NAME ........SOLUTIONS........

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# Math 220 - Midterm Examination - March 12, 2009

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**This exam has 4 pages and consists of 4 problems. To receive full credit for correct answers, it is necessary to follow instructions, show your work and provide your reasoning.**

**Problem 1** [20 points] Describe the solution set of the following system in parametric vector form.

$$x_1 + x_2 + x_3 + x_4 = -2$$

$$x_1 + x_2 + 2x_3 + 3x_4 = 2$$

$$x_1 + x_2 + 2x_3 + 2x_4 = 0$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & -2 \\ 1 & 1 & 2 & 3 & 2 \\ 1 & 1 & 2 & 2 & 0 \end{array}\right] \xrightarrow[R_3-R_1]{R_2-R_1} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & -2 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 1 & 2 \end{array}\right]$$

$$\xrightarrow{R_3-R_2} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & -2 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & -1 & -2 \end{array}\right] \xrightarrow{-R_3} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & -2 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

$$\xrightarrow[R_1-R_3]{R_2-2R_3} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & -4 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow{R_1-R_2} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -4-x_2 \\ x_2 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

($x_1, x_3, x_4$ = basic variables, $x_2$ = free variable.)

**Problem 2** [30 points] Let

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} -4 \\ -3 \\ 10 \end{bmatrix}; \ \mathbf{v}_3 = \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}.$$

(a) Is $\mathbf{v}_2$ in Span$\{\mathbf{v}_1, \mathbf{v}_3\}$? If yes, express $\mathbf{v}_2$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_3$. If no, explain why not.

$$[v_1 \ v_3 \mid v_2] = \left[\begin{array}{cc|c} 2 & 2 & -4 \\ -1 & -3 & -3 \\ 0 & 5 & 10 \end{array}\right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{cc|c} 1 & 1 & -2 \\ -1 & -3 & -3 \\ 0 & 5 & 10 \end{array}\right]$$

$$\xrightarrow{R_2+R_1} \left[\begin{array}{cc|c} 1 & 1 & -2 \\ 0 & -2 & -5 \\ 0 & 5 & 10 \end{array}\right] \xrightarrow{-\frac{1}{2}R_2} \left[\begin{array}{cc|c} 1 & 1 & -2 \\ 0 & 1 & 5/2 \\ 0 & 5 & 10 \end{array}\right] \xrightarrow{R_3-5R_2} \left[\begin{array}{cc|c} 1 & 1 & -2 \\ 0 & 1 & 5/2 \\ 0 & 0 & -5/2 \end{array}\right]$$

Since $[0 \ 0 \mid -5/2]$ is a row of the EF, $v_2$ is not in Span$\{v_1, v_3\}$.

(b) Is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ a linearly independent set? Explain.

$$\begin{bmatrix} 2 & -4 & 2 \\ -1 & -3 & -3 \\ 0 & 10 & 5 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & -2 & 1 \\ -1 & -3 & -3 \\ 0 & 10 & 5 \end{bmatrix} \xrightarrow{R_2+R_1} \begin{bmatrix} 1 & -2 & 1 \\ 0 & -5 & -2 \\ 0 & 10 & 5 \end{bmatrix}$$

$$\xrightarrow{R_3+2R_2} \begin{bmatrix} 1 & -2 & 1 \\ 0 & -5 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$ EF has pivot in every column, so $\{v_1, v_2, v_3\}$ is indeed a linearly independent set

(c) Does the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ span $\mathbb{R}^3$? Explain.

$\{v_1, v_2, v_3\}$ spans $\mathbb{R}^3$ because there's a pivot in every row of the EF in (b).

**Problem 3** [25 points] Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be a linear transformation such that

$$T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \quad T(\mathbf{e}_2) = \begin{bmatrix} 1 \\ 2 \\ h \end{bmatrix},$$

where $\mathbf{e}_1, \mathbf{e}_2$ are, respectively, the first and second column of the $2 \times 2$ identity matrix.

(a) Find the standard matrix $A$ of $T$ and use it to evaluate $T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$.

2.5 $A = \left[ T(e_1) \mid T(e_2) \right] = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ -2 & h \end{bmatrix}$

2.5 $T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = A \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ -2 & h \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ h-4 \end{bmatrix}$

10 (b) Find all $h$, if any, so that $T$ is one-to-one.

$$A \xrightarrow[R_3 + 2R_1]{R_2 - R_1} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & h+2 \end{bmatrix} \xrightarrow{R_3 - (h+2) R_2} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

There's a pivot in every column of the EF of A, so T is one-to-one for $\boxed{\text{all } h.}$

10 (c) Find all $h$, if any, so that $T$ maps $\mathbb{R}^2$ onto $\mathbb{R}^3$.

Third row of the EF of A has no pivot, regardless of h. So there is $\boxed{\text{no } h}$ for which T maps $\mathbb{R}^2$ onto $\mathbb{R}^3$.

**Problem 4** [25 points] Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

15 (a) Find the inverse of $A$, $A^{-1}$.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 2 & -1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 - 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & -2 & 1 \end{array}\right] \xrightarrow{-R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array}\right]$$

$$\xrightarrow{R_1 - 2R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -4 & 2 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array}\right]$$

$$\left( \text{Verify } \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & -4 & 2 \\ -1 & 1 & 0 \\ -1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$$

10 (b) Use $A^{-1}$ computed above to solve the system $Ax = b$.

$$Ax = b \Rightarrow x = A^{-1}b = \begin{bmatrix} 3 & -4 & 2 \\ -1 & 1 & 0 \\ -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$