A Study of Student Understandings of the Equation \( Ax = cx \)
in an IMAGEMath Undergraduate Linear Algebra Class

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April 14, 2020
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In November, 2019 I travelled to a small, elite liberal arts college in Ohio to conduct research on how undergraduates learn linear algebra in a class taught with the IMAGEMath textbook and course materials. The IMAGEMath project is a mathematics education project in development with the goal of teaching linear algebra and differential equations with inspiration from real world applications such as heat diffusion and radiography (Asaki et al., 2019). While visiting the college I observed one lecture, an in-class lab, and conducted interviews with five of the students before and after the lab.

There are two goals for this paper: To share the results of my research and to make suggestions for changes in future rounds of data collection. The suggested changes relate to the research questions and interview protocols. First I include some background information for the research. Following that I include the new research questions and the results of the research. At the end I describe how the data collection can be improved in the future.

Purpose and Context

Purpose of the Research

Linear algebra is a commonly required course for undergraduate science, technology, engineering, and mathematics majors. Linear algebra is foundational for most advanced mathematics, such as abstract algebra, algebraic geometry, and field theory. Students not studying abstract mathematics will see linear algebra used, if only implicitly, in differential equations, which is fundamental for many subfields of physics. Understanding how students learn linear algebra is important: Through research into undergraduate linear algebra education we can better serve our students and their diverse needs for linear algebra.

A popular textbook for a first semester undergraduate linear algebra course is Lay et al.’s (2016) *Linear Algebra and its Applications*. An advantage of the textbook is that it teaches students how to use computations and algorithms to answer just about any question they have about vectors, matrices, linear maps, and their properties in the finite dimensional case. One disadvantage of this algorithmic approach, however, is that students may develop concept images for concepts such as linear independence, span, and solution (to a system of equations) which depend on these computations but which do not match the concept definitions. [CITATION? I might want to add something from Simon, Tzur (2009) about "cognitive growth" and whatever they have to say about procedural tasks. Can we really say that students are learning when they churn out the results of algorithms?] See Vinner’s (1983) paper for more on concept image and concept definition.

For a brief example of concept image and concept definition, consider the following. Three column vectors $u$, $v$, and $w$ are said to be linearly independent if the equality

$$au + bv + cw = 0$$

implies $a = b = c = 0$. After being taught this definition, students may be taught the three vectors are linearly independent if the augmented matrix $[u \ v \ w]$, i.e. the matrix whose three columns are $u$, $v$, and $w$, has a pivot in every column. (A pivot is a location of a “leading 1” in the reduced row-echelon form of the matrix, which students can compute.) This fact may become part of their concept image and allows students to computationally determine if the three vectors are
linearly independent without needing to know the more complicated concept definition. In my personal experience a consequence of this type of concept image in a linear algebra class is that students may not be able to answer conceptually higher-level proof-like questions.

Disadvantages of computationally-driven linear algebra classes, along with the desire to motivate the mathematics through real world examples, lead the IMAGEMath team to write their textbook. [Confirm with IMAGEMath about their motivation.] To determine if their course is effective the team invited me to conduct research on their classes. We decided to start small by investigating how students in their classes learn about and understand eigenvalues and eigenvectors: A number $c$ is called an eigenvalue of a matrix $A$ if there is a nonzero vector $v$, called an eigenvector associated with $c$, such that $Av = cv$ (Lay et al., 2016).\footnote{Frequently I will use a letter $c$ for an eigenvalue instead of the standard $\lambda$ (lambda). The reasons are twofold: (1) To avoid alienating readers less familiar with this notation. (2) So that when I talk about eigenvalues with participants I do not have to make it known to them that we are talking about eigenvalues.}

While eigenvalues and eigenvectors have applications outside of linear algebra (they are very important for solving certain types of differential equations, for example), the choice to study how students learn about them is motivated in part by convenience. The dates I was able to travel for research aligned with when the class was taught these concepts. Now that I have some data collected, continued research in this area is a good use of my time.

**The Heat Diffusion Context**

To summarize the context of the in-class lab, the IMAGEMath students were taught about the process of heat diffusing through a rod. The rod is assumed to only diffuse heat into the environment through its ends, so for heat to disperse from one part of the rod it must flow from that point towards either end of the rod. All temperatures are given relative to the fixed ambient temperature so the ends of the rods have temperature 0. Using a finite number of sampling points to represent the rod, students could work with vector representations of a given heat state (at a specific time) in $\mathbb{R}^n$. Using differential equations which model the situation, the class constructed a matrix $E$ such that if the vector $h_t$ is the heat state of the rod at time $t$, then $Eh_t = h_{t+1}$ is the heat state of the rod one time step later. The matrix is

$$E = \begin{bmatrix} 1 - 2\delta & \delta & 0 & \cdots & 0 \\ \delta & 1 - 2\delta & \delta & & \\ 0 & \delta & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \delta & 0 \\ 0 & \cdots & 0 & \delta & 1 - 2\delta \end{bmatrix}$$

where $\delta$ (delta) is a given parameter between 0 and $\frac{1}{4}$ related to how quickly heat diffuses through the given rod.

**Theoretical Framework**

[To do...]
Literature Review and Research Questions

It is not uncommon in qualitative research to change your research questions to those the data can answer because researchers cannot always anticipate the types of responses they will get. My original research questions can be found in Appendix A. Reflecting on the original questions and the data I collected, I wrote research questions my data can answer. These questions are:

1. What ways of thinking about $Ax = cx$ do students use? More precisely:
   a. In what ways are the students conceptualizing vectors, matrix-vector and scalar-vector products, and vector equality?
   b. To the students, what does it mean to solve $Ax = cx$? What does it mean to be a solution to this equation? How do students solve $Ax = cx$ for the first time?
2. What changes in student perceptions or knowledge might we attribute to the lab?
3. How do the students use the heat state context to make arguments about linear algebra and vice-versa?

Methodology and Methods

Demographics of Participants

This research was conducted at a small private elite liberal arts college in Ohio. According to Data USA (2020), the school is predominantly white: In 2017, the school awarded nearly 400 degrees to white students and around 30 to hispanic or latinx students, the next largest group. In the same year 56% of degrees were awarded to women while 44% were awarded to men.

The five participating students were asked to share how they identified, to whatever extent they felt comfortable. One student, Student A, identified as a white female. Students B, C, and D identified as white males. Student E identified as a hispanic or latino male. Students A and B are physics majors while each of Students C, D, and E are mathematics and economics majors.

The class as a whole likely consisted of students with different majors and mathematics backgrounds, but I did not collect data on their demographics. The class instructor is one of the members of the IMAGEMath team and was not directly involved with the data collection for my observations or my interviews. However, the instructor was aware of what data I intended to collect and how.

Interview Protocols

Each of the participating students agreed to two interviews, one before and one after the in-class lab. As mentioned in the introduction the research questions for the paper are not the same research questions I went to the college to investigate. To understand my interview protocols it is important to know what questions I originally intended to answer.
Sfard’s (1991) notions of object and process conceptions motivated my original questions. These notions are still of interest to me, but they are used in my theoretical framework rather than my research questions. [Reminder: I have not written that part of the document yet!]

The original research questions were:

1. Do students have access to an object conception of \( \mathbf{A} \mathbf{v} \) and \( c \mathbf{v} \)?
2. How are the students learning about eigenvectors and eigenvalues in the lab?

The discussion includes comments about these research questions and suggestions for improvements to the protocols below.

**Pre-Lab Interview**

Within each question are sub-questions to facilitate student responses, starting from the general and moving to the specific. For example, if a student does not seem comfortable talking about vectors “in general” one can ask them the same questions with the specific matrices and vectors provided. The sub-questions were also meant to elicit responses for specific data I wanted to collect. For example, when I asked the students about the heat diffusion context I wanted to find out what they think is a vector in this context and why. The answer to this question informed how they interpreted everything else I asked them.

To avoid confusion by repeatedly switching contexts, most questions were to be asked in the heat diffusion context. If after the first question the student seemed entirely unfamiliar with the heat diffusion context, I planned to ask most of the questions without this context. Each interview ran between 25 and 50 minutes. In the future I do not expect to ask each participant each sub-question.

**Protocol.** The first four questions below relate to student ways of thinking about heat diffusion, vector equality, and each of \( \mathbf{A} \mathbf{v} \), \( c \mathbf{v} \), and \( \mathbf{A} \mathbf{v} = c \mathbf{v} \) in the heat diffusion context. These questions are meant to give me insight into what the students bring into the in-class lab. Data collected from each of these questions gives insight into the prerequisite knowledge students need before they can understand the equation \( \mathbf{A} \mathbf{x} = c \mathbf{x} \), which I am primarily interested in. The last question explicitly addresses this equation. I anticipated most students answering the first five questions relatively quickly but needing time to answer the last question.

1. Please tell me how you are using linear algebra to study heat diffusion.
   a. What kinds of things are vectors in heat diffusion? Why are they vectors?
   b. What kinds of matrices have you seen while learning about heat diffusion?
2. In heat diffusion, what can you tell me about multiplying a matrix times a vector?
   a. Can you draw me a picture that explains this idea?
   b. If \( \mathbf{A} \) is a matrix, such as the matrix \( \mathbf{E} \) from the discrete diffusion matrix equation, and \( \mathbf{v} \) is a vector, such as a heat state vector, what can you tell me about \( \mathbf{A} \mathbf{v} \)?
   c. What kind of thing is \( \mathbf{A} \mathbf{v} \)?
   d. What kind of thing do you get when you do this?
   e. For the matrix \( \mathbf{A} \) and vector \( \mathbf{v} \) provided, what can you tell me about \( \mathbf{A} \mathbf{v} \)?

\[
\mathbf{A} = \begin{bmatrix} -1 & 2 \\ -3 & 0 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}.
\]

3. In heat diffusion, what can you tell me about multiplying a number times a vector?
a. Can you draw me a picture that explains this idea?
b. If \( c \) is a number and \( \mathbf{v} \) is a vector, such as a heat state vector, what can you tell me about \( c\mathbf{v} \)?
c. What kind of thing is \( c\mathbf{v} \)?
d. What kind of thing do you get when you do this?
e. For the \( c \) and vector \( \mathbf{v} \) provided, what can you tell me about \( c\mathbf{v} \)?

\[
c = 4, \quad \mathbf{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.
\]

4. If \( E \) is the matrix appearing in the discrete diffusion matrix equation and \( \mathbf{v} \) is a heat state vector, please draw a picture to demonstrate what it would mean for \( A\mathbf{v} = 5\mathbf{v} \) to be true.

5. If \( A \) is a matrix and \( c \) is a number, what does it mean for \( x \) to be a solution to \( A\mathbf{x} = c\mathbf{x} \)?
   a. If you have a graph of \( \mathbf{x} \) and of \( A\mathbf{x} \), how might you tell if \( A\mathbf{x} = c\mathbf{x} \) for some number \( c \)? Suppose you’re working in the context of heat diffusion.
   b. If you are given a matrix \( A \) and a fixed number \( c \), how might you solve \( A\mathbf{x} = c\mathbf{x} \)?
   c. For the provided \( A \) and \( c \) below, how might you solve \( A\mathbf{x} = c\mathbf{x} \)?

\[
A = \begin{bmatrix} 2 & -2 \\ -3 & 1 \end{bmatrix}, \quad c = 4.
\]

Post-Lab Interview

Protocol. The first two questions below answer my second original research question, “How are the students learning about eigenvectors and eigenvalues in the lab?” One goal of the in-class lab is for students to visually recognize when \( A\mathbf{v} = c\mathbf{v} \) is true. It is possible that some students will enter the lab not understanding the equation \( A\mathbf{v} = c\mathbf{v} \), and I thought that after the lab they may be able to graphically interpret this equation in and out of the heat state context. The third question of this interview is the same as the last question from the first interview. I added it after conducting the first interview because I wanted to see if between the first and second interviews students feel or think differently about \( A\mathbf{x} = c\mathbf{x} \) or if they have thought or learned of new ways to solve it. The fourth question is to help frame my interpretation of their responses while the last question is to help report the demographics of my participants.

1. (Provide a copy of the lab activity, Figure B1.) In the lab you were asked to find similarities among these heat evolution scenarios. Can you tell me what kind of similarities you observed?
   a. As a class you discussed how certain heat state vectors were similar and you called them special vectors. Can you tell me what made those vectors special?
   b. How can you determine if other heat state vectors were the same special type?
2. (Provide a copy of the new activity, Figure B2.) Please look at this second collection of diagrams. Depicted in each diagram is a vector \( \mathbf{v} \) and a picture of the vector \( \mathbf{b} = A\mathbf{v} \). Please group these diagrams by any similarities you observe.
   a. Can you explain to me how you chose to group the diagrams?
   b. Can you compare how you grouped these diagrams to the way you grouped the diagrams in the lab?
c. Is there any group of diagrams here that are similar to the special vectors from the lab?

3. If A is a matrix and c is a number, what does it mean for x to be a solution to $Ax = cx$?
   a. If you have a graph of x and of Ax, how might you tell if $Ax = cx$ for some number c? Suppose you’re working in the context of heat diffusion.
   b. If you are given a matrix A and a fixed number c, how might you solve $Ax = cx$?
   c. For the provided A and c below, how might you solve $Ax = cx$?

\[
A = \begin{bmatrix}
2 & -2 \\
-3 & 1
\end{bmatrix}, \quad c = 4.
\]

4. Have you heard about eigenvalues before? If so, what do you know about them?

5. Other researchers reading my work might want to know what the demographics of my participants are. If you are comfortable doing so, please tell me how you would identify yourself? (Age, race, ethnicity, sex, gender identity, major, etc.)

Results

In this section I describe how the data I collected answers the new set of research questions. [Currently I reference instances in the interviews with the coding scheme I used in the first Data Analysis draft (e.g. A103 meaning Student A, Interview 1, Block 03). Change this to actual quotes later.]

In the pre-lab interviews each student was able to explain the heat diffusion context to some extent. Each student had an idea how to convert between heat state graphs and heat state vectors. The students knew about the heat evolution matrix E and most were able to explain something about where it came from. Students A, B, and E referred to the long term behavior of the heat states with Student E using repeated multiplication by the matrix E to contribute to his explanation [A201, B101, B104, E105]. It seems that the students entered the lab understanding the context, so that the context itself should not be an obstacle for their learning.

What ways of thinking about $Ax = cx$ do students display?

When I wrote the last question in my protocol for the pre-lab interview, I did not expect that students would generally know how to solve $Ax = cx$. I assumed that not knowing A and c explicitly might hinder the students (hence the subquestion with specific examples) but had hoped I might observe a student come to the conclusion that $(A - cI_n)x = 0$ where $I_n$ is the identity matrix. I asked the question so that I could observe if the lab prompted a change in the students, as the lab, when implemented completely, includes student discovery of this equation. The responses I got to the question were not quite what I expected. While I expected students may not have seen $Ax = cx$ before, I was surprised in some cases that the students might doubt its validity. One way of solving the equation was not expected but, in hindsight, should not have surprised me.

In what ways are the students conceptualizing vectors, matrix multiplication, and vector equality?

One reason to look at the equation $Av = cv$ in the first place is that genetic decompositions [ADD citation here or in the literature review to Salgado & Trigueros, 2015] for
STUDENT UNDERSTANDINGS OF $Ax = cx$

eigenvalues and eigenvectors include something like “students understand that each of $Av$ and $cv$ are vectors and are equal” [CITATION]. Knowing what it means for two vectors to be equal can mean knowing that as column vectors (in finite dimensions) corresponding number entries are equal. Understanding that $Av$ and $cv$ are equal vectors can come in two ways: Students might conceptualize these as the results of two separate processes which are the same or students can see these as vectors, without needing to imagine the associated process, which are equal. [Bring back to Sfard object/ process and/or Piaget.]

It was my hunch that to understand eigenvalues and eigenvectors students need to not only know that $Av$ and $cv$ are vectors which are equal, but that this equality is equality at the object level, not just the process level. I did not intend to study a causal relationship between these understandings in this research, so the interviews do not help determine if my hunch is correct. However, as a first step towards eventually answering if an object conception is necessary I gathered data on what conceptions students had available to them before the lab. [Q: Am I thinking of doing a teaching experiment? Does this need to be a quantitative question? See Simon & Tzur, 2009; Wawro, Larson, Zandieh, & Rasmussen, 2012; Kuper & Carlson, 2020.]

It is important to note that, as Sfard (1991) describes, the students are not restricted to either a process or an object conception of any topic in a math class. Students, like their professors, can leverage both types of conceptions to answer mathematical questions. Therefore I did not seek to determine which a student has but rather to which a student has access. I thought that if I asked questions which could immediately be answered with a computation, a student might naturally only give answers which indicate process conceptions. [Does this belong in the discussion? I think it’s relevant to the analysis here.]

To enable students to provide answers in which they use object conceptions I asked intentionally vague or general questions before giving students concrete matrices and vectors to work with. My thought was that if students cannot make sense of a general question without either bringing themselves to a specific example or some way of computing, they may not have access to an object conception at that time. [Add to discussion: Validity (or lack thereof) of this last thought. How might I better test for object conceptions?]

In the first round of interviews Students B, C, and D at one time or another gave responses which indicated the use of a process conception of $Av$ or $Eh$. They made claims that $Eh$ gives or produces the next heat state [B104, C105, D104]. Student B, when describing $Av$, introduced the notation $b = Av$ and described $b$ as the solution to $A$ times $v$ [B105]. This indicates that he viewed the matrix multiplication as a process to be carried out, and he gave the result of this process a new name. Student C at one point brought up spans and linear combinations and described linear combinations as “the process of using scalars and addition to produce a product between those two things. It’s a linear movement within some given plane to create something else” [C108].

Later in the first interview, when asked about $E v = 0.5v$, Student C said it would be weird to “do it” this way because there is already a scalar which gives us $0.5v$ [C108]. During the post-lab interview Student C said that when $Av = cv$ is true, “A act[s] as a scalar” [C207]. It seems that Student C relies on process conceptions to make sense of equalities like this: If we can produce $0.5v$ through a simple scalar multiplication, why would we want to use a matrix to do so? Similarly, when I asked Student B about solving $Ax = cx$, he was not sure if I wanted him to “solve for $cx$ or $x$ in general” [B113]. I believe here to solve for $cx$ he meant to compute $Ax$ to
solve for the result in the same way he described computing $A\mathbf{v}$ as solving for $b = A\mathbf{v}$ [B105]. When asked about $Ax = cx$, Student D said “$A$ times $x$...serves as a constant scalar for that $x$” [D112].

Students described $c\mathbf{v}$ as a scaling of $\mathbf{v}$ [A103] and focused on the direction and magnitude of $c\mathbf{v}$ compared to that of $\mathbf{v}$ [A106, B107, C109]. It was not always clear what students meant when they talked about “direction” [e.g. A106, B107]. When students talked about $c\mathbf{v}$ not changing direction (from $\mathbf{v}$’s direction), they may have meant that the two vectors are parallel (when $c \neq 0$) [B204] or they may have meant that $c\mathbf{v}$ pointed in the same direction as $\mathbf{v}$ when we interpret $\mathbf{v}$ as a ray pointing out from the origin. It is also possible that students seem to mean the second case when they actually mean the first case but have only considered positive scalars [C109, E206].

I did not discuss vector equality with every student because I entered the interviews erroneously assuming each student conceptualized vector equality the same way: I assumed each student, in the context of this class, would think that two vectors are equal if and only if they are the same size and their corresponding entries are the same. Students leveraged different conceptualizations of vector equality. Students B and C said two vectors are equal if they have the same magnitude and direction [B114, C102], a definition one might see in a physics course. [How does this relate to the geometric/physical interpretations of these processes? I specifically talked with one student about the "direction" of the heat states.]

To the students, what does it mean to solve $Ax = cx$? What does it mean to be a solution to this equation? How do students solve $Ax = cx$ for the first time?

Generally, the notion of “being a solution” to an equation can mean many different things to students. [CITATION - Expand on this in the literature review.] For some students “$x$ is a solution to $Ax = b$” means “when I plug in the vector $x$ into the equation, I get a true statement” [A108]. In this sense being a solution is about the properties of $x$. For other students, “$x$ is a solution to $Ax = b$” means “when I go through the algorithm of row-reducing $[A|b]$ the vector I end up with is $x$” [B112*, B113, B116, C111, C112, D112, E111]. This second interpretation is practical in the sense that students are often asked for solutions to equations without needing to interpret them [CITATION?], ergo when asked about solutions they only need to compute them. [Is there a difference between asking students to interpret the results of computations and to interpret each step of a computation? For example, in Dr. Ely's work he has had students interpret each of $f(x)dx$ and $\int_a^b f(x)dx$.]

When seeing the equation $Ax = cx$ for the first time, students who view solutions as the results of specific computations may not know how to proceed. During the interviews students did not rewrite $Ax = cx$ to the form $(A-cI_n)x = 0$. This might be explained a few ways: Students may not be used to this level of matrix algebra, they may not see the benefit or need to write the equation this way, or they may simply have not thought of it at the time. Perhaps with more examples and more time students could come to this equation on their own. [ADD: There was a point in the lab where students were expected to produce this equation (with instructor help?), but they did not reach that part of the lab while I was there. Check again with Marie if she ever collected their completed labs and if I am allowed to look at them.] However, students did write a system of equations which was equivalent to $(A-cI_n)x = 0$ in the specific example I gave them [B116, C112, D112]. Student D recognized that his method of solution could generalize to larger systems [D113]. While the data suggests students may not yet be able to use properties of matrix
algebra to solve $Ax = cx$, they are capable of identifying the equivalent system $(A-I)x = 0$ and its relevance to solving the first system.

When solving the matrix equation $Ax = b$, students are taught to form the augmented matrix $[A|b]$, row-reduce this matrix, and interpret the row-reduction. This row-reduced form tells the student if there are no solutions, exactly one solution, or infinitely many solutions. In the last case students can use the row-reduced matrix to write parametric equations for the coordinates of the solutions $x$. Given that row-reduction can completely solve this equation, it is natural to try applying it to solve $Ax = cx$. Several students wrote out the matrix $[A|cx]$ to solve the equation $Ax = cx$ during the interviews [B113, C112, E111, E208]. It is possible that the students might just be trying to apply a known tool to a new problem [Look up that “groping” blindly thing], or perhaps the students did not know why the augmented matrix works in the first case. Evidence for this second possibility includes B113 wherein the student asked if he was to “solve for $cx$ or $x$ in general” [Clarify]. Another possibility is that students in the class used MatLab, and one student claimed that the matrix $[A|cx]$ appeared in some form while using MatLab [E111]. [Check again with Marie - How was MatLab used in the course, and did a similar matrix appear?]

What changes in student perceptions or knowledge might we attribute to the lab?

[This is by far my weakest section, but that should not be surprising. The research was not designed well enough to track changes in perception. See how the pre- and post-lab interviews don’t exactly follow the same thread. The only real thing that’s the same is the one question about solving $Ax = cx$. I added that to the second interview when I thought they would have time to explore this equation during the lab, which they did not.]

During the lab students were given Figure B1 from Appendix B. The students were tasked first with describing ways of grouping the graphs by similarities they observed. [Should I describe the various similarities they observed, or only the things directly relevant to this research question?] One goal of the lab is to help students observe that in some cases multiplying by a matrix has the same effect as multiplying by a scalar and that this relationship can be expressed algebraically as $Av = cv$. The lab group consisting of Students A, D, and E realized during the lab that my questioning about $Eh = ch$ was directly related to the lab. [Expand. Rob mentioned something about “cuing students into a fact” does not detract from their learning. This might be better placed in the discussion.]

Student C explicitly attributes to the lab his understanding of $Ax = cx$, saying that his earlier doubts about the existence of solutions to the equation went away because of the lab [C208]. It is possible that the lab provided enough examples for him to interact with that the equation made more sense to him.

[EXPAND. Describe the possible shift in how students solved $Ax = cx$.]

How do the students use the heat state context to make arguments about linear algebra and vice-versa?

Of interest is whether students used this context to change their perception of linear algebra concepts, they used linear algebra to make conclusions about heat diffusion, or otherwise. In my interviews I asked the students how they used linear algebra to study heat diffusion and one student corrected me that they are only using heat diffusion to study linear
Students held diverse beliefs about the ways in which the heat diffusion context is related to vectors, what was considered a vector in this context, and why they are vectors. Student D describes (column) vectors as a way of storing data about heat states [D102]. Meanwhile, Student A described the heat states themselves as a vector space because they satisfy the vector space properties [A101]. Student C says that the rod itself is a vector because it has a heat state and mentions how the corresponding column vectors are formed [C102]. Student C also alludes to “the definition” of vectors without explicitly using it [C102]. Student C said the heat states could be a matrix [C102], possibly indicating he thinks of matrices and (column) vectors as separate objects. (Square v.s. rectangle.) Student C said a vector needs magnitude and direction, but that heat states lack those things. Several students used this physics-flavored conception of vectors.

Discussion

[First and foremost, write about the potential implications of the results for teaching and future research. Among other things, include here a discussion of how things can be improved in future rounds of data collection. See the introduction. Some of my bullet points below might fit better as part of the discussion of the results. Mention how the class did not get through most of the lab.]

The original research questions were:

1. Do students have access to an object conception of $Ax$ and $cx$?
2. How are the students learning about eigenvectors and eigenvalues in the lab?

In hindsight, these questions were not the best choice. The first question is a yes or no question which is not suitable for qualitative research. It restricts us to a shallow analysis of what we observe. Instead we might have asked about the ways in which students use process or object conceptions, or we might use object/ process viewpoints as a means of analyzing data (i.e., as part of the theoretical framework). I explain further in the last data analysis document (from 2020 February 13) what was wrong with my original questions.

The second question of the post-lab interview has a flaw I discovered after returning from Ohio: In the post-lab activity (see Appendix B, Figure B2) I provided pictures of vectors $v$ and $b = Av$. I intentionally named the second vector $b$ to avoid confusing students who did not recognize $Av$ itself as a vector, but this also allowed students to possibly circumvent thinking about the matrix $A$ at all.

Suggested Changes to the Interview Protocols

To collect more data to answer my original questions in the future, I will write interview protocols to address one or more of the following topics:
• Student ways of thinking about the heat diffusion context. The in-class lab has students explore/discover eigenvalues and eigenvectors in the heat diffusion context, so to make sense of their experience with the lab we need to know how they understand that context.
• Student ways of thinking about vector equality. In the equation \( Ax = cx \), when \( x \) is a solution we have two vectors, \( Ax \) and \( cx \), which are equal. What does it mean for two vectors to be equal? Do students think about magnitude and direction? Does their answer depend on the context in which they interpret the vectors?
• Student ways of thinking about the products \( Av \) and \( cv \). Before a student can make sense of the equation \( Ax = cx \), where \( x \) is a vector to be determined, they need to be able to make sense of the products \( Av \) and \( cv \) as well as the equation \( Av = cv \), where \( v \) is a known vector. I want to collect data on their ways of thinking about these things both in the context of heat diffusion and with matrices and vectors in general.
• Student ways of thinking about \( Ax = cx \). This is what I want to study. What do students think it means to solve this system, either computationally or geometrically? What methods do students use to solve this system?
• What prior experience, if any, do the students have with eigenvalues and eigenvectors? I will be careful not to use these terms until the end of the second interview.

**Suggested Changes to the Pre-Lab Interview**

The second interview needs the most work. On one hand it helps answer questions about the effectiveness of the lab. On the other hand, I am not sure how rich the data I collected is. I need to spend more time thinking of how to expand upon what I have here. [Further modify this.]

**Protocol.** [EDIT: I think the original protocol can still work. I would add the following questions: In the heat diffusion context, what does it mean for two vectors to be equal? If you look at two column vectors, \( u \) and \( v \), how do you know if they are equal?]

**Suggested Changes to the Post-Lab Interview**

[EDIT: Additional description of changes.]

**Protocol.** [EDIT: Should I change the activity? How far do we think the students can get through the lab in class this time? How long after the lab will I be able to interview the students? What I do here might depend on how much of the lab they have gotten through and how long they’ve had to digest the lab before I talk to them.]
References

[This will be heavily updated. I know this is pretty weak at the moment.]


Appendix A - Initial Research Plan

[I put this on this page now to save paper. Include here anything from the original version of the research which I could not include before. Maybe remove this appendix. I might want to include the lab itself as an appendix, if able. Ask IMAGEMath.]
Appendix B - Figures

Figure B1

*In-Class Lab Activity*

*Note.* Students are tasked with grouping the diagrams by observed properties.
Figure B2

Post-Lab Activity

Note. Students are tasked with grouping the diagrams by observed properties.