Solutions to Math 201 Exam 1 from spring 2007

1. The given equation is $\frac{2}{3}y + \frac{1}{2}(y-3) = \frac{y+1}{4}$.

Since the denominators all divide into 12, it will help to first multiply both sides by 12 in order to eliminate all the fractions:

original equation: $\frac{2}{3}y + \frac{1}{2}(y-3) = \frac{y+1}{4}$

BS·12: $12 \cdot \frac{2}{3}y + 12 \cdot \frac{1}{2}(y-3) = 12 \cdot \frac{y+1}{4}$

$8y + 6(y-3) = 3(y+1)$

multiply stuff out: $8y + 6y - 18 = 3y + 3$

$14y - 18 = 3y + 3$

BS $-3y$: $11y - 18 = 3$

BS $+18$: $11y = 21$

BS $/11$: $\boxed{y = \frac{21}{11}}$.

2. The given equation is $A = P + Prt$.

In order to solve for r, treat all other variables (A, P and t) as if they are constants (numbers). So you can think of the equation as if it said this:

$10 = 7 + 7 \cdot r \cdot 5$.

In order to solve this last equation, you could follow these steps:

multiply the 7 and 5 together in the $7 \cdot r \cdot 5$ term to get $10 = 7 + 35r$.;
subtract 7 from both sides to get $3 = 35r$;
divide both sides by 35 to get $\frac{3}{35} = r$.

So to solve the original equation, follow the same steps with all the variables in place:

original equation: $A = P + Prt$

multiply P and t together in the Prt term: $A = P + (Pt)r$

BS $-P$: $A - P = (Pt)r$

BS $/(Pt)$: $\boxed{\frac{A-P}{Pt} = r}$.

3. The given equation is $\sqrt{x+6} - x = 4$. Since this is a radical equation, a good strategy is to isolate the radical, then square both sides, and proceed from there:

original equation: $\sqrt{x+6} - x = 4$

BS$+x$: $\sqrt{x+6} = x + 4$

$(\text{BS})^2$: $(\sqrt{x+6})^2 = (x+4)^2$

$x + 6 = x^2 + 8x + 16$

It's now quadratic, so get a zero on one side and then try to factor:

BS$-x-6$:	$0 = x^2 + 7x + 10$
factor:	$0 = (x+2)(x+5)$
possible answers:	$x = -2$, $x = -5$

Remember that when solving any radical equation, you should check your answers, because squaring both sides can result in extraneous solutions. So we shoud check each answer in the *original* equation:

When $x = -2$, the equation says $\sqrt{(-2)+6} - (-2) = 4$, which is equivalent to $\sqrt{4} + 2 = 4$, which is true. So $x = -2$ *is* a solution.

When $x = -5$, the equation says $\sqrt{(-5)+6} - (-5) = 4$, which is equivalent to $\sqrt{1} + 5 = 4$, which is false. So $x = -5$ is *not* a solution.

So the only solution to the original equation is $\boxed{x = -2}$.

4. The given equation is $x^4 - 6x^2 + 5 = 0$. Since x^4 is the same as $(x^2)^2$, we are supposed to recognize that if we replace every x^2 in the equation by a u, the equation becomes

$$u^2 - 6u + 5 = 0\,,$$

which is a quadratic equation that we can solve for u. And of course, once we know the solution value(s) for u, we can figure out what the solution values for x have to be. So here goes. Factoring the left-hand side of the above equation gives us

$$(u-5)(u-1) = 0\,,$$

so $u = 1$ or $u = 5$. Since $u = x^2$, we need $x^2 = 1$ and $x^2 = 5$, which gives us solutions (for x) of $\boxed{x = \pm 1 \text{ or } x = \pm\sqrt{5}}$.

The use of u is not mandatory in this problem. The original equation can be factored directly as $(x^2-1)(x^2-5) = 0$, and then solved from there. Whether you use a u or not is a matter of personal preference. I (Dr. Remaley) suggest that you use whichever method makes more sense to you.

5. This is a linear inequality. Linear inequalities are solved using the same steps as linear equations. The general strategy is to get all the x terms combined on one side and all the constant terms combined on the other side, and then divide (or multiply) by an appropriate number to get the final answer. Here goes:

original inequality:	$2(4x+3) \geq 8 - 4(x-1)$
multiply stuff out:	$8x + 6 \geq 8 - 4x + 4$
simplify a little more:	$8x + 6 \geq 12 - 4x$
BS$+4x$:	$12x + 6 \geq 12$
BS-6:	$12x \geq 6$
BS/12:	$\boxed{x \geq \frac{1}{2}}$.

6. This is an absolute value inequality. There are many different ways to arrive at the correct solution. I'll go through my favorite way. I like to use the ideas shown in table 1.1 at the bottom of page 62 in your text. However, I like to restate the table as shown below. The $\boxed{}$ can be any expression.

Inequality ($d > 0$)	Equivalent statement
$\left\lvert\boxed{}\right\rvert < d$	$-d < \boxed{} < d$
$\left\lvert\boxed{}\right\rvert \leq d$	$-d \leq \boxed{} \leq d$
$\left\lvert\boxed{}\right\rvert > d$	$\boxed{} < -d$ or $d < \boxed{}$
$\left\lvert\boxed{}\right\rvert \geq d$	$\boxed{} \leq -d$ or $d \leq \boxed{}$

This particular problem fits the form given in the last line of the table. So we can rewrite the given inequality as follows:

$$\left|\frac{8-3x}{-4}\right| \geq 3 \text{ is equivalent to } \frac{8-3x}{-4} \leq -3 \text{ or } 3 \leq \frac{8-3x}{-4}\,.$$

Now, we solve the two new linear inequalities:

start with: $\dfrac{8-3x}{-4} \leq -3$ or $3 \leq \dfrac{8-3x}{-4}$

BS$\cdot(-4)$: $8-3x \geq 12$ or $-12 \geq 8-3x$

BS-8: $-3x \geq 4$ or $-20 \geq -3x$

BS$/(-3)$: $\boxed{x \leq \dfrac{-4}{3} \text{ or } \dfrac{20}{3} \leq x}$

If you were asked to give the solution set in interval notation, it would be $\boxed{\left(-\infty, \dfrac{-4}{3}\right] \cup \left[\dfrac{20}{3}, \infty\right)}$.

7. When a function rule is given, remember that the x (or whatever variable is being used) is really just a sort of place-holder. The function given here,

$$f(x) = 2x - 3$$

can be thought of as saying

$$f(\text{whatever}) = 2(\text{whatever}) - 3\,.$$

The "whatever" can be any expression. In this problem, we're asked to figure out what

$$\frac{f(x+h) - f(x)}{h}$$

is equal to. The $f(x)$ part in the numerator equals $2x-3$. So we'll replace the $f(x)$ by $2x-3$. The $f(x+h)$ part can be figured out by thinking as mentioned above. The "whatever" is the expression $(x+h)$, so we have

$$f(x+h) = 2(x+h) - 3\,.$$

So the overall calculation goes like this:

$$\frac{f(x+h)-f(x)}{h} = \frac{[2(x+h)-3]-[2x-3]}{h} = \frac{2x+2h-3-2x+3}{h} = \frac{2h}{h} = \boxed{2}\,.$$

The last equality (where $2h/h$ becomes 2) is valid as long as h is not zero.

8. $f(x) = \dfrac{x}{2x^2 - 4x}$ and $g(x) = \begin{cases} x - 4, & x < 2 \\ x^2, & x > 2 \end{cases}$.

a) Since f is a *rational* function, we know that its domain is the set of all x values which do not make the denominator become zero. To see what x avlues *do* make the denominator zero, we set the denominator equal to zero and solve:

$$2x^2 - 4x = 0 \implies 2x(x-2) = 0 \implies x = 0 \text{ or } 2\,.$$

So the domain of f is $\boxed{\text{all real numbers except 0 and 2}}$. You could just write $x \neq 0$, $x \neq 2$. If you were asked to give this answer in interval notation, it would be given like this: $(-\infty, 0) \cup (0, 2) \cup (2, \infty)$.

b) $f(-1)$ is $\dfrac{(-1)}{2(-1)^2 - 4(-1)} = \dfrac{-1}{2+4} = \boxed{\dfrac{-1}{6}}$.

c) Since g tells us how to calculate outputs for x values greater than 2 and for x values less than 2, but does *not* tell us what to do if x equals 2, the domain of g is $\boxed{\text{all real numbers except 2}}$.

d) Since -1 is less than 2, we use the upper case in the definition of g: $g(-1) = (-1) - 4 = \boxed{-5}$.

9. A third degree polynomial (in x) is any expression of the form

$$a_3x^3 + a_2x^2 + a_1x + a_0\,,$$

where only a_3 must be nonzero. So here are a few possible answers: $\boxed{x^3}$, $\boxed{2x^3 + 7x^2 - 5x + 1}$, $\boxed{10x - \dfrac{x^3}{4}}$.

10. Any expression of the form $\sqrt{\square}$ has a defined output only when $\square \geq 0$. So for the function $f(x) = \sqrt{x-1}$, the domain is $x - 1 \geq 0 \implies \boxed{x \geq 1}$. If you were asked for the answer in interval notation, it would be $[1, \infty)$.

11. a) "Break even" means (total revenue)=(total costs). So if we can get variable expressions for total revenue and total cost, then we'll have an equation set up. In this problem, since we are asked "How many figurines should be sold...", it is logical to try letting the variable be

$x =$ the number of figurines made and sold in a given month

(remember that in this course, we always assume that everything that's made is sold). Since the figurines sell for \$8 each, the total revenue per month is $8x$. Since the figurines cost \$3 each to make and fixed costs are \$1500 per month, total cost per month is $3x + 1500$. So the (total revenue)=(total cost) equation becomes

$$8x = 3x + 1500,$$

which is solved by subtracting $3x$ from both sides (which gives $5x = 1500$) and then dividing both sides by 5, which gives $x = 300$. So $\boxed{\text{300 figurines must be sold in a month to break even}}$.

For the second part, since (profit)=(total revenue)-(total cost), and we want profit to be \$6500, the new equation is

$$8x - (3x + 1500) = 6500,$$

which simplifies to $5x - 1500 = 6500$. After adding 1500 and then dividing by 5 on both sides, we have $x = 1600$. So to have a profit of \$6500, $\boxed{\text{1600 figurines must be sold in a month}}$.

11. b) If we let w represent the width of the rectangle, then $2w$ is the length. Since the area is to be 800, we have

$$(w)(2w) = 800\,, \text{ which is equivalent to } 2w^2 = 800\,.$$

If we divide both sides by 2, we have $w^2 = 400$, and the mathematical solutions to this equation are $w = \pm\sqrt{400} = \pm 20$. Of course, w represents a physical length and can not be negative, so the only acceptable answer is $w = 20$. So the dimensions of the fence should be $\boxed{\text{20 feet by 40 feet}}$, and the total amount of fencing needed is $20+40+20+40 = \boxed{\text{120 feet}}$.

12. a) By definition, $(f+g)(x) = f(x) + g(x) = (x^2 - x) + (2x) = \boxed{x^2 + x}$.

b) By definition, $(f-g)(x) = f(x) - g(x) = (x^2 - x) - (2x) = \boxed{x^2 - 3x}$.

c) Two ways to get this answer:

1) Going back to the definition of $(f-g)$: $(f-g)(4) = f(4) - g(4) = ((4)^2 - (4)) - (2(4)) = 12 - 8 = \boxed{4}$.

2) From the simplified answer in part (b): $(f-g)(x) = x^2 - 3x$, so $(f-g)(4) = (4^2) - 3(4) = 16 - 12 = \boxed{4}$.

d) By definition, $(fg)(x) = f(x) \cdot g(x) = (x^2 - x)(2x) = \boxed{2x^3 - 2x^2}$.

e) By definition, $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x^2 - x}{2x}$. Note that this output is not defined when x is 0. So if you reduce this answer like this:

$$\frac{x^2 - x}{2x} = \frac{x(x-1)}{2x} = \frac{x-1}{2}\,,$$

you technically should state the restriction that x can not be zero, since that restriction is lost when the cancellation is done. The point here is that saying

$$\frac{f(x)}{g(x)} = \frac{x^2 - x}{2x}$$

is the same as saying

$$\frac{f(x)}{g(x)} = \frac{x-1}{2}\,, \quad x \neq 0\,.$$

The $x \neq 0$ part has to be there so that the two renditions of $\frac{f(x)}{g(x)}$ have the same domain.

f) By definition, $(f \circ g)(x) = f(g(x)) = f(2x) = (2x)^2 - (2x) = \boxed{4x^2 - 2x}$.

g) By definition, $(g \circ f)(-4) = g(f(-4))$. Since $f(-4)$ is $(-4)^2 - (-4) = 16 + 4 = 20$, $g(f(-4))$ becomes $g(20) = 2(20) = \boxed{40}$.

h) As mentioned in part (e), the definition of $\frac{f}{g}$ is

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x^2 - x}{2x}\,,$$

which is undefined only when $x = 0$. So its domain is $\boxed{x \neq 0}$.

13. To find the inverse of a function, replace $f(x)$ by y, exchange x and y, re-solve the equation for y, and then replace y by $f^{-1}(x)$:

$$f(x) = 3x + 5 \implies y = 3x + 5 \implies x = 3y + 5 \implies x - 5 = 3y \implies \frac{x-5}{3} = y \implies f^{-1}(x) = \boxed{\frac{x-5}{3}}.$$

14. Relation: $5x^2 - 2xy + y^2 = 0$.

The test for x-axis symmetry is to replace every y by $(-y)$ and then see if simplifying away the negative signs can get the relation back to the exact form it started in. So here goes:

$$5x^2 - 2x(-y) + (-y)^2 = 0 \implies 5x^2 + 2xy + y^2 = 0 \ .$$

This last equation is not equivalent to the original one, because in order to change the $+2xy$ term to $-2xy$, we'd have to multiply both sides by -1, which would change the signs on the $5x^2$ and y^2 terms. So the original relation has $\boxed{\text{no } x\text{-axis symmetry}}$.

The test for y-axis symmetry is to replace every x by $(-x)$ and then see if simplifying away the negative signs can get the relation back to the exact form it started in:

$$5(-x)^2 - 2(-x)y + y^2 = 0 \implies 5x^2 + 2xy + y^2 = 0 \ .$$

This last equation is not equivalent to the original one, for the same reason as in the earlier test. So the original relation has $\boxed{\text{no } y\text{-axis symmetry}}$.

The test for origin symmetry is to replace every y by $(-y)$ and replace every x by $(-x)$ then see if simplifying away the negative signs can get the relation back to the exact form it started in:

$$5(-x)^2 - 2(-x)(-y) + (-y)^2 = 0 \implies 5x^2 - 2xy + y^2 = 0 \ .$$

This last equation is equivalent to the original one (the changes all simplified away). So the original relation $\boxed{\text{does have origin symmetry}}$.

15. Original curve: $4x^2 + y^2 = 16$

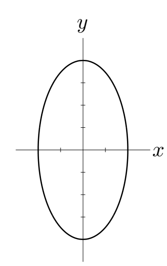
When x is set to 0, the formula gives $y^2 = 16 \implies y = \pm 4$. So the $\boxed{y\text{-intercepts are at } (0,4) \text{ and } (0,-4)}$.

When y is set to 0, the formula gives $4x^2 = 16 \implies x = \pm 2$. So the $\boxed{x\text{-intercepts are at } (2,0) \text{ and } (-2,0)}$.

When x is replaced by $(-x)$ in the formula, the negative is squared away, so this curve $\boxed{\text{does have } y\text{-axis symmetry}}$.

When y is replaced by $(-y)$ in the formula, the negative is squared away, so this curve $\boxed{\text{does have } x\text{-axis symmetry}}$.

When x and y are replaced by $(-x)$ and $(-y)$ respectively, the negatives are squared away, so this curve $\boxed{\text{does have origin symmetry}}$.



16. **a)** Turning $y = |x|$ into $y = |x+1|$ shifts its graphs one unit to the *left*. So $\boxed{\text{graph 5}}$.

b) Turning $y = x^2$ into $y = 2x^2$ vertically stretches its graph. Turning $y = 2x^2$ into $y = 2x^2 + 1$ shifts its graph one unit up. So $\boxed{\text{graph 3}}$.

c) Turning $y = \sqrt{x}$ into $y = \sqrt{x-1}$ shifts its graph one unit right. Turning $y = \sqrt{x-1}$ into $y = -\sqrt{x-1}$ reflects its graph about the x axis. So $\boxed{\text{graph 1}}$.

d) $y = \dfrac{1}{x}$ has $\boxed{\text{graph 4}}$. (no transforming involved)

e) Turning $y = x$ into $y = 3x$ is a vertical stretch of its graph. Turning $y = 3x$ into $y = 3x - 1$ shifts its graph one unit down. So $\boxed{\text{graph 8}}$. You might also know (we'll study this in chapter 3) that any relation of the form $y = ax + b$ has a straight-line graph, and there's only one straight-line graph given as an answer choice.