Lecture 20 *: Higher Order Linear Differential Equations

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Math 315
Section 03
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• Section 4.2: Higher-Order Homogeneous solutions, constant coefficients

Consider the solution to the homogeneous equation:

\[ L[y] = 0 \iff a_n y^{(n)} + a_{n-1}(t)y^{(n-1)} + a_{n-2}y^{(n-2)} + \ldots + a_2(t)y'' + a_1 y' + a_0(t)y = 0 \iff \sum_{k=0}^{n} a_k y^{(k)} = 0 \]

We can apply the same strategy as in the second-order case to obtain a characteristic equation:

Assume \( y = e^{rt} \) is a solution. Then:

\[ y = e^{rt} \implies y' = re^{rt}, y'' = r^2 e^{rt}, \ldots, y^{(n)} = r^n e^{rt} \]

\[ L[y] = 0 \iff \sum_{k=0}^{n} a_k y^{(k)} = 0 \iff \sum_{k=0}^{n} a_k (r^k e^{rt}) = 0 \iff e^{rt} \left( \sum_{k=0}^{n} a_k r^k \right) = 0 \iff \sum_{k=0}^{n} a_k r^k = 0 \]

If \( n=2 \), we get that \( a_2 r^2 + a_1 r + a_0 = 0 \), which we wrote as \( ar^2 + br + c = 0 \).

\[ \sum_{k=0}^{n} a_k r^k = 0 \]

is our characteristic equation. For \( n > 2 \), this will of course be harder to solve as there is no quadratic formula equivalent for \( n > 4 \).

For example, this is the “cubic formula” which yields the roots \( x_1, x_2, x_3 \) of:

\[ ax^3 + bx^2 + cx + d = 0 \]

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It is more realistic to consider problems polynomials $\sum_{k=0}^{n} a_k r^k$ that are factorable.

We know that there will be $n$ roots.

We have our standard toolbox of root-finding methods:

Assume $a, b, c$ are real constants and $m$ is an integer:

1. Squares:

   $$(a + b)^2 = a^2 + 2ab + b^2$$

   $$(a - b)^2 = a^2 - 2ab + b^2$$

   $$a^2 - b^2 = (a + b)(a - b)$$

   Completing the square:

   $$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = a\left[(x + \frac{b}{2a})^2 + \left(\frac{c}{a} - \frac{b^2}{4a^2}\right)\right]$$

2. Cubes:

   $$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
\[(a - b)^3 = a^3 + 3a^2b - 3ab^2 - b^3\]
\[a^3 - b^3 = (a - b)(a^2 + ab + b^2)\]
\[a^3 + b^3 = (a + b)(a^2 - ab + b^3)\]

3. Factoring by grouping for cubics:

\[acx^3 + adx^2 + cbx + bdx = ax^2(cx + d) + b(cx + d) = (ax^2 + b)(cx + d)\]

Moving on to higher powers...

4. Binomial Theorem:

\[(a + b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}x^{n-1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{n-2}a^2b^{n-2} + \binom{n}{n-1}a^1b^{n-1} + \binom{n}{n}a^0b^n\]

Where:

\[\binom{n}{k} = \frac{n!}{(n-k)!k!}\]

for \(n > k, n, k\) are positive integers

Binomial coefficients listing from \(\binom{n}{0}\) to \(\binom{n}{n}\) for \(n=0,1,2,3,4,5\):

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Pascal’s Triangle!

Application: \(n = 4 : 1, 4, 6, 4, 1 \implies (a + b)^4 = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1\)

5. Reducing convenient even degrees polynomials into quadratics:

\[ax^{2m} + bx^m + 1 = a(x^m)^2 + b(x^m) + 1\]

\[\implies x^m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\]
6. Difference of squares trick for even degree:

\[(x^{2m}+1) = (x^{2m}+2x^m+1) - 2x^m = (x^m+1)^2 - 2x^m = (x^m+1-\sqrt{2}x^{m\over 2})(x^m+1+\sqrt{2}x^{m\over 2})\]

7. Roots of unity:

\[x^n - 1 = 0 \implies x^n = 1\]

Clearly \(x = 1\) is a solution, but consider Euler’s formula:

\[e^{ix} = \cos x + i \sin x, i = \sqrt{-1}\]

\[e^{ix} = e^{i(x+2\pi)}\]

\[e^{ix} = r \iff e^{i(x+2\pi)} = r, e^{i(x+6\pi)} = r, e^{i(x+8\pi)}, \ldots e^{i(x+2m\pi)} = r\]

\[e^0 = 1 \iff e^{i(0+2m\pi)} = 1\]

\[\implies x^n = 1 \iff x = e^{i(2m\pi)/n} = \cos \left(\frac{2m\pi}{n}\right) + i \sin \left(\frac{2m\pi}{n}\right)\]

For \(m < n\), since \(e^{x+2\pi i} = e^x\)

8. Complex root factoring:

\[i^2 = -1, -i^2 = 1\]

\[\implies x^2 + a^2 = x^2 + (1)a^2 = x^2 + (-i^2)a^2 = x^2 - (ia)^2 = (x + ia)(x - ia)\]

This method is useful to find conjugates. Also, If \(a + ib\) is a root, so is \(a - ib\).

9. Long division:

Let

\[P_n(x) = \sum_{k=0}^{n} a_k x^k\]

Then all rational roots will be factors of:

\[{a_0 \over a_n}\]
Also, if \( r_k \) are roots root of \( P_n \), \( P_n \) can be written:

\[
P_n = (x - r_1)(x - r_2)\cdots(x - r_n)
\]

So if we find a root \( r \), then we can find:

\[
Q_n = \frac{P_n}{x - r}
\]

by polynomial long division or synthetic division and hopefully factor this with one of the above methods.

Trick:

\[
\frac{P_n}{x - r} = x + 1 + \frac{2x}{x + 1} = 1 + \frac{2x}{x + 1}
\]

Again we will have three types of roots: real, repeated, complex—but we can now have combinations of these.

(Ex: 1 real, 2 complex, 2 repeated for a 5th degree polynomial)

10. Calculus (Guess and Check)

Intermediate Value Theorem:

If is a root of a polynomial \( P_n(t) \) \( P(a) < 0, P(b) > 0 \) (in other words \( P(a)P(b) < 0 \)) for some \( a, b \) then \( P_n \) has a root \( c \) between \( a \) and \( b \).

Newton’s Method:

\[
x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, k = 0, 1, 2, ...
\]

Guess an \( x_0 \) and let \( f(x) = P_n(x) - r \) where \( r \) is what you think the root is. Iterate this and if you guessed well, you will produce a set of numbers that get closer to 0.

To wrap it up, the general solution will look like:

\[
\begin{align*}
    y &= \sum_{k=1}^{n} C_k e^{\alpha_k t} = 0 & r &= r_1, r_2, \ldots r_n \ n \text{ distinct roots} \\
    y &= \sum_{k=1}^{n} C_k t^{k-1} e^{\alpha t}  & r &= r, r, r, \ldots \ r \text{ is real, repeated } n \text{ times} \\
    y &= \sum_{k=1}^{n} C_k t^{k-1} e^{\alpha t} (\cos \beta t + \sin \beta t) \quad r = \alpha \pm i\beta \text{ repeated } n \text{ times} \\
    y &= \sum_{k=1}^{n/2} C_k e^{\alpha_k t} (\cos \beta_k t + \sin \beta_k t) \quad r = \alpha_1 \pm i\beta_1, \alpha_2 \pm i\beta_2, \ldots \text{distinct complex roots, } n \text{ is even}
\end{align*}
\]

And the fundamental solutions \( y_1, y_2, \ldots y_n \):

For \( k = 1, \ldots n \),
\[
\begin{aligned}
y_k &= e^{r_k t} = 0 & r &= r_1, r_2, \ldots, r_n \text{ n distinct roots} \\
y_k &= t^k e^{rt} & r &= r, r, r, \ldots \text{ r is real, repeated n times} \\
y_k &= t^{k-1} e^{\alpha t}(\cos \beta t + \sin \beta t) & r &= \alpha \pm i\beta \text{ repeated n times} \\
y_k &= C_k e^{\alpha_k t}(\cos \beta_j t + \sin \beta_j t) & r &= \alpha_1 \pm i\beta_1, \alpha_2 \pm i\beta_2, \ldots \text{ n/2 distinct complex roots and conjugates}
\end{aligned}
\]

(In the last case, \( j = 1, 2, \ldots, \frac{n}{2} \), n is even, complex roots are always in pairs!)

If the polynomial does not have roots that all fall into one of the above cases, but instead a combination:

1. Split the polynomial into a product of lower degree polynomials.
2. Apply the above scenarios to obtain fundamental solutions \( y_k \).
3. The general solution will be their linear combination \( \sum_{k=1}^{n} C_k y_k \).

**Example 2: Distinct roots**

Find the general solution:

\[3y''' - 2y'' + 6y' - 4y' = 0\]

Characteristic equation:

\[3r^4 - 2r^3 + 6r^2 - 4r = 0\]

Factor an \( r \):

\[r(3r^3 - 2r^2 + 6r - 4) = 0\]

Factor by grouping:

\[r[r^2(3r - 2) + 2(3r - 2)] = 0\]

\[r(r^2 + 2)(3r - 2) = 0\]

Complex root factoring for \( r^2 + 2 \):

\[r(r + i\sqrt{2})(r - i\sqrt{2})(3r - 2) = 0\]
$$\implies r = 0, -i\sqrt{2}, i\sqrt{2}, \frac{2}{3}$$

Fundamental solutions:

$$y_1 = e^{0t}, y_2 = \cos(t\sqrt{2}), y_3 = \sin(t\sqrt{2}), y_4 = e^{\frac{2t}{3}}$$

General solution:

$$y = C_1y_1 + C_2y_2 + C_3y_3 + C_4y_4$$

$$= C_1 + C_2\cos(t\sqrt{2}) + C_3\sin(t\sqrt{2}) + C_4e^{\frac{2t}{3}}$$

You can see that if we had initial conditions, we would need 4 of them, to have 4 equations and 4 unknowns.

**Example 3: Repeated Roots**

Find the general solution:

$$y''' + 8y'' + 16y' = 0$$

Characteristic Equation:

$$r^3 + 8r^2 + 16r = 0$$

Factor $r$:

$$r(r^2 + 8r + 16) = 0$$

Notice that: $(r + 4)^2 = r^2 + 2(r)(4) + 4^2 = r^2 + 8r + 16$

$$r(r + 4)^2 = 0$$

$$\implies r = 0, -4, -4$$

Each root is repeated twice, meaning we get a fundamental solution for each, then multiply each one by $t$ to get 2 more.
Fundamental solutions:
\[ y_1 = e^0, \quad y_2 = e^{-4t}, \quad y_3 = te^{-4t} \]
\[ \implies y = C_1 y_1 + C_2 y_2 + C_3 y_3 = C_1 + C_2 e^{-4t} + C_3 te^{-4t} \]

**Example 4: Repeated complex root.**

Find the general solution:
\[ y^{(5)} + 6y''' + 9y' = 0 \]

Characteristic Eq:
\[ r^5 + 6r^3 + 9r = 0 \iff r(r^4 + 6r^2 + 9) = 0 \iff r(r^2 + 3)^2 = 0 \iff r[(r + i\sqrt{3})(r - i\sqrt{3})]^2 \]

Roots:
\[ r = 0, i\sqrt{3}, i\sqrt{3}, -i\sqrt{3}, -i\sqrt{3} \]

We have repeated complex conjugate roots, so we need to multiply their fundamental solutions by \( t \) to get more:

Fundamental solutions:
\[ y_1 = e^0, \quad y_2 = \cos(t\sqrt{3}), \quad y_3 = \sin(t\sqrt{3}), \quad y_4 = t \cos(t\sqrt{3}), \quad y_5 = t \sin(t\sqrt{3}) \]
\[ y = C_1 e^0 + C_2 \cos(t\sqrt{3}) + C_3 \sin(t\sqrt{3}) + C_4 t \cos(t\sqrt{3}) + C_5 t \sin(t\sqrt{3}) \]

An application of higher-order systems:

A spring-mass system with multiple springs/masses attached: 2 are needed derivatives per mass.

The motion of each one depends on the motion of the one above it, so the equation for the lower springs will be needed to substituted into the equations for the spring one level above, and so on until the top is “reached.”

Here is a picture of two:
The process is much like before, just longer and more complex as \( n \) increases.

We will not go into much detail but we also have an extended Undetermined Coefficients and Variation of Parameters (covered later in Ch. 4):

- **Undetermined Coefficients:**
  
  Same guidelines for guessing, but if we reach a contradiction, the factor \( t^a \) we multiply by will now range up to \( t^{n-1} \) if the equation is degree \( n \).

- **Variation of Parameters:**
  
  Given the DE:

  \[
  L[y] = y^{(n)} + P_1(t)y^{(n-1)} + P_2(t)y^{(n-2)} + \ldots + P_{n-2}(t)y'' + P_{n-1}(t)y' + P_n(t)y = g(t)
  \]

  Now we have:

  \[
  \tilde{C} = \left( \begin{array}{c}
  C_1 \\
  C_2 \\
  C_3 \\
  \vdots \\
  C_n
  \end{array} \right) \Psi = \left( \begin{array}{cccc}
  y_1 & y_2 & \cdots & y_n \\
  y'_1 & y'_2 & \cdots & y'_n \\
  \vdots & \vdots & \ddots & \vdots \\
  y^{(n-1)}_1 & y^{(n-1)}_2 & \cdots & y^{(n-1)}_n
  \end{array} \right)
  \]

  \[
  W = \det \Psi, \; \vec{u} = \left( \begin{array}{c}
  u_1 \\
  u_2 \\
  \vdots \\
  u_n
  \end{array} \right)
  \]
We seek to find $u_1, u_2, \ldots, u_n$ such that:

$$y = y_h + y_p = C_1y_1 + C_2y_2 + \ldots + C_ny_n + u_1y_1 + u_2y_2 + \ldots + u_ny_n$$

The derivation is identical in matrix language, meaning we obtain:

$$u = \int \Psi^{-1} \tilde{f} dt = \begin{pmatrix} \int \frac{W_1g(t)}{W} dt \\ \int \frac{W_2g(t)}{W} dt \\ \vdots \\ \int \frac{W_ng(t)}{W} dt \end{pmatrix}$$

Where $W_m$ the determinant obtained from $W$ by replacing the $m$th column by the column

$$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ g(t) \end{pmatrix}$$

This result is obtained from applying Cramer’s Rule for inverting a matrix.