(a) NAME SOLUTIONS
ID BLUE EXAM

Math 220 - Midterm Examination - October 8, 2009

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This exam has 4 pages and consists of 4 problems. To receive full credit for correct answers, it is necessary to follow instructions, show your work and provide your reasoning.

Problem 1 [20 points] Describe the solution set of the following system in parametric vector form.

$$x_1 - 2x_2 + x_3 = -1$$
$$x_1 + x_2 + 2x_3 = 0$$
$$2x_1 - x_2 + 3x_3 = -1$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & -1 \\ 1 & 1 & 2 & 0 \\ 2 & -1 & 3 & -1 \end{array}\right] \xrightarrow[R_3-2R_1]{R_2-R_1} \left[\begin{array}{ccc|c} 1 & -2 & 1 & -1 \\ 0 & 3 & 1 & 1 \\ 0 & 3 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R_3-R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & -1 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{1/3 R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & -1 \\ 0 & 1 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1+2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 5/3 & -1/3 \\ 0 & 1 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x_1, x_2 : basic
x_3 : free

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1/3 - 5/3\, x_3 \\ 1/3 - 1/3\, x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1/3 \\ 1/3 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -5/3 \\ -1/3 \\ 1 \end{bmatrix}$$

Problem 2 [30 points]

Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ h \end{bmatrix}$. Find all h (if any) so that

(a) $\mathbf{v}_3$ belongs to $\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & h \end{array}\right] \xrightarrow{R_2+R_1} \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & -2 & 2 \\ 0 & 1 & h \end{array}\right] \xrightarrow{-\frac{1}{2}R_2} \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & h \end{array}\right]$$

$$\xrightarrow{R_3-R_2} \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & h+1 \end{array}\right]$$

$V_3 \in \mathrm{Span}\{v_1, v_2\}$ if and only if $h+1=0$, i.e., $\boxed{h=-1}$

(b) $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set.

if and only if $h+1 \neq 0$, i.e., $\boxed{h \neq -1}$

(c) $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ spans $\mathbb{R}^3$.

if and only if $h+1 \neq 0$, i.e., $\boxed{h \neq -1}$

Problem 3 [25 points] Let $T : \mathbf{R}^3 \longrightarrow \mathbf{R}^2$ be a linear transformation such that

$$T(x_1, x_2, x_3) = (x_1 + x_2, 2x_1 + x_2 - x_3)$$

(a) Find the standard matrix A of T.

$T(1,0,0) = (1+0, 2+0-0) = (1,2)$
$T(0,1,0) = (0+1, 0+1-0) = (1,1)$
$T(0,0,1) = (0+0, 0+0-1) = (0,-1)$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & -1 \end{bmatrix}$$

(b) Determine whether or not T is a one-to-one transformation.

$A \xrightarrow{R_2-2R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix}$ No pivot in column 3, so $\boxed{\text{NOT ONE-TO-ONE}}$

(c) Determine whether or not T maps $\mathbf{R}^3$ onto $\mathbf{R}^2$.

Pivot in every row, so

$\boxed{\text{ONTO}}$

Problem 4 [25 points] Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$.

(a) Find the inverse of A, A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 - R_1} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & -1 & 1 \end{array}\right] \xrightarrow{-R_3} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right]$$

$$\xrightarrow[R_1 - R_3]{R_2 - R_3} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right] \xrightarrow{R_1 + R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -1 & 2 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -1 & -1 & 2 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

(b) Verify that $AA^{-1} = I$.

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 & 2 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

(c) Use A^{-1} computed above to solve the system $Ax = b$.

$$x = A^{-1}b = \begin{bmatrix} -1 & -1 & 2 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$$