Value-thinking and location-thinking: Two ways students visualize points and think about graphs

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ABSTRACT

The purpose of this study is to examine the characteristics of students’ thinking about aspects of graphs in the context of evaluating statements about real-valued functions from Calculus. We conducted clinical interviews in which undergraduate students evaluated mathematical statements using graphs to explain their reasoning. From our data analysis, we found two ways students think about graphs, value-thinking and location-thinking. These two ways of thinking were rooted in students’ attention to different attributes of points on graphs we provided: either the values represented by the points or the locations of the points in space. In this paper, we report our classification of students’ thinking about aspects of graphs in terms of value-thinking and location-thinking. Our findings indicate that students’ thinking about aspects of graphs accounts for key differences in their understandings of mathematical statements. We discuss some implications of our findings for instruction and curriculum development in Calculus and beyond.

1. Introduction

Undergraduate Calculus courses, from elementary through advanced Calculus, are comprised of many definitions and theorems about real-valued functions. Often, these statements are accompanied by visual representations in the form of graphs of relevant functions. For example, the Intermediate Value Theorem (IVT) is one such statement commonly associated with a visual representation. The IVT can be stated as follows: “Suppose that \( f \) is a continuous function on \([a, b]\) with \( f(a) \neq f(b) \). Then, for all real numbers \( N \) between \( f(a) \) and \( f(b) \), there exists a real number \( c \) in \((a, b)\) such that \( f(c) = N \).” This theorem is often shown in Calculus textbooks with graphs to accompany the statement (e.g., Briggs, Cochran, & Gillett, 2011; Finney, Thomas, Demana, & Waits, 1994; Larson, Hostetler, & Edwards, 1994; Stewart, 2012). For instance, a graph such as the one in Fig. 1 (left) may be used to illustrate the IVT in the case of a monotone function. Additionally, a graph like Fig. 1 (right) may be shown to demonstrate that for a given \( N \)-value, the corresponding value of \( c \) need not be unique.

Indeed, Calculus instructors, when surveyed, reported that graphical interpretations of central ideas of Calculus comprised a significant portion of their assignments and exams (Burn & Mesa, 2015). Although research has called for the inclusion of such visual representations in undergraduate mathematics instruction (e.g., Davis, 1993; Dreyfus, 1991; Guzman, 2002; Hanna & Sidoli, 2007), most empirical studies on students’ understanding and use of graphs have focused on the population of secondary students (e.g., Bell & Janvier, 1981; Duval, 1999; Goldenberg, 1988; Knuth, 2000; Magidson, 1989; Padilla, McKenzie, & Shaw, 1986; Presmeg, 1986). The few studies that have looked at undergraduate students’ understanding of graphs have focused on their understanding of graphs...
as a whole (e.g., Frank, 2017; Moore & Thompson, 2015). However, it is not widely known what meaning undergraduate students have for various aspects of graphs, such as the input, output, and points on graphs. In other words, undergraduate students’ reasoning about graphs in visual representations, and the effectiveness of these illustrations, is not well-documented in the literature.

In undergraduate mathematics, some students may construe other properties of the given graph rather than the intended ones. For instance, there are several possible ways in which students may understand the graphs shown in Fig. 1. A student may presume that the IVT statement refers to a single value of \( N \), as only one value is depicted in each graph. A student could also presume that values of \( N \) may be selected from the entire range of the function, rather than restricting the values of \( N \) to be between \( f(a) \) and \( f(b) \). This presumption may stem from characteristics of the provided graphs, namely that the minimum and maximum of the functions in Fig. 1 are found at the end points of the domain \([a, b]\). Students’ interpretations of mathematical statements, based on the information they perceived from the provided graphs, might even hinder their subsequent mathematical activities, such as rigorous proofs (Alcock & Simpson, 2004).

The purpose of this study is to characterize students’ thinking about aspects of graphs of real-valued functions and to investigate its role in evaluating statements from Calculus. In particular, this study uses the Intermediate Value Theorem (IVT) and similar statements as a Calculus context. Through analyzing students’ evaluations and interpretations of these statements using graphs, we seek to address the following research questions:

1. How do students interpret outputs of a function on a graph, points on a graph, and a graph as a whole?
2. How are students’ thinking about graphs of real-valued functions related to their evaluations of the Intermediate Value Theorem and similar statements?

2. Review of empirical and theoretical studies

Before reviewing empirical studies related to our study, we first describe our perspective on students’ visualization of graphs, which has been a central topic of research in mathematics education. Arcavi (2003) argues that visualization has a natural role in mathematics, and gives several examples of visual representations such as graphs or diagrams. We adopt components of Arcavi’s (ibid) definition of visualization for this study:

The ability, the process, and the product of creation, interpretation, use and reflection upon pictures, images, diagrams, in our minds, on paper, or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings (p. 217).

While Arcavi’s description of visualization is broad, in this paper, we focus on investigating how students interpret, use, and think about aspects of graphs of real-valued functions and relations.

Mathematics education research has included numerous studies and discussions of the role of visualization throughout various levels of mathematics teaching and learning (e.g., Battista & Clements, 1991; Bishop, 1980; Davis, 1993; Dreyfus, 1991; Guzman, 2002; Hanna & Sidoli, 2007). Some studies have both suggested and reported the success of various instructional interventions that use visualization in undergraduate mathematics contexts such as derivatives, Riemann sums, and convergence of sequences and series (Kidron & Tall, 2015; Kowalczyk & Hausknecht, 1994; Roh, 2010; Tall, 2010; Thompson, Byerley, & Hatfield, 2013). Researchers have also examined students’ understanding of mathematical concepts using visualization tools, including their understanding of graphs of functions. (Alcock & Simpson, 2004; Kidron & Tall, 2015; Monk, 1992; Moore & Thompson, 2015; Moore, 2016; Pinto & Tall, 2002; Rasmussen, Zandieh, King, & Teppo, 2005; Roh & Lee, 2017; Thompson & Carlson, 2017). These studies provide evidence that visualization can be a powerful tool for students in advanced mathematics topics. For instance, Pinto and Tall (2002) found that a student making sense of a formal definition from his own visual imagery. Given the body of existing research in support of including visualization, graphs have earned a rightful place in the teaching and learning of Calculus.

In this section, we describe our perspective on visualization and review previous empirical and theoretical studies related to our present study. We share several instructional interventions that researchers have proposed to aid in the visualization of graphs of
functions. Then, we review some of the studies that explored students’ thinking about graphs of functions, including students’ interpretation of graphs and ways in which students may conceive of points on a graph.

2.1. Promotion of graphs in teaching undergraduate mathematics

There have been calls throughout the literature to encourage and promote the use of visual representations, such as graphs, in mathematics instruction, as an essential part of mathematical activity (Arcavi, 2003; Davis, 1993; Dreyfus, 1991; Guzman, 2002; Hanna & Sidoli, 2007). Tasks that use visual representations have been developed to support students in understanding concepts in Calculus. For example, Tall (2010) developed an intervention designed to highlight the relationship between the visual representation of continuity of a function with the analytic definition. In essence, Tall’s intervention relies on the visual process of repeatedly zooming in on a window of the graph, defined by differences of epsilon and delta from f(x) and x, respectively. Another example of a visual intervention, Roh (2010) epsilon-strip activity for sequence convergence, uses physical strips of translucent paper of a given width to illustrate which terms of a sequence lie within a certain distance from a target value. Kidron and Tall (2015) used software with a group of Calculus students to illustrate how a sequence of functions, such as the sequence of approximations given by the Taylor series representation, converge to the function represented.

Proposed instructional interventions, such as the ones described above, show promise in supporting students’ understanding (e.g., Cory & Garofalo, 2011; Kidron & Tall, 2015; Roh & Lee, 2017). However, visual representations and interventions that rely on them are not without limitations. Students may over-generalize from a single visual representation they view as “prototypical” (Harel & Sowder, 1998) or focus on irrelevant details of them (Pemseg, 1986). In particular, Alcock and Simpson (2004) found that in the context of convergent sequences and series, some students were overly confident in drawing conclusions from visual representations they created themselves rather than the formal statements. Because visual representations have the potential to support students, but may be interpreted differently by different onlookers or in different contexts, it is imperative that we investigate what sense students make of visual representations of concepts that are commonly given in textbooks or offered as instructional interventions.

2.2. Students’ thinking about graphs and points

Several studies in mathematics and science education have documented students’ creation skills, interpretations, and use of graphs (Goldenberg, 1988; Knuth, 2000; Moore & Thompson, 2015; Padilla et al., 1986; Pinto & Tall, 2002; Shah, Mayer, & Hegarty, 1999). Some studies have focused on students’ conceptions of graphs and their graphing activity (Moore & Thompson, 2015; Moore, 2016), while others have offered models for understanding points on a graph (Lakoff & Núñez, 2000; Thompson & Carlson, 2017).

2.2.1. Students’ graphing activity

Moore and Thompson (2015) highlight a distinction between static shape-thinking and emergent shape-thinking in students’ thinking about graphs of functions. In their study, students who engaged in static shape-thinking showed evidence of reasoning about graphs as though the graph itself was an object, such as a wire. In contrast, Moore and Thompson (ibid) described students who engaged in emergent shape-thinking as conceiving of a graph as a trace which emerges from the coordination of two varying quantities.

Moore and Thompson (2015) categories of student thinking are further described by Moore (2016) description of the constructs of figural and operative thought in the context of students’ interpretation of graphs. Moore (2016) explains that figural thought is dominated by visual properties of a graph, such as aspects of the shape of the graph, which override considerations about relationships of the quantities represented in the graph. As Moore explains, students who engage in figural thought think in ways which are subordinate to figurative aspects of the graph, though they may also perceive relationships the graph represents. In contrast, for students engaged in operative thought, “figurative elements of their activity are subordinate to that coordination [of covarying quantities]” (Moore, 2016, p. 3). Furthermore, Moore (ibid) aligns static shape-thinking with figural thought and emergent shape-thinking with operative thought, consistent with Piaget (2001) and Steffe (1991) use of these terms. Although students who engage in operative thought still perceive visual properties of a graph, their reasoning is informed by the relationships represented in the graph. In general, students may attend to both visual properties of graphs and the relationships of the quantities represented. However, either visual properties or the quantitative relationship guides students’ thinking.

2.2.2. Conceptions of points on graphs

While most research on students’ thinking about graphs focused on students’ conception of graphs in their entirety, other researchers have attended to students’ conception of points on a graph (e.g., Thompson & Carlson, 2017). Thompson and Carlson (ibid) contended that students need to conceive of a point on the graph of a function as representing a pair of values of two quantities simultaneously in the form of a single object. They use the term multiplicative object to refer such a conception of a point. However, Thompson et al. (2017) found that when constructing graphs, some secondary mathematics teachers did not conceive of points as multiplicative objects. Beyond not conceiving of points as multiplicative objects, it was not documented how the teachers were thinking of points, although treating coordinates as a “recipe” for a location was suspected (Thompson, Hatfield, Yoon, Joshua, & Byerley, 2017). Other researchers have also hypothesized that students may interpret the coordinates of a point as a prescription of movement to locate the point in a coordinate plane (Goldenberg, 1988; Frank, 2016).

In a different theoretical tradition, Lakoff and Núñez (2000) have offered a model for understanding points on a graph, highlighting the level of complexity involved in conceiving of a point on a graph as a pair of values. In a single dimension, Lakoff and
Núñez (ibid) argue that conceiving of numbers as points on a number line requires the use of what they call a conceptual metaphor. In their description, a conceptual metaphor is “an inference preserving cross-domain mapping” (p. 6). In this metaphor, the first domain is the set of numerical values, such as the set of real numbers, and the second domain is the positions on a line. Lakoff and Núñez (ibid) explain that thinking of representing numbers on a number line entails a cross-domain mapping of numerical values to positions on a line. The idea of mapping numerical values to physical locations on a line preserves properties of values such as ordering. For example, numerically, 3 is greater than 2, but less than 4. This numerical ordering among these three values is preserved on a number line, as the smallest number 2 is positioned on the far left on the number line, the largest number 4 is positioned on the far right on the number line, and 3 is positioned between the two points for the numerical values 2 and 4 on the number line. In Lakoff and Núñez’s (ibid) model, the conception of plotting a point in two dimensions relies on using this conceptual metaphor for each value of the pair represented.

We note that Lakoff and Núñez (2000) model for understanding points is distinct from a radical constructivist perspective. Lakoff and Núñez (ibid) describe a possible conception of points without reference to a context, implying that points may exist outside of a specific context. In contrast, a researcher adopting a radical constructivist perspective would typically describe an individual’s meaning for points as an active construction in the mind of the individual within a specified context. Each perspective may offer different advantages. For instance, Lakoff and Núñez’s (ibid) idea of a conceptual metaphor may offer an explanation for the cognitive roots of the representation of pairs of values in a two-dimensional Cartesian coordinate system. On the other hand, Thompson et al. (2017) findings of individuals who did not conceive of points as multiplicative objects may account for some difficulties students face when interpreting points on graphs.

Although the theoretical discussion in this section can help distinguish certain aspects of students’ thinking about graphs, current theories do not account for various aspects of student thinking about graphs. Students may have alternative conceptions of points on the graph, which might be different from conceptions of points as ordered pairs or as multiplicative objects. Thus, it is informative to study students’ interpretation of outputs of functions and points on graphs in undergraduate contexts, such as those from Calculus.

3. Methods

As part of a larger study, we conducted two-hour clinical interviews (Clement, 2000) with nine undergraduate students from a public southwestern university in the United States. We selected three undergraduates who had just completed one of the following three mathematical courses: Calculus I, Introduction to Proof, and Advanced Calculus, as students may be introduced to the IVT in at least one of these courses. At the institution where we collected data, Calculus I refers to a first course in Calculus typically covering limits, derivatives, and an introduction to integrals. Introduction to Proof refers to a course that introduces proof techniques, also known as Transition-to-Proof. Advanced Calculus refers to an introductory course in Real Analysis.

The research team consisted of four members who served in various roles. During each interview, one of the researchers served as the interviewer with the other three acting as witnesses—two in the room which were needed to operate the cameras, and one in another room watching the interview via video chat. Each interview was recorded with two cameras, one to capture the entire frame and the other to record student work on the desk. A third camera was set up to capture the interview on video chat for the third witness in the other room to limit the number of witnesses present with the student. All four researchers used laptops, whose screens the student could not see, to discuss the researchers’ current models of the student’s thinking in real-time via group chat. This arrangement allowed all three non-interviewing researchers to actively observe the interview and offer clarifying questions for the interviewer to pose to the student to test these models.

3.1. Interview tasks

During the interview, the interviewer asked students to evaluate each of the four mathematical statements in Table 1 and to provide justifications for their evaluations. The second statement is a statement of the Intermediate Value Theorem (IVT). This was the only true statement we presented. The remaining three statements (1, 3, and 4), all of which are false, were created from the IVT by reordering the quantifiers (for all, there exists) and/or the variables \(N, c\). To clarify, we did not give the students any information regarding the truth-values of the statements nor did we mention that Statement 2 is the IVT. These four statements were selected to serve the purposes of the larger study examining students’ interpretations of mathematical representations of concepts in Calculus (Sellers, Roh, & David, 2017). Although the four statements were chosen as part of the larger study, we found that students’ responses

<table>
<thead>
<tr>
<th>Table 1 Statements Presented to Students.</th>
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<tbody>
<tr>
<td>Statement 1 Suppose that (f) is a continuous function on ([a, b]) such that (f(a)=f(b)). Then, for all real numbers (c) in ((a, b)), there exists a real number (N) between (f(a)) and (f(b)) such that (f(c) = N).</td>
</tr>
<tr>
<td>Statement 2 (IVT) Suppose that (f) is a continuous function on ([a, b]) such that (f(a)=f(b)). Then, for all real numbers (N) between (f(a)) and (f(b)), there exists a real number (c) in ((a, b)) such that (f(c) = N).</td>
</tr>
<tr>
<td>Statement 3 Suppose that (f) is a continuous function on ([a, b]) such that (f(a)=f(b)). Then, there exists a real number (N) between (f(a)) and (f(b)) such that for all real numbers (c) in ((a, b)), (f(c) = N).</td>
</tr>
<tr>
<td>Statement 4 Suppose that (f) is a continuous function on ([a, b]) such that (f(a)=f(b)). Then, there exists a real number (c) in ((a, b)) such that for all real numbers (N) between (f(a)) and (f(b)), (f(c) = N).</td>
</tr>
</tbody>
</table>
to these four statements, allowed us to examine students' interpretation of graphs and evaluation of statements. Table 1 lists the statements in the order presented to the students.

After students evaluated the four statements, the interviewer presented the six graphs in Fig. 2 and asked them if they could use any of these graphs to explain their evaluation of each statement. Students were allowed to change their evaluations at any point in the interview. They were also asked to explain how they interpreted various aspects of each graph and to label relevant points and values on the graphs where appropriate.

The graphs in Fig. 2, which were selected to represent a spectrum of possible functions, relations, and relevant counterexamples, include: a polynomial whose minimum value is smaller than the output values of the endpoints of the displayed function (Graph 1), a vertical line segment (Graph 2), a continuous sinusoidal function (Graph 3), a monotone increasing function (Graph 4), a constant function (Graph 5), and a function that is discontinuous on \([a, b]\) (Graph 6). Graphs 1, 3, and 4 were chosen to represent three cases of continuous functions where \(f(a) \neq f(b)\). In Graph 1, the set of values between \(f(a)\) and \(f(b)\) does not coincide with the range of the function on the interval \([a, b]\), as there are output values which are not between the values of \(f(a)\) and \(f(b)\). Graph 3 is similar to the graph in Fig. 1 (right), as the range of the function is the same as the set of all values between \(f(a)\) and \(f(b)\). Additionally, there are multiple values of \(c\) for each chosen value of \(N\) such that \(f(c) = N\). Graph 4 was chosen as the range of the function is the same as the set of values between \(f(a)\) and \(f(b)\), and each \(N\)-value chosen has a unique \(c\) value associated with it (i.e., it is a one-to-one function). Graphs 2, 5, and 6 were chosen because each graph does not meet one of the conditions of the four statements: Graph 2 does not represent a function, Graph 5 does not satisfy the condition \(f(a) = f(b)\), and Graph 6 is not continuous on \([a, b]\). However, we did not presume that students would consider Graph 2 to not represent a function, that Graph 5 does not meet the condition that \(f(a) = f(b)\), or Graph 6 to not represent a continuous function on \([a, b]\). From our perspective, each of these six graphs may provide an opportunity to further examine students' interpretations of graphs.

3.2. Data analysis

We emphasize that this study, including the data collection and data analysis, is grounded in a constructivist perspective. In particular, we adopt von Glasersfeld (1995) view that students’ knowledge consists of a set of action schemes that are increasingly viable given their experience. This perspective implies that we, as researchers, do not have direct access to students’ knowledge and can only model their interpretation of aspects of graphs based upon their observable behaviors. Thus, our analysis reflects our best attempt at creating hypothetical models of students’ thinking about graphs grounded in observable behaviors, including their words, gestures, and markings on graphs.
Our data analysis was consistent with Corbin and Strauss (2014) description of grounded theory, in which categories of students' thinking about graphs emerged from the data analysis. This analysis consisted of three phases to analyze students' thinking about aspects of graphs while students evaluated the IVT and similar statements.

3.2.1. Phase I: Preliminary analysis

Initial data analysis began while each interview took place as well as immediately following the interview. During each interview, we, as a research team, actively formed initial conjectures based on the student’s verbal responses, gestures, and markings on graphs. These initial conjectures were used to guide the interviewer’s follow-up questions, which acted as a way of testing our conjectures. Based on the student's response to these follow-up questions, the initial conjectures were refined or changed accordingly. Additional questions were asked until sufficient data was collected. After each interview, the research team also met to debrief the current interview in light of the previous interviews. As more interviews were conducted and analyzed, differences in students' thinking about aspects of graphs emerged. Specifically, after conducting several interviews, the research team noticed that students had different meanings for the phrase “N between f(a) and f(b)” in the four statements (Table 1). In the subsequent interviews, the researchers were increasingly attentive to student language and work that indicated a meaning for “N between f(a) and f(b)” and asked questions to clarify students’ meanings for this phrase.

3.2.2. Phase II: Open coding

After conducting the nine interviews, we closely analyzed the video data student-by-student for patterns in student thinking that could explain their evaluations of the four statements. Through our initial analysis of students’ responses via open coding (Corbin & Strauss, 2014), students’ interpretations of various aspects of the graphs (e.g., outputs, points) emerged as highly relevant to their reasoning and subsequent evaluation of each statement. We analyzed the first video by developing codes to describe interesting and relevant aspects of the first student’s thinking about aspects of graphs. Next, we used these codes to analyze the second video, adding to, refining, and expanding the initial coding system as needed to describe the second student’s thinking about graphs. This process continued until we had analyzed all nine interviews, using the codes developed and refined in the previous analyses for each following interview. By the end of this process of open coding, two prominent codes, value-thinking and location-thinking, emerged that broadly characterized our students’ thinking about aspects of graphs.

3.2.3. Phase III: Axial coding

Once these value-thinking and location-thinking categories were established in Phase II, we refined these categories using axial coding (Corbin & Strauss, 2014). To refine our definitions of value-thinking and location-thinking, we compared characteristics of students’ thinking about three aspects of graphs (outputs, points, and graphs as a whole) within each category to check for consistency in our description of that category, adjusting the descriptions as needed. Additionally, we contrasted students in different categories as a means of further confirming our descriptions of the constructs. Through this process, we developed a description of each category of thinking about aspects of graphs which emerged in our analysis. Finally, we re-coded the video interview data using these refined codes. In the next section, we use these descriptions to frame our findings about students’ thinking about graphs.

4. Results I: Students’ interpretation of aspects of graphs

Our purpose in this study is to characterize students’ thinking about aspects of graphs related to statements from Calculus contexts. In our data, we observed that some students primarily focused on the values represented by the coordinates of points on graphs. In contrast, we found that other students primarily attended to the spatial location of points on graphs. In short, we observed that one property of points, either the values of the coordinates, or the points' locations, became foci of a student’s attention while reasoning about graphs. When students focused on the values represented by the coordinates of a point, we refer to their thinking as value-thinking. We use the term location-thinking to refer to thinking that primarily attends to the spatial location of the point. We noticed these two distinct ways that students interpreted each of the following aspects of graphs: outputs, points, and graphs as a whole.

In this section, we report five illustrative episodes from both categories, value-thinking and location-thinking, in order to highlight the defining characteristics of each category of student thinking about aspects of graphs. Episodes I and II illustrate characteristics of value-thinking and Episodes III–V illustrate characteristics of location-thinking. In each episode, we provide students’ gestures, labeling on graphs, and verbal explanations of their interpretation of the graphs to describe details of each student’s thinking about graphs, characteristic of the type of thinking they exhibited.

4.1. Characteristics of value-thinking

By value-thinking, we mean thinking about graphs that relies on the input and output values represented by the coordinates of a given point on a graph of a function. One of the key characteristics of value-thinking is distinguishing between the output of a function and the point on a graph. Students who engage in value-thinking label points as ordered pairs, e.g., (a, f(a)), and speak about points as representing both input and output values simultaneously. When considering the output of a function, students who think in this way tend to label relevant output values on the output axis of the graph of a function, and specifically speak about the values of the output. Students who engage in value-thinking, then, treat graphs as a collection of ordered pairs that relate corresponding input and output values.
In this section, we describe these characteristics of value-thinking in terms of students’ meanings for outputs, points, and “\(N\) between \(f(a)\) and \(f(b)\).” To highlight these characteristics, we primarily draw on episodes from one student, Jay, whom we categorized as a student who engaged in value-thinking. We present his verbal explanations along with his labeling on graphs. The first episode highlights Jay’s interpretation of outputs of a function as values, and points on a graph as ordered pairs. The second episode further illustrates Jay’s value-thinking through his interpretation of the phrase, “\(N\) between \(f(a)\) and \(f(b)\).”

4.1.1. Episode I: Outputs as values & points as ordered pairs

Even before the interviewer provided him with graphs, Jay drew his own graphical representations to explain his understanding of the statements. For instance, when the interviewer asked Jay to evaluate Statement 2 (IVT, see Table 1), he responded by saying that the statement is true, and drew a graph to explain his reasoning. Fig. 3 shows Jay’s hand-drawn graph.

Jay first plotted two points to represent the endpoints of a graph of a function that he envisioned. After labeling these endpoints as the ordered pairs, \((a,f(a))\) and \((b,f(b))\), Jay drew a horizontal line, which he labeled ‘\(N\),’ to represent a value of \(N\) between \(f(a)\) and \(f(b)\). Jay explained that, from his perspective, Statement 2 must be true because any graph of a continuous function whose endpoints are \((a,f(a))\) and \((b,f(b))\) must intersect the horizontal line \((y=N)\) that he labeled \(N\). He then labeled \(c\) on the \(x\)-axis and treated \(c\) as the \(x\)-coordinate of the intersection of the graph of the function and the horizontal line \(y=N\). Jay explained that if the function crossed the line \(y=N\) at the value \(c\), then \(f(c)\) equals \(N\). Jay then drew a graph from \((a,f(a))\) to \((b,f(b))\) as an example of a continuous function that he was speaking of. Transcript 1 contains his verbal explanation and our description of his gestures and labels on the graph.


<table>
<thead>
<tr>
<th>Statement 2</th>
<th>Hand-Drawn Graph (Fig. 3)</th>
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<tbody>
<tr>
<td>Interviewer: Can you explain why you said this statement was true?</td>
<td>Jay: …This right here is (y) equals (N), this line is (y) equals (N) ((points to horizontal line which he labeled (N)).) If (f) crosses this line ((points to horizontal line (N))), then there exists a real number (c) in the open interval such that (f(c)) is (N) because if I cross this line say here ((marks c on x-axis directly below)). This is my value of (c), right? Because that’s where (f(c)) is (N). Okay, so the only way for like this [(Statement 2)] not to be true is if you can draw… a continuous graph from (a) to (b) ((traces imaginary graph with finger between (a, f(a)) and (b, f(b)))) with (f(a)) being this ((points to (a, f(a))), f(b)) being this ((points to (b, f(b)))), such that you skip (N). And there’s no way to do that.</td>
</tr>
<tr>
<td>Interviewer: Can you re-explain what this statement means?</td>
<td>Jay: … (f) is a continuous function, right? …What this means is this ([function]) doesn’t skip over any values between (f(a)) and (f(b)) …The function passes through all of those lines, every single one of them…at some point ((traces finger over graph he drew)) and that point is ((c, f(c)))…and that’s really what the statement says.</td>
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Both Jay’s graph in Fig. 3 and his verbal justification in Transcript 1 provide evidence of Jay’s value-thinking, as described in Table 2. Specifically, Jay treated outputs of the function as values and points on the graph as ordered pairs.

4.1.1.1. Outputs as values. Jay drew a graph of a horizontal line \(y=N\) and claimed that where the function “cross[es] this line” is “where \(f(c)\) is \(N\).” Jay’s purpose in drawing this horizontal line was to identify a particular output value of the function, namely, \(N\).

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1 We note that within the transcript excerpts we provide in this paper, an ellipse indicates words omitted from a student’s transcript, text in brackets indicates explanation added by the authors to clarify what the student was referencing if that was made clear in the interview, student gestures are italicized in parentheses, and words from students’ transcripts are italicized for added emphasis by the authors.
Table 2
Comparison of Value-Thinking and Location-Thinking.

<table>
<thead>
<tr>
<th>Aspects of a Graph</th>
<th>Value-Thinking</th>
<th>Location-Thinking</th>
</tr>
</thead>
<tbody>
<tr>
<td>Output of Function</td>
<td>The resulting value from inputting a value in the function</td>
<td>The resulting location in the Cartesian plane from inputting a value in the function</td>
</tr>
<tr>
<td>Point on Graph</td>
<td>The coordinated values of the input and output represented together</td>
<td>A specified spatial location in the Cartesian plane</td>
</tr>
<tr>
<td>Graph as a Whole</td>
<td>A collection of coordinate pairs of values of the input and output</td>
<td>A collection of spatial locations in the Cartesian plane associated with input values</td>
</tr>
</tbody>
</table>

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<thead>
<tr>
<th></th>
<th>Interprets</th>
<th>Evidence</th>
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We take both Jay’s graph of the horizontal line $y = N$, and the way he used this line in his explanation as evidence that he was attending to the value of outputs, similar to Fig. 1(left). Indeed, Jay consistently treated outputs of the function as values. For instance, Jay confirmed his attention to the values of outputs of the function when he explained that a continuous function does not “skip over any values between $f(a)$ and $f(b)$.” We thus take Jay’s language and graph labels as evidence of his consideration of outputs of the function as values, a characteristic of value-thinking.

4.1.1.2. Points as ordered pairs. Jay labeled the endpoints of the graph as ordered pairs, $(a, f(a))$ and $(b, f(b))$, as shown in Fig. 3. Additionally, he referred to the intersection point of a continuous function, $f$, and any horizontal line, $y = N$, as an ordered pair, $(c, f(c))$. Jay’s labeling and description of points as ordered pairs indicate that he attended to both the values of the input and output represented at each point. We thus take Jay’s consistent treatment of points as ordered pairs as evidence of value-thinking.

4.1.2. Episode II: $N$ as a value “between $f(a)$ and $f(b)$.”
Jay’s interpretation of “$N$ between $f(a)$ and $f(b)$” in the later portion of the interview provides further evidence of Jay’s value-thinking. Earlier in the interview, Jay had already evaluated Statement 1 (Table 1) correctly as false. Later, when presented with the graphs we provided (see Fig. 2), Jay again explained his evaluation of Statement 1 while looking at Graph 1. Fig. 4 contains Jay’s work with Graph 1.

When explaining why he concluded that Statement 1 is false, Jay referenced the interval for possible $N$-values that he drew on Graph 1, as shown in Fig. 4. He approximated this interval to be $(0, 4)$ and explained that although 0 was an input value between $a$ and $b$, i.e., in the interval $(-3, 4)$ in Fig. 4, its corresponding output, $f(0)$, was not between $f(a)$ and $f(b)$, i.e., in the interval he approximated to be $[0, 4]$. Transcript 2 provides Jay’s verbal explanation.

Transcript 2. Jay’s Explanation of his Evaluation of Statement 1

**Interviewer:** Can you use this graph to explain why the statement is false?

**Jay:** This [Statement 1] is false in this case because it would say that in the open interval $(a, b)$ (points to $a$ and $b$ on x-axis), okay, so for all numbers between these two numbers, $-3$ and $4$, (gestures along x-axis between $-3$ and $4$) there exists a real number $[N]$ between, and this is the important part, between $0$ and $4$, or $0$ and $3$, so between these two values $f(a)$ and $f(b)$, (forms hand as a c-shape on graph, parallel to y-axis, to indicate the interval he marked off using the curly brace above) such that when I evaluate the function here (gestures along x-axis between $-3$ and $4$), I get the value in here (gestures to indicate interval on y-axis from $f(a)$ to $f(b)$). Okay so if I am looking at $f(0)$, right? $f(0)$ is $-7$ which is outside of the interval of $[0, 4]$, which means that there is no $N$ between $0$ and $4$ such that… $-7$ which is… $f(c)$ is equal to $N$, that value between $0$ and $4$.

4.1.2.1. $N$ as a value “between $f(a)$ and $f(b)$.” Jay marked off what he took to be the interval of possible $N$ values between $f(a)$ and $f(b)$ using a curly brace along the y-axis. Jay also gestured to indicate that he was considering values on the y-axis between the values of $f(a)$ and $f(b)$, which he approximated to be $4$ and $0$, respectively. Using $f(0) = -7$ as an example, Jay explained the input value, 0, was between $a$ and $b$, but the output value, $-7$, was not between $f(a)$ and $f(b)$. This example supported Jay’s conclusion that Statement 1 is false. We thus take Jay’s curly brace drawn in Fig. 4, gestures, and words as indicating his consideration of $N$ as a value between $f(a)$ and $f(b)$, which we consider evidence that Jay engaged in value-thinking.

4.2. Characteristics of location-thinking

Unlike the graphical interpretations illustrated by Jay above, some students did not consider outputs as values, points as ordered pairs, or $N$ as values “between $f(a)$ and $f(b)$.” Instead, these students labeled outputs at locations on the graph rather than the y-axis, labeled points as outputs rather than ordered pairs, and considered $N$, not as a value, but as a spatial location between the endpoints.
of the graph. Zack was one such student who labeled $f(a)$, $N$, and $f(b)$ not on the $y$-axis, but on the graphs that he drew, as shown in Fig. 5. As a result, Zack did not label points as ordered pairs.

Examples of treating outputs as locations, as shown in Zack’s work in Fig. 5, were not isolated occurrences. Rather, we repeatedly observed similar interpretations in almost half of the students we interviewed. In this section, we describe characteristics of this type of thinking, distinct from value-thinking, which we refer to as location-thinking.

By location-thinking, we mean thinking about graphs that relies on the spatial locations of points in a Cartesian plane. Students engaged in location-thinking focus on the location of points, while the values of the coordinates are either in the background of their reasoning or absent from it. In contrast with value-thinking, one of the key characteristics of location-thinking is treating the output of the function as indistinguishable from the location of the coordinate point. Accordingly, students engaged in location-thinking often label points on the graph as outputs, e.g., $f(a)$, rather than ordered pairs and speak about points in terms of their location in the coordinate plane. While students engaged in value-thinking label outputs on the output axis, students engaged in location-thinking frequently place the output label at the location of the point on the graph. Instead of speaking about output values, these students speak about points on the graph as the result of an input value. Students engaged in location-thinking, then, treat graphs as a collection of locations in space associated with inputs.

To highlight the characteristics of location-thinking, we present three illustrative episodes from two students, Zack and Nate. We present their verbal explanations along with their gestures and labels on graphs to illustrate defining characteristics of location-thinking. In Episodes III, IV, and V, we highlight specific aspects of location-thinking in terms of outputs, points, and the phrase “$N$ between $f(a)$ and $f(b)$,” respectively.

### 4.2.1. Episode III: Outputs as locations

While students who engaged in value-thinking considered outputs of the function to be values, students whom we classified as engaging in location-thinking considered outputs to be locations on graphs. We provide an episode from Zack’s interview to highlight this characteristic of location-thinking. Earlier in the interview, Zack had evaluated Statement 4 incorrectly as true, and then clarified that Statement 4 is true for some graphs and false for other graphs. Thus, he then concluded that Statement 4 is sometimes true. Later, Zack explained that in the case of Graph 3, Statement 4 is true. Fig. 6 contains Zack’s labels on Graph 3, in which he placed an output label at the location of the point.
Zack first pointed to the endpoints of the graph, and explained that the premise of the statement is true for this function, specifically that \( f(a) \) does not equal \( f(b) \). He then pointed to the \( x \)-axis and explained that there exists a real number \( c \) on the \( x \)-axis, between \( a \) and \( b \), for which he would get an \( N \) between \( f(a) \) and \( f(b) \). When describing \( c \) between \( a \) and \( b \), Zack marked off a portion of the \( x \)-axis with his hands and pointed to the axis. When describing \( N \) between \( f(a) \) and \( f(b) \), he first pointed to the endpoints of the graph and then swept his pen along the entire graph of the function. Finally, he gave \(-2\) as an example of a possible value of \( c \), labeled this value on the \( x \)-axis as "\( c = -2 \)" and explained that \( f(-2) \), which would be \( N \), would be on the graph. Zack then plotted a dot on the graph to represent what he considered to be the resulting \( N \), and labeled the dot "\( f(-2) = N \)". Transcript 3 contains Zack's explanation and gestures in this episode.

Transcript 3. Zack's Explanation of Statement 4 with Graph 3

Interviewer: Do you think it is possible to explain why this statement [Statement 4] is sometimes true using this graph [Graph 3]?

Zack: … So umm when I input \( a \) (points to left endpoint of the graph) I know that’s not going to be \( f(b) \) (points to right endpoint of the graph), so \( f(a) \) does not equal \( f(b) \) (points to left and right endpoint of the graph), which is correct. Umm following that, I know that there exists a real number \( c \) so from (points to \( a \) on \( x \)-axis), in between here (places hands on graph at \( a \) and \( b \) on the \( x \)-axis, palms facing each other), or on this \( x \)-axis (points to \( x \)-axis) there exists a real number \( c \) so that I’ll get a number \( N \) between \( f(a) \) and \( f(b) \) (points to both endpoints of the graph) on this curve (sweeps finger along entire graph) that proves the statement is true. So…if I let my \( c = -2 \) (labels "\( c = -2 \)" on the \( x \)-axis) I know that my \( N \) would be right here. (moves pen straight down from where he labeled \( c = -2 \) to graph and labels "\( f(-2) = N \)" on the graph).

4.2.1.1. Outputs as locations. Unlike the students engaged in value-thinking, who placed output labels on the \( y \)-axis, Zack, in Graph 3, placed an output label at a location on the graph as shown in Fig. 6. Zack's labeling in Fig. 6 was not the only instance of such label placement. In fact, Zack consistently placed output labels at points on the graphs of the functions he worked with throughout the interview (see Fig. 5 above and Fig. 7 in Episode IV). Zack’s placement of an output label at a location on the graph, rather than on the \( y \)-axis, indicates that he considered these outputs to be locations on the graph, rather than values on the \( y \)-axis. Additionally, Zack called the endpoints of the graph “\( f(a) \)” and “\( f(b) \)”, again indicating that he conceived of outputs as locations on the graph. Furthermore, when Zack referenced possible \( N \)'s between \( f(a) \) and \( f(b) \), he swept along the entire graph of the function, rather than along the \( y \)-axis. His gesture along the graph when describing \( N \)'s between \( f(a) \) and \( f(b) \) is also consistent with his conception of outputs as locations along the graph. We thus take Zack’s label on the graph, words, and gestures as evidence of his consideration of outputs of the function as locations, indicative of location-thinking.

Fig. 6. Zack’s output labels at a location on the graph.

Fig. 7. Zack’s labeling of \( f(a) \), \( f(b) \), and possible \( N \)'s as points on Graph 5 in which he claimed \( f(a) \neq f(b) \).
4.2.2. Episode IV: points as locations

In Episode III, we highlighted Zack’s meaning for outputs as locations, rather than values, an indication of location-thinking. More striking evidence of Zack’s location-thinking was observed when he was working with a constant function in Graph 5 (see Fig. 7) and claimed that \( f(a) \) is not equal to \( f(b) \). While Zack’s claim further supports his interpretation of outputs as locations, in this episode, we focus on Zack’s conception of points on a graph as locations, another characteristic of location-thinking.

When the interviewer presented Zack with Graph 5, Zack confirmed that the function is continuous on the interval \([a, b]\), pointed to the endpoints of the graph, and stated that \( f(a) \) is not equal to \( f(b) \). Zack read off the remainder of Statement 3, and again pointed to the endpoints of the graph when reading the phrase “\( N \) between \( f(a) \) and \( f(b) \).” Next, he pointed to a spot on the graph, which he explained was an example of \( N \) between \( f(a) \) and \( f(b) \), plotted a dot there, and labeled the dot on the graph as “\( N \).” He also labeled the endpoints of the graph as \( f(a) \) and \( f(b) \), respectively. Transcript 4 contains Zack’s explanation of Statement 3 with Graph 5.

Transcript 4. Zack’s explanation of his evaluation of Statement 3 with Graph 5

<table>
<thead>
<tr>
<th>Zack</th>
<th>Statement 3</th>
<th>Graph 5 (Fig. 7)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interviewer: Let’s do one more graph. [Can you use this graph to explain why the statement is true?]</td>
<td>Zack: Um (reading Statement 3) it’s continuous on ( a ) to ( b ), ( f(a) ) does not equal ( f(b) ) (points to endpoints), so I am good all the way up to there [the premise of Statement 3], and then there exists a real number ( N ) between ( f(a) ) and ( f(b) ) (points to endpoints) such that for all real numbers ( c ) in the interval ( a ) to ( b ), ( f(c) ) equals ( N ) (finishes reading statement). So if I start with my ( N ) first um let’s say it’s right here (points to a point on the graph, marks it with a dot, and labels ( f(a) ) and ( f(b) ) at the endpoints of the graph). This is my ( N ) (labels the point he just marked as ( N )). So I know there exists a real number ( N ) between ( f(b) ) and ( f(a) )....</td>
<td></td>
</tr>
</tbody>
</table>

4.2.2.1. Points as locations. We take Zack’s claim that \( f(a) \) and \( f(b) \) are not equal for the constant function \( f \) as evidence that he attended to the different spatial locations of the endpoints, rather than the pairs of input and output values represented at each point. For Zack, the point \( N \) that he labeled on the graph was in between the locations of the endpoints, which he referred to as \( f(a) \) and \( f(b) \).

We also note that Zack labeled points as \( f(a) \), \( f(b) \), and \( N \), rather than as ordered pairs. In essence, for Zack, there was no difference between outputs of the function and points on the graph, as both referred to spatial locations on the graph. We thus conclude that Zack conceived of points as locations, an indication of location-thinking.

4.2.3. Episode V: \( N \) as a location “between \( f(a) \) and \( f(b) \)”

Our final episode illustrating location-thinking comes from Nate’s interview. Nate had completed an advanced Calculus course and seemed to understand the difference among the four statements. Nate claimed that the four statements “all say different things.” For instance, he claimed that Statement 1 “says that for every input \( c \)…there exists an output \( N \) on the function such that the input maps to that output.” When describing Statement 2, Nate correctly interpreted the logical structure of the statement as, “for every output, there exists an input such that the input maps to that output.” Although Nate had correctly interpreted the logical difference in Statements 1 and 2 and had evaluated Statement 2, he had incorrectly evaluated Statement 1. In order to understand why his evaluation for Statement 1 was incorrect, despite his explanation above, we asked him to explain his reasoning for his evaluation.

By analyzing Nate’s response, we found that, like Zack and other students who engaged in location-thinking, Nate also considered outputs as locations, rather than values, and points as locations, rather than ordered pairs. Nate first labeled the endpoints of the graph as \( f(a) \) and \( f(b) \), respectively. Then, Nate highlighted the \( x \)-axis with his pen, and explained that for every \( c \) on this axis, he could find an \( N \) on the curve that \( c \) maps to. He labeled several \( c \)'s on the \( x \)-axis as he explained this. Nate’s labeling on Graph 1 is shown in Fig. 8. He also motioned from the \( x \)-axis vertically to the graph when describing that \( c \)'s mapped to \( N \)'s on the graph. Similarly, when describing \( N \)'s, he swept along the entire graph of the function from what he marked as \( f(a) \) to \( f(b) \). Like the other students who engaged in location-thinking, Nate interpreted outputs of the function and points on the graph as referring to spatial locations in the Cartesian plane, as evidenced by his labels and explanations.

Fig. 8. Examples of \( N \)'s Nate claimed were between \( f(a) \) and \( f(b) \), labeled on the graph.
location of the endpoints, which indicates that he conceived of the output of the function as the location of the point itself. Nate's labels above provide evidence that his interpretation of aspects of graphs was characterized by location-thinking.

While Nate's explanation above was intriguing, we were most surprised by what Nate did next. Nate had labeled possible N's on the graph that are, from our perspective, not between \( f(a) \) and \( f(b) \) (see Fig. 8). Noticing Nate's placement of N labels and his sweeping motion along the graph, we hypothesized that Nate interpreted “N between \( f(a) \) and \( f(b) \)” to mean all the points on the graph between the points that he labeled \( f(a) \) and \( f(b) \). To further examine Nate's meaning for this phrase, the interviewer extended the graph to the right and marked a point on this extension of the graph, at approximately \((5, 1)\), whose output value, 1, is between the values of \( f(a) \) and \( f(b) \) (see Fig. 9). The interviewer then asked Nate if this output was between \( f(a) \) and \( f(b) \). After thinking about the question briefly, Nate stated that the output was not between \( f(a) \) and \( f(b) \). Transcript 5 contains Nate's explanation of Statement 1 and Fig. 9 contains his corresponding labels on the graph.

Transcript 5. Nate's Explanation of Statement 1

\[
\begin{align*}
\text{Nate: } \text{Statement 1} & \quad \text{Graph 1 (Fig. 9)} \\
\text{Interviewer: } \text{...Okay. Let's say we picked a point over here (points to a point beyond endpoints of the graph, on portion of extended graph, the rightmost marked point on Fig. 9). Would you say that that output would be between } f(a) \text{ and } f(b) \text{?} \\
\text{Nate: (pauses) I would not say it's between } f(a) \text{ and } f(b). \text{ Even though the, yeah. This is the confusing part, where the actual numbers are 2.5 and 0 (marks the endpoints 2.5, and 0, boxed on Fig. 9). This would be, if you are looking at numbers 2.5, 0, this would be in between that interval. But it's in between that number interval. But it's not in between the functional interval in this case ...The interval refers to all these points between } f(a) \text{ and } f(b) \text{ (sweeps pen along the entire original graph). All points of the function. That's what I am interpreting.}
\end{align*}
\]

4.2.3.1. \( N \) as a location “between \( f(a) \) and \( f(b) \).” The interviewer’s prompt allowed Nate to more carefully consider his meaning for “\( N \) between \( f(a) \) and \( f(b) \).” Nate explained that there are two possible interpretations of this phrase, which he described as a “number interval” and a “function interval.” By “number interval,” Nate referred to the set of all output values between 0 and 2.5. In contrast, Nate used “function interval” to refer to the set of all points on the graph between the endpoints which Nate labeled \( f(a) \) and \( f(b) \). While Nate said the point that the interviewer marked was not between \( f(a) \) and \( f(b) \), he acknowledged that this point was “in the number interval” between 2.5 and 0 (the values of \( f(a) \) and \( f(b) \)). Nate clarified that although the output value of this point was between 0 and 2.5, the point was not between \( f(a) \) and \( f(b) \) because it was not “in the function interval.” As he described the “function interval,” Nate motioned along the entire graph between the points which he had labeled \( f(a) \) and \( f(b) \). Although Nate acknowledged the numerical interval of output values, [0, 2.5], he considered his notion of the “function interval” as more relevant for interpreting the phrase “\( N \) between \( f(a) \) and \( f(b) \).” We thus take Nate’s interpretation of “\( N \) between \( f(a) \) and \( f(b) \)” in terms of the spatial location of the points, rather than the values of the outputs, as indicative of location-thinking.

4.3. Summary of value-thinking and location-thinking

In this paper, we described five episodes from three students to illustrate the characteristics of each of the two ways of thinking, value-thinking and location-thinking, which we observed in this study. While we offer these episodes as exemplars, our findings of the two ways students think about graphs were not only found in these episodes. Rather, our description of these ways of thinking are based on our analysis of many episodes from all nine students whom we interviewed.

We categorized students as engaging in value-thinking if, like Jay, they considered outputs as values, points as coordinates of input and output values, and \( N \) as a value, as evidenced by their words, gestures, and markings on graphs. Consequently, they conceived of graphs as a collection of ordered pairs of values. For instance, Mike, who we also categorized as engaged in value-thinking, explained that “\( f(a) = f(b) \)” in the statements meant to him that “\( f(a) \) and \( f(b) \) is not the same value.” Another student
engaged in value-thinking, Nikki, drew a graph with a maximum value greater than the output values at the endpoints. She referred to the maximum value as “\(f(c)\)” and explained, “Number-wise, \(f(c)\) will be a higher number than \(f(a)\) and \(f(b)\).” In comparing \(f(c)\) to \(f(a)\) and \(f(b)\), Nikki focused on the values of these outputs. Similarly, James drew a graph and compared the values of the outputs, saying “there could be a c over here where it’s [the output is] clearly higher than \(f(a)\) and \(f(b)\)” In most cases, these students consistently labeled outputs on the \(y\)-axis and points as ordered pairs. Additionally, they referred to the \(y\)-axis when speaking about possible values of \(N\) in the various statements.

On the other hand, we identified students as engaging in location-thinking who considered both outputs and points as locations on the graph of the function, and, consequently, \(N\) as a location between the endpoints of the graph. These students, including Nate and Zack, consistently labeled outputs at locations along the graph rather than on the \(y\)-axis, and pointed to the graph when speaking about outputs rather than the \(y\)-axis. Additionally, they placed output labels at locations of points and did not use ordered pairs to label points. For instance, in the later portion of her interview, Marie consistently pointed to the endpoints of the graph when describing the statements and claimed, “this is \(f(a)\)” and “this is \(f(b)\)” while pointing to the left and right endpoints, respectively. For these students, all outputs, such as \(f(a)\), \(f(b)\), and \(N\), referred to distinct locations along the graph; consequently, all points on the graph would be possible \(N\)’s between the endpoints \(f(a)\) and \(f(b)\).

We detail characteristics of both categories of thinking about aspects of graphs in Table 2 by listing the interpretations for aspects of the graph for each category and observable behaviors indicative of these interpretations. The three aspects of graphs in Table 2 are named from our perspective as researchers, whereas the interpretations explicited under each type of thinking about graphs are from the perspective of the student. To be clear, the distinction between these two ways of reasoning lies in the place of the student’s attention when reasoning about graphs.

### 5. Results II: Relationship in students’ interpretation of aspects of graphs and evaluation of statements

We found that students’ interpretations of aspects of graphs were related to their evaluations of the four statements we presented. We highlight this relationship by reporting both students’ ways of thinking about graphs and their evaluation of the statements together.

To classify students’ ways of thinking about graphs, we used the description of our constructs in Table 2. Most of the students we interviewed consistently demonstrated either value-thinking or location-thinking. However, as students familiarized themselves with the statements, and the differences between them, they appeared to settle on their interpretations of the statements. For instance, Hannah and Marie exhibited moments of both value-thinking and location-thinking, engaging in the former first, then using the latter in their final interpretations. While Hannah and Marie exhibited both value-thinking and location-thinking, they both consciously chose to use interpretations associated with location-thinking (i.e., interpreting outputs as points) in their final interpretations of the statements. Using students’ final interpretations of the statements, we found five students engaged in value-thinking and four students engaged in location-thinking.

Because students may have changed their evaluations throughout the interview, we report students’ final evaluations of the four statements, labeled S1–S4, as true, false, or sometimes true. Table 3 summarizes our classification of each student’s thinking (Value-Thinking or Location-Thinking), along with each student’s mathematical level (Calculus, Introduction to Proof, and Advanced Calculus), and final evaluations of the four statements (True, False, or Sometimes True). Among the five engaged in value-thinking, three students evaluated all four statements correctly as FTFF. In contrast, no student engaged in location-thinking evaluated all four statements correctly.

In the following subsections, we describe our results in terms of the relationship between value-thinking and location-thinking with students’ evaluations of the four statements. We also highlight the relationship between students’ interpretations of aspects of graphs with their evaluation by describing how students’ way of thinking about graphs we presented changed when their evaluations changed.

### Table 3

Students’ interpretation of graphs, mathematical level, & final evaluations of statements.

<table>
<thead>
<tr>
<th>Students Observed Engaging in…</th>
<th>Student Name</th>
<th>Math Level</th>
<th>Final Student Evaluations</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>S1(T)</td>
</tr>
<tr>
<td><strong>Value-Thinking</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Jay</td>
<td>Advanced Calculus</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>James</td>
<td>Advanced Calculus</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>Nikki</td>
<td>Introduction to Proof</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>Ron</td>
<td>Introduction to Proof</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>Mike</td>
<td>Introduction to Proof</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td><strong>Location-Thinking</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Zack</td>
<td>Calculus</td>
<td>ST</td>
<td>ST</td>
</tr>
<tr>
<td>Nate</td>
<td>Advanced Calculus</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>Hannah</td>
<td>Calculus</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>Marie</td>
<td>Calculus</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

Shaded cells indicate mathematically incorrect evaluations.
5.1. The impact of students’ ways of thinking about graphs on their evaluations

In our study, we observed that both value-thinking and location-thinking impacted students’ evaluations of the four statements we provided. Notably, all three students who evaluated all four statements correctly were engaged in value-thinking. Thus, we consider value-thinking as a necessary component for correctly evaluating the four statements. However, not all of these students evaluated the statements correctly.

One such student, Ron, was classified as engaged in value-thinking. However, he understood the four statements differently than the other students in the same category. Ron even evaluated statement 1 incorrectly due to his inattention to the restriction of the statements correctly. However, not all of these students evaluated the statements correctly.

One such student, Ron, was classified as engaged in value-thinking. However, he understood the four statements differently than the other students in the same category. Ron even evaluated statement 1 incorrectly due to his inattention to the restriction of the statements correctly.

One such student, Ron, was classified as engaged in value-thinking. However, he understood the four statements differently than the other students in the same category. Ron even evaluated statement 1 incorrectly due to his inattention to the restriction of the statements correctly. When considering possible values of N, Ron referenced all values along the y-axis. For instance, Ron highlighted the entire y-axis on Graph 1 when referencing values of N, drawing arrows at both ends of the y-axis, as shown in Fig. 10.

Ron explained that, with Graph 1, he was thinking about values of N as “all real numbers” on the y-axis, as he illustrated in Fig. 10. Although he failed to attend to the restriction that N be “between f(a) and f(b),” Ron’s interpretation of this graph was characterized by value-thinking, as he attended to both input and output values at points, and treated outputs of functions as values found on the y-axis.

Ron’s failure to attend to the restriction on the values of N led him to evaluate statement 1 as true. In some cases, though, other factors in addition to students’ interpretation of aspects of graphs contributed to students’ incorrect evaluations. For example, Mike, another student who engaged in value-thinking, evaluated all four statements as false due to unconventional meanings for the logical quantifiers (for all, there exists) involved in the statements (see Sellers et al., 2017). Thus, we view value-thinking as necessary, but not sufficient, for correctly evaluating the IVT and similar statements.

While some students who engaged in value-thinking correctly evaluated the statements presented, no students who engaged in location-thinking evaluated all four statements correctly. We take this as an indication that location-thinking does not support students in correctly evaluating the statements we presented. Even Nate, a student with a more advanced mathematical background, incorrectly evaluated Statement 1 as true. His location-thinking was the main factor in his incorrect evaluation. In some cases, though, other factors in addition to students’ interpretation of aspects of graphs contributed to these students’ incorrect evaluations. For instance, Zack also mistakenly concluded that each statement was sometimes true and that all four statements were equivalent. We thus conclude that, in the context of the IVT and similar statements, location-thinking prohibits students from correctly evaluating these statements. Both value-thinking and location-thinking, as ways of thinking about graphs, impacted students’ evaluations of the four statements.

5.2. The impact of students’ evaluations on their ways of thinking about graphs

For most students whom we interviewed, their ways of thinking about the graphs influenced their evaluations of the statements, as reported in Sections 4.1. On the other hand, there were some students whose evaluations of the statements were not informed by, but rather influenced their ways of thinking about graphs. One of these students, Hannah, had originally evaluated all four statements as false, but decided to change all her evaluations to true, due to a belief she had about these statements resembling theorems she had learned as true in Calculus. When she changed her evaluations, Hannah explained that she consequently changed her way of thinking about the graphs. For example, when Hannah first evaluated Statement 4, she said it was false and labeled an interval of possible N values between f(a) and f(b) on the y-axis, similar to other students who engaged in value-thinking (see Fig. 4). Later in the interview, Hannah re-examined Statement 4, described her hesitation in evaluating the statement as false, and then changed her evaluation of the statement to true. Hannah also changed her interpretation of “N between f(a) and f(b)” to refer to all points along the graph, similar to students who engaged in location-thinking. Hannah looked at Graph 1 and explained how the change in her evaluation of Statement 4 changed her way of thinking about the graph as follows:

The way I was looking at it [Graph 1] makes it [Statement 4] false but I don’t think it is false. So I am trying to figure out a way to prove it true... I think I figured out where I messed up... So I thought about these, f(a) and f(b), as values, like say this [f(a)] is –1
and this \([f(b)]\) is like 5…but in between here \([f(a)\) and \([f(b)]\) you could have an \(f(c)\) that’s –10 and that would still be in there if you consider them \([f(a)\) and \([f(b)]\) as more like points on a graph, and not as like numbers…[Now] I am interpreting it \([N\) between \([f(a)\) and \([f(b)]\) as …all of these coordinates, every single point on this graph (sweeps along entire graph on Graph 1) from this point (points to left endpoint) to this point (points to right endpoint).

Hannah’s thinking about points on the graph changed from value-thinking to location-thinking, which was a result of the change in her evaluation of Statement 4 from false to true. Initially, when Hannah had evaluated Statement 4 as false, she had interpreted “\(N\) between \([f(a)\) and \([f(b)]\)” as referring to values between the values of \([f(a)\) and \([f(b)]\), as indicated in her words and graph labels. Later, Hannah explained that she thought all the statements, including Statement 4, must be true and explained that “\(N\) between \([f(a)\) and \([f(b)]\)” must instead refer to all the locations between the endpoints of the graph and gestured along the entire graph. While most students’ ways of thinking about the graphs informed their evaluations, Hannah adjusted her way of thinking about the graphs to accommodate for her change in evaluations of the statements. From our results, we recognize that students’ ways of thinking about graphs can inform their evaluations of the statements and in some cases, their evaluations of the statements can influence their ways of thinking about graphs.

6. Discussion

Our findings in this study, value-thinking and location-thinking, reveal critical distinctions in students’ interpretations of aspects of graphs. Furthermore, the different ways in which students interpret graphs have significant implications for how students understand important mathematical ideas, such as the Intermediate Value Theorem (IVT). In this section, we discuss the significance of our findings in this study and situate our findings relative to existing literature about students’ interpretation of graphs. We finally discuss the implications of our findings for practitioners, both in the classroom and in curricular design.

6.1. Significance of findings and use of constructs for future research

The emergence of value-thinking and location-thinking as two ways of thinking about points in a Cartesian coordinate system aligns with the two aspects of points as they are used in mathematics. Points in a two-dimensional Cartesian system are intended to represent the concurrence of the values of two quantities, conventionally notated as an ordered pair \((x, y)\). When referencing the coordinates of a point, as in the case of points on the graph of a real-valued function, the point is typically labeled with its ordered pair. This pair of values \(x\) and \(y\) also has a spatial location in the Cartesian plane found by coordinating two signed distances. To graph a point \((x, y)\) in the Cartesian plane, one marks the spatial location that is a distance of \(x\) units from the origin along the \(x\)-axis (conventionally left or right) and a distance of \(y\) units from the origin along the \(y\)-axis (conventionally up or down). Thus, we see a point on a graph as dual-natured, simultaneously representing a pair of values, as well as a specified location in the space of the Cartesian plane. When referring to the location of the point, such as in geometric contexts, the point is typically labeled with a single character, such as \(P\). The students in our study appeared to attend to one of these two aspects of a point, either the pair of values (value-thinking) or the location of the point (location-thinking).

We summarize our findings of the essential characteristics of value-thinking and location-thinking using the following two examples of different labels on the same graph. We also use these examples to illustrate the utility of the constructs of value-thinking and location-thinking to categorize students’ interpretations of aspects of any graphs in a Cartesian coordinate system, beyond the graphs used in this study. In Fig. 11, the two sets of labels illustrate distinctive characteristics of value-thinking (left) and location-thinking (right), respectively.

In Fig. 11 (left), input values are labeled on the input axis, and output values are labeled on the output axis. Furthermore, points on this graph are labeled as ordered pairs. In Fig. 11 (right), while the input values are labeled on the input axis, the output values are not labeled on the output axis. In contrast to Fig. 11 (left), points on the graph in Fig. 11 (right) are labeled with only output notation,

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\begin{align*}
\text{Fig. 11. Sample labels indicative of value-thinking (left) vs. location-thinking (right).}
\end{align*}
\]
rather than ordered pairs. Additionally, the placement of output labels differs between the two figures. For example, in Fig. 11 (left), $f(c)$ is a label of a specific value on the output axis, whereas in Fig. 11 (right), $f(c)$ is a label on a particular spatial location.

In our development and application of the constructs of value-thinking and location-thinking in this study, we relied on evidence found in the student's (repeated) actions, including their words, gestures, and labels on graphs, rather than their abilities to think in various ways. Thus, the classification of a student as engaged in location-thinking does not imply that the student is unable to consider values of the output. On the contrary, it is likely that students may be classified as engaged in location-thinking and may still at times refer to or find specific values of the output, as we saw in the case of Nate. Rather, students engaged in location-thinking focus on the location of the relevant points in their reasoning, as revealed through their words and actions.

One may conjecture that students engaged in location-thinking in this study may have been directed to consider locations due to the spatial connotations of the word “between” in English. While we acknowledge the challenge of linguistic differences in words in English and mathematics (Schleppergrell, 2007), we highlight that our findings are not solely based on students' meaning for this phrase, but are grounded in multiple aspects of students' interpretation of graphs. For instance, Zack's claim that $f(a)$ did not equal $f(b)$ on a constant function (see Transcript 4), did not refer to the phrase “$N$ between $f(a)$ and $f(b)$.” Although the phrase “$N$ between $f(a)$ and $f(b)$” in the statements presented may have influenced certain aspects of students' interpretations of graphs, we maintain that these constructs may characterize student thinking in graphical contexts beyond the statements used in this study.

Accordingly, no single piece of evidence (e.g., one labeled point, one gesture) was viewed as sufficient for categorizing a students' way of thinking. For example, consider the case of a student who at first labels points as ordered pairs on a graph, a behavior associated with value-thinking in Table 2. If this student later repeatedly attends to locations of points when reasoning about outputs of the function, the student would be classified as engaged in location-thinking, despite an action that may otherwise indicate value-thinking. We also recognize that the same student, when presented with different contexts, may engage in value-thinking or location-thinking, depending on what he or she focuses on. Furthermore, we acknowledge that a student may, within the same context, engage in each of the two ways of thinking at different times, as Hannah and Marie did in this study. We thus emphasize that in our view, the distinction between value-thinking and location-thinking is one of the placement of students' attention while reasoning about the graph, rather than an exclusive way of thinking.

We view these two constructs of value-thinking and location-thinking, which we summarize in Table 2, as having potential use for further investigating students' interpretations of aspects of graphs in a variety of contexts and their relationship to their understanding of that context. While in this study, we found the distinction in value-thinking and location-thinking in the context of the IVT, we conjecture that these two ways of thinking occur in a variety of contexts involving graphs in Cartesian coordinate systems. Furthermore, each way of thinking may support students to understand different mathematical contexts that rely on graphs. For instance, any graph of a real-valued function, such as those found throughout high school and undergraduate curricula, might also elicit both value-thinking and location-thinking in students, though value-thinking may be preferable to understanding the given context. On the other hand, location-thinking may better support students in thinking about contexts from geometry, such as polygons placed in a Cartesian plane. Furthermore, understanding some graphical representations as intended may require a coordination of both value-thinking and location-thinking. For example, the graph of a parametric curve may represent both coordinates of pairs of values, as well as a path traveled along the curve, including directionality, often indicated by arrows on the curve. Further research may build on these constructs to better understand the ways in which students use each of these ways of thinking to interpret graphs in their study of mathematics.

### 6.2. Relationship to existing literature

Based on our study, we conclude that the differences in the place of a student's focus when reasoning about points on a graph have significant implications on their understanding of related mathematical ideas. The constructs of value-thinking and location-thinking, which we described and illustrated in the results section, provide a new level of insight into distinguishing characteristics of students' interpretations of graphs not previously detailed in existing literature.

We emphasize the potential generalizability of our constructs as presented generally in Table 2 to other contexts involving graphs in Cartesian coordinates, as describe in Section 5.1. Thus, although we illustrated our constructs in the context of the IVT in this paper, they may also be used to characterize students' interpretation of graphs in other contexts across K-16 settings. Our results highlight and explain significant aspects of students' interpretations of graphs not previously accounted for by current theories and studies on students' thinking about graphs. Thus, the use of our constructs of value-thinking and location-thinking may progress the depth of analysis in the field of students' understanding of graphs.

Our findings contribute to the literature that exists on students' interpretations of graphs. We view our constructs as related to Moore (2016) work in describing students' interpretations of graphs as a whole. Value-thinking and location-thinking, as defined in this study, align with Moore (2016) constructs of operative and figurative thought, respectively. In our description, the visual perception of the graph of a student engaged in value-thinking, specifically the spatial location of the points, is subordinate to their meanings for output values. Thus, value-thinking, aligns with Moore (2016) description of operative thought. Students engaged in location-thinking, on the other hand, may be considered to have also engaged in figurative thought. The visual cues from the graph, such as the spatial location of the points did dominate their thinking about some aspects of the graph.

The results of this study also extend this literature on students' interpretations of graphs. While Moore and Thompson (2015) categories of static and emergent shape-thinking consider students' interpretations of graphs as a whole, our constructs also include students' interpretations of outputs of a function and points on graphs, which was not previously documented. Thus, our findings help to further explain how students understand the outputs and points that comprise an entire graph.
Our findings in this study also build on the literature on students’ interpretations of points. For example, our construct of location-thinking provides insight into how students who do not conceive of points as multiplicative objects (Thompson & Carlson, 2017) are interpreting points. While students engaging in location-thinking focus on the location of the point in space, they may also recognize values at a given point, as observed in the case of Nate. Furthermore, a student may engage in value-thinking and location-thinking at different times, as observed in the case of Hannah. Thus, we do not view students’ interpretations of points as multiplicative objects as a simple binary classification, but acknowledge the flexibility in students’ interpretation of points between different contexts or at different times. Additionally, we view our findings as providing empirical data to support and further explain hypotheses about students’ cognitive activity when conceiving of points on graphs. Specifically, our finding of students who engaged in location-thinking supports previous researchers’ theories that students may treat ordered pairs as a recipe to identify a location and thus conceive of points as locations (Frank, 2016; Goldenberg, 1988; Thompson et al., 2017). Similar to Frank’s (2016) hypothesis, the process of plotting points by an action of beginning at the origin and moving over x amount and up y amount to locate a point may promote location-thinking. Relative to the existing literature on this subject, we thus consider our constructs of value-thinking and location-thinking, which are grounded in empirical data, as further describing students’ interpretations of aspects of graphs.

6.3. Implications for curriculum and instruction

The findings of this study have implications for instruction and curriculum development at both the K-12 and the undergraduate level. As students’ interpretations and use of graphs is an essential component of mathematics education across grade levels, the findings of our study provide a foundation for the improvement of classroom teaching to support students in this area.

Our study has direct implications for the teaching of the IVT. In the context of the IVT, the use of images, such as the graphs in Fig. 1, alone, may not be sufficient for students to reflect on the intended meaning of “N between f(a) and f(b)” as referring to values, as the endpoints of the interval are also the absolute minimum and maximum values of the output. Thus, reasoning with these images via value-thinking and location-thinking produces the same result and misses an opportunity to distinguish them. We suggest that in the context of the IVT, both instructors and textbook authors incorporate images such as Graph 1 that we used in our study, in which the function includes output values which are not between the values of f(a) and f(b).

This study also has implications for the use of graphs broadly throughout K-16 curriculum and instruction. The results of this study indicate that overcoming various perceptual cues found in graphs, beyond conceiving of the graph as a static shape, is a nontrivial achievement, even for advanced students. Teachers utilizing graphical representations may seek means to support students in overcoming adherence to visual cues. Because this study sheds light on the various ways in which students may interpret graphs, instructors may find the constructs of location-thinking and value-thinking beneficial in differentiating students’ understanding in the classroom. Furthermore, the placement of labels on graphs of functions, especially output labels, may significantly influence the way in which a student understands points on a graph, as observed in labeling of students engaged in location-thinking. For instance, students may benefit from teachers’ attention to labeling output values on the y-axis as shown in Fig. 11 (left), as opposed to labeling outputs at points on the graph as shown in Fig. 11 (right). Additionally, gesturing from the x-axis to the graph, as we observed in students engaged in location-thinking, may also support students in incorrectly conceiving of a graph as a set of input values mapped to particular spatial locations.

While in this study, value-thinking helped students to correctly evaluate the IVT and similar statements, in other contexts, such as diagrams in geometric settings, location-thinking may be preferable. Additionally, other contexts, like graphs of parametric curves, may require both location and value-thinking for various aspects of the same image. Ideally, students should possess the ability to think in both ways, focusing on the values represented at a point and the point’s spatial location, as well as the discernment for when to use each way of thinking. To support students in recognizing these two ways of thinking, instructors and curriculum developers may consider providing students with opportunities to think both ways and bring to light this distinction. We hope that our findings increase practitioners’ awareness of the subtle yet significant details of students’ interpretations of aspects of graphs and may thus inform decisions in curriculum design and instruction.

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