(b) NAME SOLUTIONS
ID PINK EXAM

Math 220 - Midterm Examination - October 8, 2009

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This exam has 4 pages and consists of 4 problems. To receive full credit for correct answers, it is necessary to follow instructions, show your work and provide your reasoning.

Problem 1 [20 points] Describe the solution set of the following system in parametric vector form.

$$\begin{aligned} x_1 - 3x_2 - 2x_3 + 5x_4 &= 2 \\ x_2 - x_3 + 2x_4 &= 0 \\ -x_1 + 3x_2 + 7x_3 \qquad &= 3 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & -3 & -2 & 5 & 2 \\ 0 & 1 & -1 & 2 & 0 \\ -1 & 3 & 7 & 0 & 3 \end{array}\right] \xrightarrow{R_3+R_1} \left[\begin{array}{cccc|c} 1 & -3 & -2 & 5 & 2 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 5 & 5 & 5 \end{array}\right]$$

$$\xrightarrow{1/5 R_3} \left[\begin{array}{cccc|c} 1 & -3 & -2 & 5 & 2 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right] \xrightarrow[R_1+2R_3]{R_2+R_3} \left[\begin{array}{cccc|c} 1 & -3 & 0 & 7 & 4 \\ 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R_1+3R_2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 16 & 7 \\ 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right]$$

x_1, x_2, x_3 basic

x_4 free

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7-16x_4 \\ 1-3x_4 \\ 1-x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -16 \\ -3 \\ -1 \\ 1 \end{bmatrix}$$

Problem 2 [30 points] Consider the set $S = \left\{ \begin{array}{c} v_1 \\ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix} \end{array}, \begin{array}{c} v_2 \\ \begin{bmatrix} 7 \\ -4 \\ 5 \\ 1 \end{bmatrix} \end{array}, \begin{array}{c} v_3 \\ \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \end{array} \right\}$.

(a) Determine whether or not S is linearly independent.

$$\begin{bmatrix} 1 & 7 & -1 \\ 0 & -4 & 2 \\ 1 & 5 & 0 \\ 2 & 1 & 1 \end{bmatrix} \xrightarrow[R_4-2R_1]{R_3-R_1} \begin{bmatrix} 1 & 7 & -1 \\ 0 & -4 & 2 \\ 0 & -2 & 1 \\ 0 & -13 & 3 \end{bmatrix} \xrightarrow{-\frac{1}{4}R_2} \begin{bmatrix} 1 & 7 & -1 \\ 0 & 1 & -1/2 \\ 0 & -2 & 1 \\ 0 & -13 & 3 \end{bmatrix}$$

$$\xrightarrow[R_4+13R_2]{R_3+2R_2} \begin{bmatrix} 1 & 7 & -1 \\ 0 & 1 & -1/2 \\ 0 & 0 & -1 \\ 0 & 0 & -7/2 \end{bmatrix} \xrightarrow{-R_3} \begin{bmatrix} 1 & 7 & -1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \\ 0 & 0 & -7/2 \end{bmatrix}$$

$$\xrightarrow{R_4+\frac{7}{2}R_2} \begin{bmatrix} 1 & 7 & -1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ EF}$$

pivot in every column so
S is linearly independent

(b) Determine whether or not S spans $\mathbf{R}^4$.

The EF above does not have a pivot in every row, so S does not span $\mathbb{R}^4$

(c) Is the vector $\begin{bmatrix} 8 \\ -4 \\ 6 \\ 3 \end{bmatrix}$ in Span(S)? Note that $v_1 + v_2 = \begin{bmatrix} 8 \\ -4 \\ 6 \\ 3 \end{bmatrix}$ or trace the steps in part (a):

$$b = \begin{bmatrix} 8 \\ -4 \\ 6 \\ 3 \end{bmatrix} \xrightarrow[R_4-2R_1]{R_3-R_1} \begin{bmatrix} 8 \\ -4 \\ -2 \\ -13 \end{bmatrix} \xrightarrow{-\frac{1}{4}R_2} \begin{bmatrix} 8 \\ 1 \\ -2 \\ -13 \end{bmatrix} \xrightarrow[R_4+13R_2]{R_3+2R_2} \begin{bmatrix} 8 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

So the system $[v_1\ v_2\ v_3 \mid b]$ is consistent and hence

$$\begin{bmatrix} 8 \\ -4 \\ 6 \\ 3 \end{bmatrix} \in \text{Span}(S)$$

Problem 3 [25 points] Let $T : \mathbf{R}^2 \longrightarrow \mathbf{R}^3$ be a linear transformation such that

$$T(x_1, x_2) = (x_1 + x_2,\ x_1 - x_2,\ h(x_1 + x_2))$$

(a) Find the standard matrix A of T.

$T(e_1) = T(1,0) = (1+0,\ 1-0,\ h(1+0)) = (1, 1, h)$

$T(e_2) = T(0,1) = (0+1, 0-1, h(0+1)) = (1, -1, h)$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ h & h \end{bmatrix}$$

(b) Find all h (if any) so that T is a one-to-one transformation.

$$A \xrightarrow[R_3 - hR_1]{R_2 - R_1} \begin{bmatrix} 1 & 1 \\ 0 & -2 \\ 0 & 0 \end{bmatrix}$$

pivot in every column, regardless of h

$\boxed{\text{ALL } h}$

(c) Find all h (if any) so that T maps $\mathbf{R}^2$ onto $\mathbf{R}^3$.

No pivot in row 3, regardless of h.

$\boxed{\text{No } h}$

Problem 4 [25 points] Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$.

(a) Find the inverse of A, A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 - R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 + R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \xrightarrow[R_1 - R_3]{R_2 - R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & -1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ -1 & 1 & 1 \end{bmatrix}$$

(b) Verify that $A A^{-1} = I$.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

(c) Use A^{-1} computed above to solve the system $Ax = b$.

$$x = A^{-1} b = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 3 \end{bmatrix}$$