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Consequences of FOIL for undergraduates

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FOIL is a well-known mnemonic that is used to find the product of two binomials. We conduct a large sample \(n = 252\) observational study of first-year college students and show that while the FOIL procedure leads to the accurate expansion of the product of two binomials for most students who apply it, only half of these students exhibit conceptual understanding of the procedure. We generalize this FOIL dichotomy and show that the ability to transfer a mathematical property from one context to a less familiar context is related to both procedural success and attitude towards math.

1. Introduction

There is a world of difference between a student who can summon a mnemonic device to expand a product such as \((a + b)(x + y)\), and a student who can explain where the mnemonic comes from. The student who can explain the rule understands the mathematics, and may have a better chance to succeed at a less familiar task such as expanding \((a + b + c)(x + y)\).[1,p.4]

The mnemonic that is referred to in this section of the common core state standards (CCSS) for mathematics is FOIL. This well-known mnemonic is used to multiply two binomials by multiplying the first terms, the outer terms, the inner terms, and then the last terms (see Figure 1).

Internet searches quickly turn up people who love FOIL and people who hate FOIL. The most common criticism of FOIL is that it masks the use of the distributive property; students can apply the FOIL procedure and not understand why it works. Procedures that lack understanding are criticized in the CCSS for mathematics:

Students who lack understanding of a topic may rely on procedures too heavily... In short, a lack of understanding effectively prevents a student from engaging in the mathematical practices.[1,p.8]

A primary goal of our research is to analyse the effects of FOIL on first-year college students. This population is interesting because most of them first learned algebra within the previous decade. Significant time has passed since first exposure, yet not so much time has passed that all has been forgotten.

The intent of this paper is not to participate in the historical debate on conceptual understanding versus procedural skill, which has weighed heavily to each side throughout

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history.[2,3] We would be remiss, however, to not acknowledge our concurrence with mathematics educators like Wu, [4] Schmittau,[3] and Kieran [2], who refer to the dichotomy between conceptual understanding and procedural skill as being false. Wu concludes his article *Basic Skills Versus Conceptual Understanding: A Bogus Dichotomy in Mathematics Education* with the remark, ‘Let us teach our children mathematics the honest way by teaching both skills and understanding’. [4,p.19] Schmittau, a proponent of a mathematics curriculum created by V.V. Davydov and his colleagues which is grounded in Vygotskian cultural-historic theory, summarizes the methods used in Davydov’s curriculum to teach multi-digit multiplication and long division with the statement: ‘Thus, the procedural is rendered conceptual and the conceptual becomes procedural; it is impossible to say where one ends and the other begins’. [3,p.31] Currently, the writers of the CCSS for mathematics indicate, ‘The Standards for Mathematical Content are a balanced combination of procedure and understanding’. [1,p.8] We will revisit the CCSS for mathematics in Section 10 and suggest that the success of this balance is not as transparent in the Algebra standards as it is in the pre-Algebra standards. This general phenomenon is described by Kieran:

> Although recent reform efforts have partially shifted the focus in algebra, at least during the earlier years of high school, from procedural work to real-world problem solving and multiple representations for these problems, the issue remains, especially during the later years of high school. When students are eventually faced with the literal-symbolic, transformational activity of algebra, the cleavage between the procedural and the conceptual reappears; the teaching of algebraic procedures is approached by and large as a skills-based endeavor, where the conceptual is generally absent. This way of thinking about algebra must change. [2,p.169]

Unquestionably, FOIL is a procedure, and very few educators question whether or not FOIL leads to an accurate expansion of the product of two binomials. We question, however, whether or not first-year college students attach any conceptual understanding to the mnemonic, and whether or not the lack or presence of conceptual understanding has further implications.

Maintaining our stand on the false dichotomy between procedural skill and conceptual understanding, we ask whether or not the FOIL procedure allows for students to develop both procedural skill and conceptual understanding. Based on information gathered from our large data set, we determine where, in Figure 2, FOIL is located.

In this paper, we test whether or not FOIL yields procedural skill by determining whether or not students can correctly expand the product of two binomials. We test conceptual understanding by determining whether or not FOIL is a transferable skill. Seeley, NCTM’s 2004–2006 President, has given several talks in which she states that all the mathematics students need ‘deep transferable skills for versatilizing’, a phrase she takes from Friedman,
who writes about the need for workers to be able to carry skills learned in one context or job to a new context or job.\cite{5} We call a mathematical skill transferable if it can be learned in one mathematical context and applied to a new, less familiar mathematical context. If a mathematical skill is transferable, then conceptual understanding is exhibited. We test whether or not students understand FOIL conceptually by determining whether students transfer their knowledge of FOIL to the product of a binomial and a trinomial.

In her doctoral dissertation titled *College Students’ Concepts of Multiplication*,\cite{6} Schmittau collected data both from written surveys (subjects were asked, in part, to multiply a monomial and binomial, to multiply two binomials, and to square a trinomial) and from flexible clinical interviews with each subject. Her sample consisted of 10 Cornell University students who were drawn from a pool of volunteers, where the basis for selection was to maximize diversity across major fields and mathematics backgrounds. Schmittau made several summarizing statements about FOIL, including the following.

The ‘FOIL’ method for multiplying binomials seemed to be a major factor in obscuring a subject’s own meaning. (p.102)

Finally, the pitfalls resulting from the ubiquitous use of the ‘FOIL’ method cannot be ignored. The ‘FOIL’ method contributes nothing to understanding; its favor in classrooms seems to reside in the fact that it is easily remembered. (p.107)

Again, the testimony of subjects points to the questionable value of the FOIL method, considering the conceptual confusion which it introduces for them. (p.137)

We add to Schmittau’s discussion by analysing data obtained from a large data set ($n = 252$). In Sections 2 and 3, we describe our data collection process and discuss the frequently occurring data. In Section 4, we show why FOIL is popular but not adequate. In Section 5, we show that while FOIL is both procedurally and conceptually understood by a subset of subjects, it is procedurally but not conceptually understood by an equally large subset of subjects. We call this ‘The FOIL Dichotomy’ in Section 5, which we generalize in Section 6 as we compare the group of subjects that exhibits an ability to transfer mathematical concepts to the group of subjects that is unable to transfer mathematical concepts. We show that the former group outperforms the latter group, giving credence to our strategy of testing conceptual understanding by determining whether or not FOIL is a transferable skill. Sections 7 and 8 explore some unanticipated connections between a subject’s attitude towards math and his procedural skill and conceptual understanding of

---

**Figure 2.** A $2 \times 2$ grid comparing procedural skill and conceptual understanding.

<table>
<thead>
<tr>
<th>Procedural Skill</th>
<th>Yes</th>
<th>No</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conceptual Understanding</td>
<td>Yes</td>
<td>GOAL</td>
</tr>
<tr>
<td>No</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

---
polynomial multiplication. Moreover, we find that the group of subjects that is willing to perform polynomial multiplication but is not willing and/or able to explain the reasoning behind their computations is not homogenous with respect to mathematical ability; half of this group produced correct calculations and the other half did not. They are homogenous, however, with respect to their attitude towards math; they do not like it. In Sections 9 and 10, we discuss the box method referred to by a small number of subjects in our sample, and we also discuss its connection to the area model, a model that can expose the progression of the multiplication of real numbers throughout the mathematics curriculum. In light of our general findings, we revisit FOIL in Section 11 and conclude, like Schmittau, that FOIL does not belong in the algebra classroom.

2. Data collection

Our research is based on data collected from an observational study at the University of Maine at Farmington (UMF). UMF is Maine’s public liberal arts college, with an enrollment cap of 2000 full-time students. Each semester at UMF, almost all first-year students take a first-year seminar, FYS 100, or a first-year English class, ENG 100. In Spring 2013, there were a total of 24 sections of FYS 100 and ENG 100, and we attended 23 of these sections to administer our survey. Of the 265 students, who were present in class to complete the survey, 252 of them consented to participation in our research. Of these 252 subjects, 96% were first-year students. The other 4% were second- and third-year students who were in these classes for various reasons.

The survey was administered in two parts. First, the subjects were asked to complete Figure 3.

Second, the subjects were asked to turn the paper over and answer the following questions. They were not allowed to turn the paper back over to the first side.

- What year are you? (First Year, Second Year, Third Year, Fourth Year)
- What is your major?
- What is the highest level of Math you took in high school?
- What is the highest level of Math you have taken at UMF (include courses you are currently in)?
- Did you learn the ‘FOIL’ acronym in previous courses?

<table>
<thead>
<tr>
<th>Write the product as a sum.</th>
<th>What mathematical method, property or idea did you use to solve this problem?</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2(x + 6)$</td>
<td></td>
</tr>
<tr>
<td>$(x + 3)(x + 5)$</td>
<td></td>
</tr>
<tr>
<td>$(x^2 + 2x + 4)(x + 3)$</td>
<td></td>
</tr>
</tbody>
</table>

Figure 3. Part 1 of our survey.
On a scale of 1 to 5, how would you rate your relationship to math (1 = hate it, and wish I never had to do it again; 5 = love it, and I have fun doing it)?

Figure 4 displays the results from the first page of the survey.

We refer to $2(x + 6), (x + 3)(x + 5)$, and $(x^2 + 2x + 4)(x + 3)$ as Q1, Q2, and Q3, respectively. In parentheses, we give the per cent of subjects who gave each answer. Some of the data from the second page of the survey is summarized in Figure 5.

Throughout this paper, we refer to an answer sequence and a property sequence. The subject whose survey appears in Figure 6(a) has an answer sequence of YYN because he got questions one and two correct and question three incorrect. Consulting Figure 4, we see that his property sequence is DFQ because his properties for Q1, Q2, and Q3 are ‘distributive,’ ‘FOIL,’ and ‘no answer,’ respectively. Similarly, the subject whose survey is shown in Figure 6(b) has an answer sequence of YYN and a property sequence of FFQ.

We note that an answer sequence gives information about whether or not the multiplication in Q1, Q2, and Q3 were done correctly or not. The answer sequence YYN, for example, could mean that the polynomial multiplication in Q3 was not performed, that it
was performed correctly except for a minor error (saying that $4 \times 3 = 10$, for example), or that a major error (like not distributing correctly) occurred. These three different reasons that cause Q3 to take on the value of $N$ are discernable by looking at the value of the variable Q3 error. The possible values of this variable are listed in Figure 4. Regarding the three situations described above, Q3 error would equal E, O, and D, respectively. Because the purpose of this paper is to analyse a large data set, we focus very little on oddities like $4 \times 3 = 10$. This information, nonetheless, has been maintained.

3. Common data

As shown in Figure 7, both in tabular form and in a Venn diagram, there are four answer sequences that are most common: YYY, YYN, YNN, and NNN. That is, 93.6% of our subjects either expanded all three problems correctly (YYY), expanded Q1 and Q2 correctly but not Q3 (YYN), expanded only Q1 correctly (YNN), or expanded none of the products correctly (NNN). That the other four possible answer sequences are not common is to be expected. The answer sequence YNY, for example, indicates that the subject correctly multiplied a constant and binomial and also correctly multiplied a binomial and trinomial,
but incorrectly expanded the product of two binomials. The 16 responses that fall outside of the four common answer sequences are the oddities referred to in Section 2. We do not claim that these oddities are all ‘silly mistakes’ that are not conceptual; of the 10 NYY, NYN, and NNY subjects, for example, six of them wrote that \(2(x + 6) = 2x + 6\). We only claim that these results are not typical, and are not part of this analysis.

Figure 8 shows the distribution of property sequences for each of the four common answer sequences. Note that the other answer sequence category consists of answer sequences that occurred at most six times. Similarly, the other property sequence category consists of property sequences (like DAA, DQD, FFF, and AFQ) that occurred at most six times.

As stated in Figure 4, if, for the property, a subject gave either no answer or a really general answer like ‘multiplication’, we coded that property as Q. Thus, the property sequence QQQ yields no information about subject’s understanding of properties. There are six most prominent property sequences (DDD, DQQ, DFF, DFD, QFQ, and DFQ) that do give information about properties. The property ‘D’ is ‘distributive property’ and ‘F’ is ‘FOIL’. Notice that with the exception of five subjects, the subjects with one of these six property sequences got at least one of Q1, Q2, or Q3 correct, and most of the data is concentrated in the YYY, YYN, and YNN answer sequences.

Figure 9 includes the data for the most common property and answer sequences. The bar graph shows how the property sequences are distributed within the answer sequences. It is very interesting to observe the property sequence modes for these answer sequences (see Figure 10). The mode of YYY is DDD (DFF is also common), the mode of YYN is DFQ (QFQ is also common), and the mode of YNN is DQQ. Notice that all Ds and Fs in a mode property sequence correspond to Ys in its answer sequence, and the Qs in the mode

<table>
<thead>
<tr>
<th>Answer Sequence</th>
<th>Frequency (n = 252)</th>
<th>Relative Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>YYY</td>
<td>94</td>
<td>37.3%</td>
</tr>
<tr>
<td>YYN</td>
<td>61</td>
<td>24.2%</td>
</tr>
<tr>
<td>YNY</td>
<td>6</td>
<td>2.4%</td>
</tr>
<tr>
<td>YNN</td>
<td>62</td>
<td>24.6%</td>
</tr>
<tr>
<td>NYY</td>
<td>4</td>
<td>1.6%</td>
</tr>
<tr>
<td>NYN</td>
<td>5</td>
<td>2%</td>
</tr>
<tr>
<td>NNY</td>
<td>1</td>
<td>0.4%</td>
</tr>
<tr>
<td>NNN</td>
<td>19</td>
<td>7.5%</td>
</tr>
</tbody>
</table>

Figure 7. A tabular and Venn diagram illustration of the four most prevalent answer sequences.

<table>
<thead>
<tr>
<th></th>
<th>YYY</th>
<th>YYN</th>
<th>YNN</th>
<th>NNN</th>
<th>OTHER</th>
<th>TOTAL</th>
</tr>
</thead>
<tbody>
<tr>
<td>DDD</td>
<td>20</td>
<td>2</td>
<td>5</td>
<td>1</td>
<td>3</td>
<td>31</td>
</tr>
<tr>
<td>DQQ</td>
<td>2</td>
<td>3</td>
<td>16</td>
<td>3</td>
<td>1</td>
<td>25</td>
</tr>
<tr>
<td>DFF</td>
<td>17</td>
<td>5</td>
<td>5</td>
<td>2</td>
<td>2</td>
<td>29</td>
</tr>
<tr>
<td>QQQ</td>
<td>20</td>
<td>11</td>
<td>22</td>
<td>11</td>
<td>3</td>
<td>67</td>
</tr>
<tr>
<td>DFD</td>
<td>9</td>
<td>2</td>
<td>2</td>
<td></td>
<td></td>
<td>13</td>
</tr>
<tr>
<td>QFQ</td>
<td>5</td>
<td>10</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>19</td>
</tr>
<tr>
<td>DFQ</td>
<td>6</td>
<td>11</td>
<td>2</td>
<td>5</td>
<td>2</td>
<td>24</td>
</tr>
<tr>
<td>OTHER</td>
<td>15</td>
<td>17</td>
<td>3</td>
<td>1</td>
<td></td>
<td>44</td>
</tr>
<tr>
<td>TOTAL</td>
<td>94</td>
<td>61</td>
<td>62</td>
<td>19</td>
<td>16</td>
<td>252</td>
</tr>
</tbody>
</table>

Figure 8. The joint frequency distribution for the answer and property sequence variables.
4. Is FOIL successful?

On Q2, the question that directly assesses the procedural success of FOIL, 105 subjects (almost 42% of our sample) listed FOIL as the mathematical property. Of these 105 subjects, more than 83% correctly multiplied the two binomials. This gives credence to why 85% of the subjects in our study said they have been taught FOIL; procedural skill is attained for the particular type of problem for which FOIL was designed.

There were 47 subjects (almost 19% of our sample) who said FOIL was their property on Q2 but who gave no relevant method, property, or idea for Q3. These students have an answer sequence of \*FQ, where \* can take on any value of P1 in Figure 4. Figure 11 shows how many of these subjects got Q2 and Q3 correct. We find that such students are more likely to get Q2 right than wrong (\(p\)-value < .0001). However, they are also more likely to
get Q3 wrong than right ($p$-value = .014). For this population, FOIL is not a transferable skill.

We note that of the subjects with *FQ as their property sequence, 26% (12/47) of them correctly expanded all three products on the survey; but this value increases to 40% (82/205) when all other subjects (even those who listed no mathematical property) are pooled. The *FQ subjects are unable to transfer FOIL from Q2 to the less familiar problem in Q3. It appears that to them, Q1, Q2, and Q3 are three unrelated problems. The breakdown of the *FQ group is 24 DFQ, 19 QFQ, 3 AFQ, and 1 FFQ. Schmittau [6] makes reference to the group of subjects that we call the DFQ group, and comments that FOIL is not transferable:

Half the subjects reported using different methods in each instance of polynomial multiplication. Generally, they used the distributive property (though most were not able to name it) for the product of a monomial and a binomial, and then used the FOIL method to obtain the product of two binomials. When confronted with the trinomial task, subjects either acknowledged that they could not extend the FOIL method to trinomials, and therefore, did not know what to do, or they squared each term of the trinomial, or else they actually squared the trinomial correctly, but had very little confidence in their answers. (p.136)

5. The FOIL dichotomy

Figure 12 tracks the number of FOIL occurrences (the number of times a subject listed FOIL as the mathematical method, property, or idea on the survey), distributed by answer sequence. We see that of the subjects who listed FOIL exactly once, only 35% correctly expanded all three products. This value increases to 51%, however, for the subjects who listed FOIL as a property on at least two of the problems. In addition, among the subjects who listed FOIL exactly once, 43% had YYN as their property sequence, and this value decreases to 28% among the subjects who listed FOIL at least twice.

Referring to FOIL more than once suggests the presence of a conceptual understanding of FOIL that is absent if FOIL is listed only once. Reflecting back on the six most common property sequences, DDD, DQO, DFF, DFD, QFQ, and DFQ, four of these include reference to FOIL. The property sequences DFF and DFD, however, exhibit different conceptual understanding than the property sequences QFQ and DFQ. The first pair, DFF and DFD, involves FOIL but suggests an understanding that the underlying process of FOIL is transferable to problems other than the product of two binomials (DFF) or that ‘FOIL’ is a special case of the distributive property (DFD). The QFQ and DFQ property sequences, however, show an inability to transfer FOIL; FOIL is restricted to the case of multiplying two binomials.

We perform a hypothesis test for the difference of two population proportions. The first population consists of students who would produce DFQ or QFQ as their property sequence. The second population consists of students who would produce DFF or DFD. The data in Figure 13 show that of our DFF/DFD subjects, nearly 62% (26/42) answered all three questions correctly, but of the DFQ/QFQ subjects, less than 26% (11/43) answered
all three questions correctly. A student is more likely to get all three questions correct if he has DFF/DFD understanding instead of DFQ/QFQ understanding (p-value = .00037).

We now revisit the discussion in the introduction about whether FOIL leads to procedural skill and/or conceptual understanding (see Figure 14). We have found that there are two very different groups of subjects who referenced FOIL: the DFD/DFF subjects and the DFQ/QFQ subjects. These two groups are approximately equal in size (42 subjects vs. 43 subjects). FOIL leads to the procedural skill for the product of two binomials for both groups. Of the DFD/DFF subjects, 77% multiplied correctly in Q2; and of the DFQ/QFQ subjects, 74% multiplied correctly in Q2. However, a majority of the DFD/DFF subjects were able to transfer FOIL to a less familiar problem, thus exhibiting conceptual understanding, whereas three-fourths of the DFQ/QFQ subjects were unable to transfer the procedural skill.

6. The more general dichotomy

In the previous section, we exposed a difference in conceptual understanding between subjects with a DFD/DFF property sequence, and those with a DFQ/QFQ property sequence. We broaden this dichotomy now, and divide the six common (non-QQQ) property sequences (DDD, DFF, DFD, DFQ, QFQ, and DQQ) into two groups: the property sequences that contain a repeated (non-Q) property and the property sequences where all the (non-Q) properties are distinct. This combines DDD, DFF, and DFD into one group and...
DFQ, QFQ, and DQQ into another group. We see in Figure 15 that of the YYY answer sequences with these six property sequences, 78% have property sequence DDD/DFF/DFD, whereas among the subjects who did not get all three questions correct, only 33% had a DDD/DFF/DFD property sequence.

It is evident that there is a difference in conceptual understanding between the DDD/DFF/DFD and the DFQ/QFQ/DQQ groups. Thus, we now generalize the notion of repeated properties even more by analysing all non-QQQ property sequences, not just the six most popular ones. We create a variable called MaxSame, which is defined as the most repetitions of a recurring (non-Q) property. For example, MaxSame equals 3 for DDD, 2 for FFQ, and 1 for DFQ (see Figure 16).

The table in Figure 17 is more general than the one in Figure 15. Note that DDD/DFF/DFD property sequences are located in either the ‘MaxSame = 2’ or the ‘MaxSame = 3’ category and DFQ/QFQ/DQQ property sequences are in the ‘MaxSame = 1’ category. We see that 3 is the mode number of correctly multiplied problems among subjects with MaxSame equalling 2 or 3, but 2 is the mode number of correctly multiplied problems among subjects with MaxSame equalling 1. However, again the relationship between procedural skill (as measured by correct polynomial multiplication) and conceptual understanding (as measured by transferring mathematical properties from one problem to a related one) is exposed.

The results are a bit less dramatic in Figure 17 than when looking at only the six most common property sequences in Figure 15, but there is still a difference. For the YYY answer sequence, 74% had a property sequence with a repeated property, whereas among the subjects who did not get all three questions correct, only 43% had a property sequence with a repeated property. When comparing the population of students with ‘MaxSame > 1’ understanding to those with ‘MaxSame = 1’ understanding, the proportion of YYY answer sequences is significantly greater for the former group (p-value < .0001). Being able to transfer a mathematical method, property, or idea from one problem to another relates to improved performance.

Figure 18 gives the per cent of correct responses to each question for the ‘MaxSame > 1’ and ‘MaxSame = 1’ groups. Not only do the percent of correct responses drop as the questions get less familiar, but the gap between the two groups gets wider. We can practically see subjects falling within the cracks as the material gets harder, hindered by an inability to transfer mathematical properties from one problem to a less familiar problem.
7. Connections to math attitude

We asked the subjects in our study ‘On a scale of one to five, how would you rate your relationship to math (1 = hate it, and wish you never had to do it again; 5 = love it, and have fun doing it)?’ The comparative boxplots in Figure 19 compare the results of the LikeMath variable among the groups whose MaxSame variable equals 1, 2, and 3. We see that of the subjects who refer to the same property for all three problems, roughly three-fourths have a LikeMath variable equalling 3 or above; of the subjects who gave the same property for two (but not three) of the problems, roughly half have a LikeMath variable equalling 3 or above; and of the subjects who saw no property connections among the problems, roughly only one-quarter have a LikeMath variable equalling 3 or above. (The actual values are 80%, 68%, and 43%) There is a relationship between subjects being able to transfer mathematical properties and their positive relationship with math; the more transference, the more positive the mathematical relationship.

8. Learning from the ‘no properties’ group

We have omitted data from the subjects with property sequence QQQ for the analysis, thus far in this paper. They did not give us information about their understanding of mathematical properties. Some of these subjects might dislike providing mathematical justification for their work. Other subjects might not have known what to call the mathematical property or
might not have known that a relevant mathematical property exists. We will see, however, that we can learn from the QQQ property-sequence group.

Figure 20 shows the answer sequence distribution for the QQQ group. The bar graph in Figure 21 organizes this information according to the variable NumCorrect, which equals the number of problems (among Q1, Q2, and Q3) that were multiplied correctly. We see that the subjects who have QQQ as their property sequence are not consistently strong or weak in polynomial multiplication. In fact, 50.7% got zero or one questions correct, and 49.3% got two or three questions correct. So the ability/willingness to provide mathematical ideas, properties, or methods does not relate to mathematical ability, defined by the number of survey problems done correctly; they are all over the spectrum of possible answers. However, the dot plot in Figure 21 shows that the ability/willingness to provide mathematical ideas, properties, or methods does relate to attitude towards math. The ‘non-verbal’ QQQ group is not conflicted with respect to liking math or not. The distribution is skewed right, with 76% (51/67) of these subjects giving a rating of 3 or less.

In the comparative boxplots in Figure 22, the QQQ group (labelled as MaxSame = 0) is similar to the MaxSame = 1 group, regarding both the number of problems done correctly and attitude towards math. These subjects are outscored, both in terms of the number of problems done correctly and in attitude towards math, by the subjects who exhibited the ability to transfer a mathematical property from familiar to less familiar polynomial multiplication.
Figure 19. Comparative boxplots comparing attitude towards mathematics and the ability to transfer mathematical properties.

<table>
<thead>
<tr>
<th>Answer Sequence for QQQ group</th>
<th>YYY</th>
<th>YYN</th>
<th>YNY</th>
<th>YNN</th>
<th>NYY</th>
<th>NYN</th>
<th>NNY</th>
<th>NNN</th>
</tr>
</thead>
<tbody>
<tr>
<td>frequency n = 67</td>
<td>20</td>
<td>11</td>
<td>22</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>11</td>
<td></td>
</tr>
</tbody>
</table>

Figure 20. Answer sequence distribution for the QQQ group.

Figure 21. Graphs describing the variables NumCorrect (bar graph) and LikeMath (dot plot) for the QQQ subjects.

Figure 22. Comparing the QQQ group (MaxSame = 0) to the other subjects in terms of number of correct survey responses and attitude towards math.
9. The box method

We will suggest in the next session that the use of area models in mathematics helps unify the progression of the multiplication of real numbers from pre-Algebra to Calculus. In 2004, Schmittau reacts positively to its incorporation into the US curriculum. ‘The distributive property constitutes the underlying conceptual structure in the algorithms for multiplication of multi-digit whole numbers and polynomials. Some U.S. textbooks are now incorporating rectangular models that illustrate this fact, a commendable improvement over pre-reform methods’. [3,p.29]

No subject in our study referenced an area model. However, of the 252 subjects in this research project, 8 subjects mentioned ‘box’ or drew a rectangular figure at least once when answering, ‘What mathematical method, property or idea did you use to solve this problem?’ Of these eight subjects, all had an answer sequence of YYY (two subjects) or YYN (six subjects). An in depth analysis of the work of these subjects illuminates what they understand about the ‘box’.

Of the six YYN subjects, four of them produced a picture and two of them did not. The two subjects who did not produce a picture got Q3 wrong because they made a distribution error. The four subjects who did produce a picture got Q3 wrong, but not because of a distribution error. Figure 23 is the picture produced by one of these subjects. This subject left his answer as $x^2, 5x^2, 10x, 12$; he did not add the terms. Two of these subjects incorrectly calculated the area of one of the smaller rectangles: one wrote $5x$ inside the rectangle with dimensions $2x$ and $3$, and the other wrote $3x$ inside the rectangle with dimensions $2x$ and $x$. The fourth of these subjects wrote $3x$ inside the rectangle with dimensions $3$ and $x^2$, but it is reasonably clear that this subject’s error was due to messy handwriting. The three non-handwriting errors here suggest that the ‘box’ may lack a connection to the area concept, where the areas of four smaller rectangles are added to obtain the area of the larger rectangle. In fact, the mere reference to a ‘box’ method causes great concern since boxes are three-dimensional objects, yet the pictures these subjects drew are two dimensional. Teaching a ‘box method’ that does not focus on area is simply a procedure with no conceptual understanding.

Even the two YYY subjects demonstrated a lack of conceptual understanding of the model. One commented ‘I used a chart to help me multiply.’ The other subject wrote ‘box’ for Q3 and drew a subdivided rectangle, but for the properties to the first two questions he wrote ‘couldn’t tell yah’ and ‘don’t remember.’ Though visual, a subdivided rectangle that is conceptually void is no better than FOIL.
10. The area model

‘It is significant that the subject who gave evidence of the highest degree of conceptual integration across the category expressed multiplication not only in terms of cardinality but of measurement (rectangular area) as well’. [6,p.62]

Area model references abound in the mathematics education literature, yet in practice the model is not emphasized and used consistently. This is illustrated in the CCSS for Mathematics. The table in Figure 24 lists topics in the elementary, middle, and high school curriculum that pertain to the multiplication of real numbers. It gives the grade level, where the topic appears in the CCSS, and whether or not the CCSS connects modelling with the topic.

Kieran states, ‘Algebra has traditionally been viewed as a domain of school mathematics that is dominated by procedures of symbol manipulation and where the presence of the conceptual has been considered all but an oxymoron’. [2,p.154] We see evidence of this by the lack of modelling that appears in the CCSS relevant to algebra. The extension of models, such as the area model, to algebraic topics like polynomial multiplication, factoring, completing the square, and the quadratic formula is natural;[7] it is a shame for an algebra curriculum to not explicitly make the connection to modelling in the pre-Algebra curriculum:

The distributive property is, after all, a fairly simple and transparent property of the real numbers under addition and multiplication. Throughout considerable experience in mathematics education, the author cannot recall a single instance in which a student failed to understand it. And yet, subjects failed almost totally to grasp its essential role in the extension of multiplication to the abstraction of the algebra of the real numbers. Even when subjects employed it correctly, they were often plagued by doubts about the accuracy of their methods’. [6,p.137]

In order to add conceptual understanding to algebraic procedures, we hope for a modified CCSS that emphasizes modelling in the Algebra standards that expands on the modelling in the pre-Algebra standards.

<table>
<thead>
<tr>
<th>Topic</th>
<th>Grade Taught</th>
<th>CCSS reference to modeling?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Multiplying Whole Numbers</td>
<td>3</td>
<td>Yes</td>
</tr>
<tr>
<td>Standard Multiplication Algorithm</td>
<td>4</td>
<td>Yes</td>
</tr>
<tr>
<td>Multiplying Fractions</td>
<td>5</td>
<td>Yes</td>
</tr>
<tr>
<td>Multiplying Decimals</td>
<td>5</td>
<td>Yes</td>
</tr>
<tr>
<td>Distributive Property</td>
<td>3, 6</td>
<td>Yes, with whole numbers; No, with variables</td>
</tr>
<tr>
<td>Multiplying Polynomials</td>
<td>High School</td>
<td>No</td>
</tr>
<tr>
<td>Factoring</td>
<td>High School</td>
<td>No</td>
</tr>
<tr>
<td>Completing the Square</td>
<td>High School</td>
<td>No</td>
</tr>
<tr>
<td>Quadratic Formula</td>
<td>High School</td>
<td>No</td>
</tr>
</tbody>
</table>

Figure 24. References to modelling in the CCSS for multiplicative topics.
The area model can be used to teach all the topics in Figure 24. Moreover, it is important to note that the use of the area model in the progression of the multiplication of real numbers does not cease with Algebra. Students can be taught the product rule in Calculus using the area model,[8, Section 3.3] and this approach demystifies the addition and subtraction of a mystical term.[13, Section 3.2] that leaves students with the fear, ‘I could never have thought of that’. To dwell on the usefulness of the area model even more, students who study multi-dimensional Calculus learn that there are two types of vector multiplication: a dot product (or scalar product) and a cross product (or vector product). A very common way of introducing these products is to give definitions like: the dot product of two vectors is the sum of the products of the corresponding entries in the vectors. That is

\[ \langle 3, 1 \rangle \cdot \langle 1, 2 \rangle = 3 \cdot 1 + 1 \cdot 2 = 5. \]

This definition is procedural and does not lead to a conceptual understanding of vector multiplication.[9, Section 11.3] Given the connection between the multiplication of real numbers and area, we try to connect the multiplication of vectors to area. The vectors \( \langle 3, 1 \rangle \) and \( \langle 1, 2 \rangle \) are shown in Figure 25. Let \( \theta \) be the angle between these vectors. If we are thinking about area, it is natural to connect the product of these vectors with the area of the parallelogram that is created by these two vectors. Denote the length of the vector \( \langle x, y \rangle \) as \( |\langle x, y \rangle| \). The area of the parallelogram is \( |\langle 3, 1 \rangle| \cdot |\langle 1, 2 \rangle| \cdot \sin \theta \), which is in fact the length of the cross product vector. It should not seem unreasonable, therefore, to consider the dot product definition \( \langle 3, 1 \rangle \cdot \langle 1, 2 \rangle = |\langle 3, 1 \rangle| \cdot |\langle 1, 2 \rangle| \cdot \cos \theta \), which is equivalent to the algebraic definition above. See 10 for a related discussion.

Students who only learn multiplication procedures like ‘FOIL’ and ‘to multiply fractions, multiply the tops and then multiply the bottoms’ and ‘to multiply vectors, multiply the components and then add’ are left to conclude that multiplication is a collection of disconnected rules depending on what is being multiplied. If we teach our students to ‘think area model’ when they see multiplication, then multiplication becomes a unified concept.
11. Should FOIL exist in the algebra classroom?

We learn from this research that a student who can transfer mathematical properties to a less familiar setting achieves a higher level of procedural skill and has a more positive attitude towards math. Should we teach FOIL? We have shown that FOIL is not consistently transferable. Some educators think that teaching FOIL is fine as long as we teach students why it works. The permission to teach FOIL in this situation, however, assumes that students will retain the conceptual understanding. This assumption is dangerous:

Many students are already in the habit of ‘cognitively dumping’ information that they feel is no longer of importance. Once a chapter is covered, the test taken, students dump what, in their minds, is unnecessary. This usually occurs because they fail to see the connections. Students instead view each topic in mathematics as an entirely new concept with its own properties and theorems that must be memorized. Since there is not enough ‘space’ to retain everything when viewed as an individual element, students have to pick and choose what to keep and what to dump. Mnemonics are an easy way to recall information and they take up very little space, thus the mnemonic is remembered while the mathematics behind the mnemonic is not.[11,p.25]

There is only one way to avoid the possibility of students abandoning the conceptual understanding of the distributive property in favour of a procedural understanding of the FOIL mnemonic, and that is to not teach FOIL. FOIL does not belong in the Algebra classroom.

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References