12, 20, 21

$$A = \begin{pmatrix} 1 & 2 & -4 & 3 & 3 \\ 5 & 10 & -9 & -7 & 8 \\ 4 & 8 & -9 & -2 & 7 \\ -2 & -4 & 5 & 0 & -6 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -4 & 3 & 3 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{R_1 + 4R_2} \begin{pmatrix} 1 & 2 & 0 & -5 & 3 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{-\frac{1}{5}R_3} \begin{pmatrix} 1 & 2 & 0 & -5 & 3 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{R_1 - 3R_3} \begin{pmatrix} 1 & 2 & 0 & -5 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow$$ columns 1, 3 and 5 are basic, i.e. lin. independent $\Rightarrow$ basis for the column space is

$$\underline{B_{colA}} = \left\{ \begin{pmatrix} 1 \\ 5 \\ 4 \\ -2 \end{pmatrix}, \begin{pmatrix} -4 \\ -9 \\ -9 \\ 5 \end{pmatrix}, \begin{pmatrix} 3 \\ 8 \\ 7 \\ -6 \end{pmatrix} \right\}.$$

Clearly, $\dim B_{colA} = 3$.

Now, solution space of the homogeneous matrix equation $Ax = 0$ gives $N(A)$:

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -2x_2 + 5x_4 \\ x_2 \\ 2x_4 \\ x_4 \\ 0 \end{pmatrix} =$$

$$= x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 5 \\ 0 \\ 2 \\ 1 \\ 0 \end{pmatrix}.$$

So we see that $N(A)$ is spanned by the set $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \\ 2 \\ 1 \\ 0 \end{pmatrix} \right\}$, which forms ~~the~~ a basis for $N(A)$.

$\dim N(A) = 2$: so that $5 = \dim N(A) + \operatorname{rank}(A) = \#$ of columns of A.

20. By Rank - Nullity theorem, rank $A = 5 - \dim N(A)$.

Since $\dim N(A) = 3$, rank $A = 2$.

21. The dimension of the solution space

$Ax = 0$ is the dimension of the null-space

of A. Since rank $A = 4$, by Rank - Nullity

then $\dim N(A) = 6 - 4 = 2$.

§3.1 2, 4, 20, 22

2. $$\begin{vmatrix} 0 & 5 & 1 \\ 4 & -3 & 0 \\ 2 & 4 & 1 \end{vmatrix} = 0 \cdot \begin{vmatrix} -3 & 0 \\ 4 & 1 \end{vmatrix} - 5 \begin{vmatrix} 4 & 0 \\ 2 & 1 \end{vmatrix} + 1 \cdot \begin{vmatrix} 4 & -3 \\ 2 & 4 \end{vmatrix} =$$

$$= -5 \cdot (4 - 2 \cdot 0) + 1 \cdot (16 + 5) = -20 + 21 = 1.$$

4. $$\begin{vmatrix} 1 & 3 & 5 \\ 2 & 1 & 1 \\ 3 & 4 & 2 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & 1 \\ 4 & 2 \end{vmatrix} - 3 \cdot \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} + 5 \begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} =$$

$$= (2 - 4) - 3 \cdot (4 - 3) + 5 (8 - 3) =$$

$$= -2 - 3 + 25 = 20.$$

do.

(4)

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow{k \cdot R_2} \begin{pmatrix} a & b \\ kc & kd \end{pmatrix} = A^*$$

$\det A = ad - cb$

$\det A^* = kad - k \cdot cb = k(ad - cb) = k \cdot \det A$:

Scaling a row by a factor scales the determinant of the original matrix by the same factor.

(22)

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow{R_1 + kR_2} \begin{pmatrix} a + kc & b + kd \\ c & d \end{pmatrix} = A^*$$

$\det A = ad - bc$

$\det A^* = (a + kc)d - c(b + kd) = ad + kcd - cb - kcd =$

$= \det A$.

So: scaling a row by a factor and adding to other row leaves the determinant of the matrix fixed.

§ 2.2. 2, 4, 16, 18, 22, 24, 32, 33, 40.

2. Scaling row 1 by 2 scales the determinant by 2.

4. Scaling row 1 by -3 and adding to row 3 leaves the determinant unchanged.

16.
$$\begin{vmatrix} a & b & c \\ 3d & 3e & 3f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} 3e & 3f \\ h & i \end{vmatrix} - b\begin{vmatrix} 3d & 3f \\ g & i \end{vmatrix} + c\begin{vmatrix} 3d & 3e \\ g & h \end{vmatrix} =$$

$$= a(3ei - 3fh) - b(3di - 3gf) +$$

$$+ c(3dh - 3eg) =$$

$$= 3[a(ei - fh) - b(di - gf) + c(dh - eg)] =$$

$$= 3\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 3 \cdot 7 = 21.$$

18.
$$\begin{vmatrix} g & h & i \\ a & b & c \\ d & e & f \end{vmatrix} = -\begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix} = -\left(-\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}\right) = 7.$$

22. $$\begin{vmatrix} 5 & 0 & -1 \\ 1 & -3 & -2 \\ 0 & 5 & 3 \end{vmatrix} = 5\begin{vmatrix} -3 & -2 \\ 5 & 3 \end{vmatrix} - 0\begin{vmatrix} 1 & -2 \\ 0 & 3 \end{vmatrix} + (-1)\begin{vmatrix} 1 & -3 \\ 0 & 5 \end{vmatrix} =$$

$$= 5(-9+6) + (-1)(5+3\cdot 0) = 5 - 5 = 0 \Rightarrow$$

$\Rightarrow$ matrix not invertible.

24. Let: $\vec{v}_1 = \begin{pmatrix} 4 \\ 6 \\ -7 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} -7 \\ 0 \\ 2 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} -3 \\ -5 \\ 6 \end{pmatrix}$.

We form the matrix $A = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3]$.

By Invertible Matrix theorem, if $\det A \neq 0$, then A is invertible $\Rightarrow$ cols of A, i.e. vectors $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ are lin. independent.

So we check whether $\det A \neq 0$:

$$\begin{vmatrix} 4 & -7 & -3 \\ 6 & 0 & -5 \\ -7 & 2 & 6 \end{vmatrix} = -6\begin{vmatrix} -7 & -3 \\ 2 & 6 \end{vmatrix} - (-5)\begin{vmatrix} 4 & -7 \\ -7 & 2 \end{vmatrix} =$$

$$= -6(-42+6) + 5(8-49) =$$

$$= -6\cdot(-36) + 5(-41) = 216 - 205 = 11 \neq 0.$$

32. $\det(rA) = r^n \cdot \det(A)$:

intuitively, since every entry in A is a multiple of r (and none of the entries are $= 0$) then expanding along some row we can factor r out. When finding det's of co-factors of the row entries (their dimensions are $n-1 \times n-1$) we again expand along some row and can factor r out so that we get r^2 in front of our expression. Continuing this way we get the formula.

33. We know, by theorem 6

$$\det(AB) = \det(A) \cdot \det(B)$$

where $\det(A), \det(B) \in \mathbb{R}$ that commute.

<u>So:</u> $\det(AB) = \det(A) \cdot \det(B) =$

$$= \det(B) \cdot \det(A) = \det(BA).$$

40. (8)

a. -2

b. $2^5 = 32$

c. $2^4(-1) = -16$

d. 1

e. -1

1. $$\begin{pmatrix} s-2t \\ s+t \\ 3t \end{pmatrix} = s\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} \Rightarrow$$

$$B_{\text{subspace}} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} \right\} \quad \text{since}$$

$$\left\{ \begin{pmatrix} s-2t \\ s+t \\ 3t \end{pmatrix} \middle| \; s, t \in \mathbb{R} \right\} = \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} \right\}.$$

$$\dim(\text{subspace}) = \dim(B) = 2.$$