Solutions to Math 201 Exam 2 from spring 2007

1. The formula for finding the slope of the line through points (x_1, y_1) and (x_2, y_2) is $m = \dfrac{y_2 - y_1}{x_2 - x_1}$. Basically, this formula says that you subtract the y coordinates to get the slope's numerator, and you subtract the x coordinates *in the same order* to get the slope's denominator. The points given in this problem are $(-2, 4)$ and $(-3, 1)$, so here's one way to get the right slope in this problem:

$$m = \frac{(4) - (1)}{(-2) - (-3)} = \frac{3}{1} = \boxed{3} \, .$$

The other way would be to do

$$m = \frac{(1) - (4)}{(-3) - (-2)} = \frac{-3}{-1} = 3 \, .$$

2. Making use of the point-slope form, the equation of any nonvertical line passing through the point $(-1, 2)$ can be written as

$$y - 2 = m(x - (-1)) \, , \quad \text{same as} \quad y - 2 = m(x + 1) \, .$$

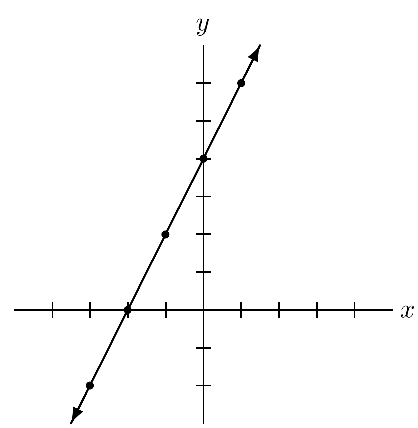
All we have to do now is figure out what the slope of this line is supposed to be. But the problem tells us that it's parallel to the line $y = 2x - 4$, which has a slope of 2. So our new line must also have a slope of 2, and so its equation is

$$y - 2 = 2(x + 1) \, .$$

In order to put the equation in slope-intercept form, we need to multiply out the right hand side and solve for y:

$$\boxed{y = 2x + 4} \, .$$

We know this line has a slope of 2, which means that when we jump from one point to another on this line, the rise (change in y) divided by the run (change in x) will always come out to be 2. We also know the y-intercept is 4, because when we insert 0 for x in the equation, we get $y = 4$. So we can graph this line by putting a point at the y-intercept $(0, 4)$, and then generating a few new points by moving up 2 and right 1 a few times. Also, if we move down 2 and left 1, we should still get points that are on this line. So here it is:



3. The given equation is $y = f(x) = x^2 - 4x - 5$.

To find the x-intercept(s), set $y = 0$ and solve for x:

$$0 = x^2 - 4x - 5 \implies 0 = (x-5)(x+1) \implies x = 5 \text{ or } x = -1 \text{ . So } \boxed{x \text{ intercepts are } (5,0) \text{ and } (-1,0)}.$$

To find y-intercept, set $x = 0$ and solve for y:

$$y = (0)^2 - 4(0) - 5 \implies y = -5 \text{ . So the } \boxed{y\text{-intercept is } (0,-5)}.$$

The x-coordinate of the vertex is often called h, and it's always equal to $-\dfrac{b}{2a}$:

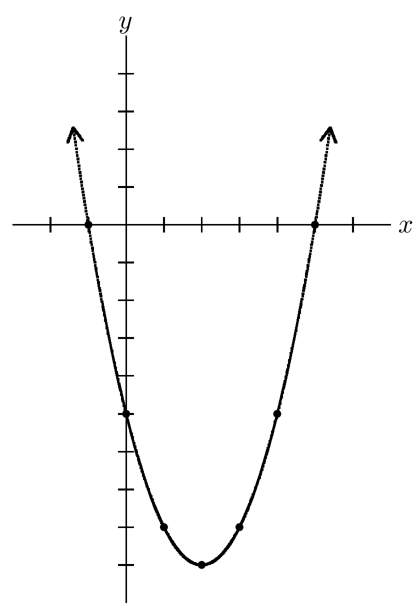
$$h = -\frac{b}{2a} = -\frac{(-4)}{2(1)} = 2 \text{ .}$$

The y-coordinate of the vertex is always $f(h)$:

$$f(h) = f(2) = (2)^2 - 4(2) - 5 = 4 - 8 - 5 = -9 \text{ . So the } \boxed{\text{vertex is at } (2,-9)}.$$

Since this is a quadratic function, we know its graph is a parabola, and since a is positive (a=1 here), it opens up. So we know the range (which is the set of all y values that are involved on the entire graph) will be from the lowest y value of -9 (which occurs at the vertex) up to ∞. That is, $\boxed{\text{the range of } f \text{ is } [-9,\infty)}$. We also know that because $a = 1$, this graph is not stretched or compressed vertically when compared to the graph of $y = x^2$. That is, this graph and the graph of $y = x^2$ have exactly the same shape. Putting all this info together, along with a few extra ordered pairs, gives us a nice graph:

x	y
0	-5
5	0
-1	0
2	-9
3	-8
4	-5
6	7



4. We should recognize the given revenue function, $R = 1800q - 3q^2$, as a *quadratic* function with a negative a value ($a = -3$). This means that if we were to graph this relation with q as input and R as output, we'd see a parabola which opens down. This means that the biggest possible R value would occur at the vertex. So when they ask us to find the q value that maximizes the revenue, they are simply asking us to find the q value at the vertex:

$$h = -\frac{b}{2a} = -\frac{(1800)}{2(-3)} = \boxed{300} \text{ (units) .}$$

5. This is a linear system. There are two methods of elimination that can be used to solve these systems: **substitution** and **addition**. The method of **substitution** might go like this:

original system: $\begin{cases} 2x - \ \ y = 5 & \boxed{\text{A}} \\ 3x + 2y = 18 & \boxed{\text{B}} \end{cases}$

solve eqn $\boxed{\text{A}}$ for y: $2x - y = 5 \implies 2x = 5 + y \implies 2x - 5 = y \quad \boxed{\text{C}}$

use eqn $\boxed{\text{C}}$ to substitute for y in $\boxed{\text{B}}$: $3x + 2(2x - 5) = 18$

solve for x: $3x + 4x - 10 = 18 \implies 7x - 10 = 18 \implies 7x = 28 \implies x = 4.$

go back to $\boxed{\text{C}}$ and solve for y: $y = 2(4) - 5 = 8 - 5 = 3 \implies y = 3.$

So the solution to the system is $\boxed{x = 4 \text{ and } y = 3}$, or you can say $(x, y) = (4, 3)$.

The method of addition might go like this:

original system: $\begin{cases} 2x - \ \ y = 5 & \boxed{\text{A}} \\ 3x + 2y = 18 & \boxed{\text{B}} \end{cases}$

multiply eqn $\boxed{\text{A}}$ by 2: $4x - 2y = 10$

rewrite eqn $\boxed{\text{B}}$ underneath it: $3x + 2y = 18$

add the previous two eqns: $7x + 0y = 28 \implies 7x = 28$

solve for x: $7x = 28 \implies x = 28/7 \implies x = 4.$

go back to $\boxed{\text{A}}$ (or $\boxed{\text{B}}$) to solve for y: $2(4) - y = 5 \implies 8 - y = 5 \implies y = 3.$

So the solution obtained is (again) $\boxed{x = 4 \text{ and } y = 3}$.

6. This is a *non*linear system. It's best to try to solve a nonlinear system using substitution. Since the first equation is already solved for p, we can use it to substitute for p in the second equation, which yields

$$\sqrt{q} = q^2 \ .$$

This is a radical equation, and the radical expression is already isolated, so we should square both sides to get

$$q = q^4 \ .$$

Now, since it has become a polynomial equation, we should move all terms to one side (in order to obtain a zero on the other side) and then attempt to factor:

$$q = q^4 \implies q - q^4 = 0 \implies q(1 - q^3) = 0 \implies q = 0 \text{ or } 1 - q^3 = 0 \implies q = 0 \text{ or } q = 1 \ .$$

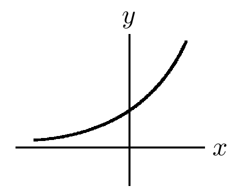
Note that the solution $q = 1$ came from the fact that $1 - q^3 = 0$ is equivalent to $1 = q^3$, which is equivalent to $\sqrt[3]{1} = q$, and $\sqrt[3]{1}$ is 1.

Now, since we are trying to find (p, q) ordered pairs which solve the system, we need a p value to go with each q value. Remembering that the first equation in the system says $p = \sqrt{q}$, we have:

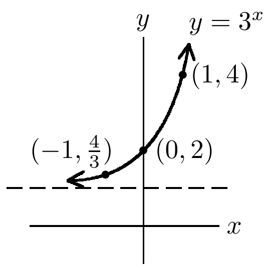
when $q = 0$, $p = \sqrt{0} = 0$; when $q = 1$, $p = \sqrt{1} = 1$.

So the solution pairs for this system are $(p, q) = \boxed{(0, 0) \text{ and } (1, 1)}$.

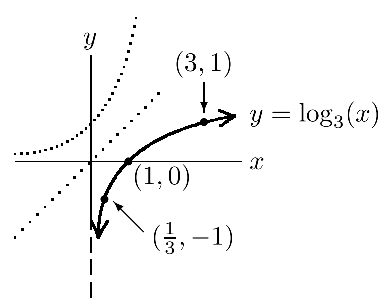
7. a) Recall that any function of the form $y = f(x) = a^x$ (with $a > 0$) is called an "exponential" function. We should know that when we have an exponential function whose a value is greater than 1, its graph looks like the sketch on the right. This generic graph of $y = a^x$ (for $a > 1$) shows that the negative y-axis serves as a *horizontal asymptote*, which is a line that the graph gets infinitely close to as we move away from the origin. When an asymptote does not happen to lie on top of one of the axes, it should be drawn as a dashed line. The graph of $y = a^x$ also goes through the point $(0, 1)$, has a domain of $(-\infty, \infty)$, and a range of $(0, \infty)$.



The graph of $y = 3^x$ looks like the sketch above, and has all the properties listed above. The graph of $y = f(x) = 3^x + 1$ is a shift (one unit up) of the graph of $y = 3^x$. So we now know that the graph of $y = 3^x + 1$ should look like the one on the left. Note how the horizontal asypmtote is now drawn as a dashed line, so that anyone who is looking knows that the graph gets closer and closer to the dashed line as it moves left. Also, three points on the graph are labeled with their coordinates, as requested.



b) The logarithmic functions are the inverses of the exponential functions. In particular, $y = \log_3(x)$ is the inverse of $y = 3^x$, which we dealt with in part (a). Since the inverse of any curve can be obtained by reflection about the line $y = x$, we should know that the graph of $y = \log_3(x)$ is just the reflection (about the line $y = x$) of the graph of $y = 3^x$. So the graph of $y = \log_3(x)$ looks like the graph below. The graphs of $y = 3^x$ and the line $y = x$ are drawn as dashed curves so that the inverse relation between $y = 3^x$ and $y = \log_3(x)$ can be seen. Again, three points are labeled with their coordinates, as requested. The lower y-axis is dashed to indicate that it serves as a vertical asymptote.



FYI, when generating input/output pairs for a logarithmic graph, it can be useful to remember that

$\log_b(x) = y$ is equivalent to $b^y = x$

and that logarithms can be thought of like this:

$\log_b(x) =$ (the power you put on b to get x) .

So, for example, $\log_3(9) =$ (the power you put on 3 to get 9) $= 2$, and $\log_3(\frac{1}{3}) =$ (the power you put on 3 to get $\frac{1}{3}$) $= -1$.

8. Before trying problems like these, it's best if you know the following properties of logarithms:

1) $\log_b(AB) = \log_b(A) + \log_b(B)$

2) $\log_b(A/B) = \log_b(A) - \log_b(B)$

3) $\log_b(A^r) = r\log_b(A)$

4) $\log_b(x) = \dfrac{\log_a(x)}{\log_a(b)}$ ("change-of-base formula")

5) $\log_b(1) = 0$

6) $\log_b(b) = 1$

7) $\log_b(b^r) = r$ for any real number r

8) $b^{\log_b(r)} = r$ for any positive real number r

9) $\log_b(x) = y$ is equivalent to $b^y = x$

10) **log** means $\mathbf{log_{10}}$ and **ln** means $\mathbf{log_e}$.

a) The statement $\log_b(1) = 1$ is $\boxed{\text{FALSE}}$, because $\log_b(1)$ is zero.

b) The statement $\frac{1}{2}\log(x) = \log(\sqrt{x})$ is $\boxed{\text{TRUE}}$ by property (3) above, since $\sqrt{x} = x^{1/2}$.

c) The statement $\log_b(m+n) = \log_b(m) + \log_b(n)$ is $\boxed{\text{FALSE}}$ (this is a messed up version of property (1) above). As a counter-example, $\log_2(4+8)$ is not the same as $\log_2(4) + \log_2(8)$:

$$\log_2(4+8) = \log_2(12) = 3.\text{something}; \qquad \log_2(4) + \log_2(8) = 2 + 3 = 5\,.$$

d) The statement $\ln(x) - \ln(y) = \dfrac{\ln(x)}{\ln(y)}$ is $\boxed{\text{FALSE}}$ (this is a messed up version of property (2) above). As a counter-example, $\ln(1) - \ln(e)$ is not the same as $\dfrac{\ln(1)}{\ln(e)}$:

$$\ln(1) - \ln(e) = 0 - 1 = -1\,; \qquad \frac{\ln(1)}{\ln(e)} = \frac{0}{1} = 0\,.$$

9. By property (8) above, with r replaced by $x+1$ and $b = 2$, we have $2^{\log_2(x+1)} = \boxed{x+1}$.
Note that $x+1$ has to be positive for the original expression to be defined at all. This problem can also be done by remembering the verbal definition of a logarithm. The thought process would go something like this:

The expression $\log_2(x+1)$ represents the power you'd put on 2 to get $x+1$, so if we put this expression on 2 as a power, we get $x+1$:

$$2^{\log_2(x+1)} = 2^{(\text{the power you put on 2 to get } x+1)} = x+1\,.$$

10. Here's one way to think through this problem:

By property (7) above, the 3 and the 4 can be moved inside the logs as exponents:

$$3\ln(x) + 4\ln(y) - \ln(z) = \ln(x^3) + \ln(y^4) - \ln(z)\,.$$

Now, by property (1) above, the first two logs can be rewritten as one:

$$\ln(x^3) + \ln(y^4) - \ln(z) = \ln(x^3y^4) - \ln(z)\,.$$

Finally, by property (2), these two logs can be rewritten as one:

$$\ln(x^3y^4) - \ln(z) = \boxed{\ln\left(\frac{x^3y^4}{z}\right)}.$$

11.
$$\ln\left(\frac{x^3}{(x+1)(x-2)}\right) = \ln(x^3) - \ln((x+1)(x-2)) \quad \text{(by property (2))}$$
$$= \ln(x^3) - [\ln(x+1) + \ln(x-2)] \quad \text{(by property (1))}$$
$$= \boxed{3\ln(x) - \ln(x+1) - \ln(x-2)} \quad \text{(by property (3) and distributing the negative)}$$

12. a) There are several ways of thinking through this equation. Some of them involve property (9) mentioned before the solutions to problem 8. That property says that an expression of the exponential form $\mathrm{b}^{\boxed{}} = k$ can always be rewritten in the logarithmic form $\boxed{} = \log_{\mathrm{b}}(k)$, and vice-versa. All the following solution methods are equivalent in the end, of course.

Method 1:

	original equation:	$3^{2x} = 81$
	rewrite the statement logarithmically:	$2x = \log_3(81)$
	simplify the right side (because you can!):	$2x = 4$
	divide by 2:	$\boxed{x = 2}$.

Note that when we "rewrite" in the second step, we're really just writing down the statement "$2x$ is the power you put on 3 to get 81". Also note that if you did not realize that $\log_3(81)$ is 4, your answer would read $x = \dfrac{\log_3(81)}{2}$, which is correct, though not fully simplified.

Method 2:

	original equation:	$3^{2x} = 81$
	take $\log_3$ of both sides:	$\log_3(3^{2x}) = \log_3(81)$
	rewrite left side using property (7):	$2x = \log_3(81)$
	simplify right side (because you can):	$2x = 4$
	divide by 2:	$\boxed{x = 2}$.

The only difference between method 2 and method 1 is in the thought process of the second lines. In method 1, a "rewriting" is done, whereas in method 2, we took a "do the same thing to both sides" approach. The results are the same, but the thought process is different.

Method 3:

	original equation:	$3^{2x} = 81$
	take ln of both sides:	$\ln(3^{2x}) = \ln(81)$
	rewrite the left side using property (3):	$2x\ln(3) = \ln(81)$
	divide by $2\ln(3)$:	$\boxed{x = \dfrac{\ln(81)}{2\ln(3)}}$.

Note that this answer can be shown to equal 2, because $\ln(81) = \ln(3^4) = 4\ln(3)$, so $\dfrac{\ln(81)}{2\ln(3)} = \dfrac{4\ln(3)}{2\ln(3)} = \dfrac{4}{2} = 2$. Note also that using ln instead of $\log_3$ leaves the answer more calculator-ready, since most (if not all) calculators have only ln and log buttons on them.

Method 4:

	original equation:	$3^{2x} = 81$
	take log of both sides:	$\log(3^{2x}) = \log(81)$
	rewrite the left side using property (3):	$2x\log(3) = \log(81)$
	divide by $2\ln(3)$:	$\boxed{x = \dfrac{\log(81)}{2\log(3)}}$.

The only change here (compared to method 3) is the use of log instead of ln. This answer is also calculator-ready, but this answer can also be shown to equal 2, as was done following method 3 above.

b) Since 2^5 is 32, it follows that $\log_2(32)$ is 5, so the answer they're after is $\boxed{y = 5}$. This is really just a simplification problem. Or, we could rewrite it using property (9): $\log_2(32) = y \implies 2^y = 32 \implies y = 5$.

12. **c)** In a logarithmic equation like this, when there are more than one log term, it's best to try to combine them all into one log term. This is accomplished by getting all the log terms on one side of the equation and then combining them using the properties of logs:

original equation:	$\log_2(x+5) = 3 + \log_2(x-3)$
subtract $\log_2(x-3)$ from both sides:	$\log_2(x+5) - \log_2(x-3) = 3$
combine the logs on LHS using property (2):	$\log_2\left(\dfrac{x+5}{x-3}\right) = 3$

After that, the equation can be rewritten using property (9) so that the log is no longer present, and the variable (in this case x) is exposed:

rewrite without the log (using/property (9)):	$\dfrac{x+5}{x-3} = 2^3$
both sides times $(x-3)$:	$x+5 = 8(x-3)$
distribute on right side:	$x+5 = 8x-24$
both sides $-x$, both sides $+24$:	$29 = 7x$
divide BS by 7:	$\boxed{x = 29/7}$.

Answers to logarithmic equations should always be checked (at least roughly), because log functions have restricted domains. Remember that the output of a log function is only defined if the input number is positive. So we should at least ask ourselves whether plugging in 29/7 (which equals $4\frac{1}{7}$) for x in the original equation will cause the inside of any of the logs to become negative. If we were to plug $4\frac{1}{7}$ in for x in the original equation, we'd get

$$\log_2(4\tfrac{1}{7}+5) = 3 + \log_2(4\tfrac{1}{7}-3) \implies \log_2(9\tfrac{1}{7}) = 3 + \log_2(1\tfrac{1}{7})\ .$$

Since the insides of both logs are positive, this answer should check, provided the previous work is all good.

d) This is another logarithmic equation, since the variable x is present inside of a log. This time, there is only one logarithmic term and it is already isolated on one side of the equation, so we can "rewrite" the equation and go from there:

original equation:	$\log_x(x+6) = 2$
rewrite using property (9):	$x^2 = x+6$
get 0 on one side:	$x^2 - x - 6 = 0$
factor the left side:	$(x-3)(x+2) = 0$
set each factor equal to 0:	$x-3=0$ or $x+2=0$
finish:	$x=3$ or $x=-2$

Before boxing the answer(s), remember that answers to logarithmic equations should be checked. Note that if we plug $x = 3$ into the original equation, we get

$$\log_3(9) = 2\ ,$$

which is a true statement. However, if we plug $x = -2$ into the original equation, we get

$$\log_{-2}(4) = 2\ .$$

While it is true that $(-2)^2$ is 4, logs are *never* permitted to have negative bases. So, since x is the base of the logarithm in the original equation, the solution $x = -2$ is not permitted, and the only solution to the original equation is $\boxed{x = 3}$.

e) We are being asked to solve the equation, and then try to put your answer in terms of ln(2), ln(9) and ln(7). So here goes:

original equation:	$9(2^x) = 7$
divide both sides by 9:	$2^x = 7/9$
take ln of both sides:	$\ln(2^x) = \ln(7/9)$
rewrite left side:	$x\ln(2) = \ln(7/9)$
divide both sides by ln(2):	$x = \dfrac{\ln(7/9)}{\ln(2)}$

Now that it's solved, there's a little work left to write the answer in terms of ln(2), ln(9) and ln(7). The ln(7/9) in the numerator of the large fraction can be broken into two logs using property (3): $\ln(7/9) = \ln(7) - \ln(9)$. So the whole answer can be written as

$$x = \frac{\ln(7/9)}{\ln(2)} = \frac{\ln(7) - \ln(9)}{\ln(2)}.$$

Finally, we're told that ln(2) is called a, ln(9) is called b, and ln(7) is called c. So we put these letters in for the quantities they represent, and we're done:

$$x = \frac{\ln(7) - \ln(9)}{\ln(2)} = \boxed{\frac{c-b}{a}}.$$

13. When they say they want the investment to triple, this means that A (the end value of the investment) is supposed to be equal to $3P$ (three times the initial value of the investment). So:

equation:	$3P = P(1.075)^t$
divide both sides by P:	$3 = (1.075)^t$
take ln of both sides:	$\ln(3) = \ln((1.075)^t)$
rewrite the right side using property (3):	$\ln(3) = t\ln(1.075)$
divide both sides by ln(1.075):	$\boxed{\dfrac{\ln(3)}{\ln(1.075)} = t}.$