal submatrices are $P_0$-matrices.
To summarize

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If $A$ is almost semimonotone, then

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**Conjecture**
If $A$ is almost semimonotone, then

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2. $A^{-1} \preceq 0$ with non-diagonal entries strictly less than zero
3. Every proper principal submatrix of $A$ is a $P_0$-matrix
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**Conjecture**

If $A$ is almost semimonotone, then

1. $\det A < 0$
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4. The matrix $SAS$ is semimonotone for every signature matrix $S \neq \pm I$. 
To summarize I would really like to prove all of the following about almost semimonotone matrices.

**Conjecture**

If $A$ is almost semimonotone, then

1. $\det A < 0$
2. $A^{-1} \leq 0$ with non-diagonal entries strictly less than zero
3. Every proper principal submatrix of $A$ is a $P_0$-matrix
4. The matrix $SAS$ is semimonotone for every signature matrix $S \neq \pm I$.
5. $A$ cannot reverse the sign of a vector with both positive and negative entries (but no zero entries).
Some results

Lemma

If $A$ does not completely reverse the sign of any vector of mixed sign, then for any signature matrix $S \neq \pm I$, $SAS$ is strictly semimonotone. (Note this implies that all proper principal submatrices are $P$-matrices.)

Proposition

Suppose $A$ has all proper principal submatrices semimonotone and suppose that $A$ does not reverse the sign of a vector of mixed sign. Then either $A$ is a $P$-matrix or $A$ is almost semimonotone (and almost-$P$-matrices).

Proposition

Suppose $A$ is an almost semimonotone matrix which is also an almost $P_0$ matrix. Then $A^{-1} \leq 0$. 

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*If A does not completely reverse the sign of any vector of mixed sign, then for any signature matrix $S \neq \pm I$, $SAS$ is strictly semimonotone. (Note this implies that all proper principal submatrices are $P$-matrices.)*

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Suppose $A$ has all proper principal submatrices semimonotone and suppose that $A$ does not reverse the sign of a vector of mixed sign. Then either $A$ is a $P$-matrix or $A$ is almost semimonotone (and almost-$P$).

**Proposition**

Suppose $A$ is an almost semimonotone matrix which is also an almost $P_0$ matrix. Then $A^{-1} \leq 0$. 
Outline

1 Introduction
- The definition of semimonotone & an example
- Some observations and previous results
- Questions

2 Some Results
- What kinds of matrices are semimonotone?
- Properties of semimonotone matrices

3 Conjectures

4 Future Directions
Prove all these conjectures or find counterexamples.
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- Prove all these conjectures or find counterexamples.
- Is it true that a semimonotone matrix is the sum of a $P_0$ matrix and a nonnegative matrix, or something similar?
- Find a way to create generic semimonotone matrices or test whether or not a matrix is semimonotone.
Thank you.