NULLSPACES OF MATRICES AND THEIR COMPOUNDS

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Abstract

We consider compound matrices and exterior products in order to generalize a fundamental relation between the eigenspace of a matrix A corresponding to a simple eigenvalue λ and certain minors of $A - \lambda I$ of order $n - 1$. By means of the Laplace Expansion Theorem, we show that if the geometric multiplicity of λ is $k > 1$, then the exterior product of k linearly eigenvectors corresponding to λ is uniquely (up to scalar multiples) determined by certain minors of $A - \lambda I$ of order $n - k$. Additional results are included on the nullspace of a compound matrix.

1 Introduction

Let $A = (a_{ij}) \in \mathcal{R}^{n\times n}$ be a matrix of rank $n-1$ and let $\langle n\rangle = \{1, 2, \ldots, n\}$. Expanding the determinant of A with respect to the i–th row of A one obtains

$$detA = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} detA(i\,|\,j) = 0, \tag{1.1}$$

where $A(i\,|\,j)$ represents the submatrix of A obtained by deleting the i–th row and j–th column. Furthermore, for any row $m \neq i$ we have that

$$\sum_{j=1}^{n} (-1)^{m+j} a_{mj} detA(i\,|\,j) = 0, \tag{1.2}$$

since the left hand side of (1.2) represents the expansion with respect to the m–th row of the determinant of a matrix with two rows identical (this is called an expansion by *alien* cofactors in [A]). Let

$$w = \left[-detA(i\,|\,1), detA(i\,|\,2), \ldots, (-1)^n detA(i\,|\,n)\right]^T. \tag{1.3}$$

From equations (1.1) and (1.2) we have that

$$Aw = 0. \tag{1.4}$$

If row i is chosen so that rows $\langle n\rangle \setminus \{i\}$ form a basis of the row space of A, then $w \neq 0$ and therefore w is the unique (up to scalar multiples) eigenvector of A corresponding to 0. We can use this approach to find an eigenvector corresponding to any eigenvalue λ of geometric multiplicity 1 by considering the matrix $A - \lambda I$ of rank $n - 1$.

Equations (1.3) and (1.4) give a fundamental relation between certain cofactors of A and the nullspace of A, denoted by $N(A)$. *We are interested in generalizing this relation to the case where the nullity is $k > 1$.* For that purpose we shall consider some of the basic tools of multilinear algebra. In Section 2 we establish the necessary notation which is mainly drawn from [F] and [M], with some exceptions, and we present fundamental results for reference purposes. Section 3 contains our main result (Theorem 3.1), which links the nullspace of the k–th compound of A to an exterior product of vectors in the row space of A, denoted by $R(A^T)$. Further results on the relation between the nullspace of a matrix and the nullspaces of its compounds are obtained in Section 4.

In the remainder of this paper we take our matrices to be real but, as it will be noted, our results also apply to matrices over the complex field.

2 Compounds and Exterior Products

Given positive integers $n,\ \ell$ with $\ell \leq n$ we denote by $Q_{\ell,n}$ the ℓ–tuples of $\langle n\rangle$ with elements in increasing order. $Q_{\ell,n}$ has $\binom{n}{\ell}$ members which we will assume are ordered lexicographically. For any $\emptyset \subseteq \alpha \subseteq \langle n\rangle$, we denote by α^c the complement of α with respect to $\langle n\rangle$, by $|\,\alpha\,|$ the cardinality of α and by $s(\alpha)$ the sum of the indices in α, e.g., if $\alpha = \{1,2,3\}$ then $s(\alpha) = 6$. For any matrix $B \in \mathcal{R}^{m\times n}$ and $\emptyset \subset \alpha \subseteq \langle m\rangle,\ \emptyset \subset \beta \subseteq \langle n\rangle$, we let $B[\alpha\,|\,\beta]$ denote the submatrix of B in rows and columns indexed by α and β, respectively. We write $B[\alpha]$ for $B[\alpha\,|\,\alpha]$ and $B(\alpha\,|\,\beta)$ for $B[\alpha^c\,|\,\beta^c]$. Given an integer $0 < \ell \leq \min(m,n)$, the *ℓ–th compound* of B is defined as the $\binom{m}{\ell} \times \binom{n}{\ell}$ matrix

$$B^{(\ell)} = (det B[\alpha\,|\,\beta])_{\alpha\in Q_{\ell,m},\beta\in Q_{\ell,n}}.$$

If α is lexicographically the i–th ℓ–tuple of $\langle m\rangle$ and β the j–th ℓ–tuple of $\langle n\rangle$, then $det B[\alpha\,|\,\beta]$ is the (i,j)–th entry of $B^{(\ell)}$, which we denote by $B^{(\ell)}_{\alpha,\beta}$.

For $u_i \in \mathcal{R}^n,\ i = 1,2,\ldots,\ell$, the *exterior product* $u_1 \wedge \ldots \wedge u_\ell$ is defined in [F] to be the $\binom{n}{\ell}$–component vector equal to the ℓ–th compound of the matrix $U = [u_1\,|\,u_2\,|\,\ldots\,|\,u_\ell]$, that is,

$$u_1 \wedge \ldots \wedge u_\ell = \bigwedge_{i\in\langle\ell\rangle} u_i \equiv U^{(\ell)} = (det U[\alpha\,|\,\langle\ell\rangle])_{\alpha\in Q_{\ell,n}}.$$

Similarly, we now introduce for notational purposes the *supplementary exterior product* of the $u_i,\ i = 1,2,\ldots,\ell$, as the $\binom{n}{\ell}$–component vector

$$u_1 \wedge' \ldots \wedge' u_\ell = {\bigwedge_{i\in\langle\ell\rangle}}' u_i \equiv \Big((-1)^{s(\alpha)+s(\langle\ell\rangle^c)} det U(\alpha\,|\,\langle\ell\rangle^c)\Big)_{\alpha\in Q_{n-\ell,n}}.$$

The following fundamental properties of the exterior product can be found in Chapter 6 of [F]. All vectors are in $\mathcal{R}^n$.

$$\text{If } \sigma \in \mathcal{S}_n, \text{ then } u_1 \wedge \ldots \wedge u_\ell = sgn(\sigma)(u_{\sigma(1)} \wedge \ldots \wedge u_{\sigma(\ell)}). \tag{2.1}$$

$$u_1 \wedge \ldots \wedge u_\ell = 0 \iff u_1 \ldots, u_\ell \text{ are linearly dependent.} \tag{2.2}$$

$$(u + u_1) \wedge u_2 \wedge \ldots \wedge u_\ell = (u \wedge u_2 \wedge \ldots \wedge u_\ell) + (u_1 \wedge u_2 \wedge \ldots \wedge u_\ell). \tag{2.3}$$

$$\text{If } \mu \in \mathcal{R}, \text{ then } (\mu u_1) \wedge u_2 \wedge \ldots \wedge u_\ell = \mu(u_1 \wedge u_2 \wedge \ldots \wedge u_\ell). \tag{2.4}$$

$$A^{(\ell)}(u_1 \wedge \ldots \wedge u_\ell) = Au_1 \wedge \ldots \wedge Au_\ell. \tag{2.5}$$

A vector $u \in \mathcal{R}^{\binom{n}{\ell}}$ is called *decomposable* if there exist $u_i \in \mathcal{R}^n,\ i = 1, 2, \ldots, \ell$, such that

$$u = u_1 \wedge \ldots \wedge u_\ell.$$

We refer to $u_1, \ldots, u_\ell$ as the factors of u. Not all vectors are decomposable. For example, $u = [1, 0, 0, 0, 0, 1]^T$ is not decomposable because

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix}^{(2)} = [1, 0, 0, 0, 0, 1]^T$$

gives an inconsistent system of 6 equations in 8 unknowns.

Consider now an ℓ–dimensional subspace S spanned by $\{x_1, \ldots, x_\ell\}$. By (2.2) we have that

$$S = \{x \in \mathcal{R}^n\ ;\ x \wedge x_1 \wedge \ldots \wedge x_\ell = 0\}, \tag{2.6}$$

(see also Theorem 4.1.6 in [M]), and therefore two ℓ–dimensional subspaces spanned by $\{x_1, \ldots, x_\ell\}$ and $\{y_1, \ldots, y_\ell\}$, respectively, are the same if and only if for some nonzero $\mu \in \mathcal{R}$ we have

$$x_1 \wedge \ldots \wedge x_\ell = \mu y_1 \wedge \ldots \wedge y_\ell. \tag{2.7}$$

If $X = \{x_1, \ldots, x_n\}$ is a basis for $\mathcal{R}^n$, then the set

$$Y = \{\bigwedge_{i \in \alpha} x_i\ ;\ \alpha \in Q_{\ell,n}\} \tag{2.8}$$

constitutes a basis of $\mathcal{R}^{\binom{n}{\ell}}$ since, by (2.7) and the comments preceding it, Y consists of $\binom{n}{\ell}$ linearly independent vectors. Furthermore, the members of Y are precisely the columns of the ℓ–th compound of the matrix $[x_1 \,|\, \ldots \,|\, x_n]$. As a consequence, by Theorem 6.16 in [F], if X is an orthogonal basis of $\mathcal{R}^n$, then Y is also an orthogonal basis of $\mathcal{R}^{\binom{n}{\ell}}$.

Similar properties and results as the ones described in (2.1)–(2.8) hold for the supplementary exterior product as well.

We now present an example, in order to illustrate these definitions and also motivate our analysis in the following section.

Example 2.1 Consider the following matrix and its second compound

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}, \quad A^{(2)} = \begin{bmatrix} 0 & -1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 & -1 & 0 \end{bmatrix}.$$

The nullspace of A is spanned by $v_1 = [1, -1, 0, 0]^T$ and $v_2 = [0, 0, 1, -1]^T$. The exterior product of v_1 and v_2 is

$$v_1 \wedge v_2 = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}^{(2)} = [0, 1, -1, -1, 1, 0]^T.$$

In addition, the supplementary exterior product of the first and second row of A is given by

$$w \ = \ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \ \wedge' \ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \ = \ [0, 1, -1, -1, 1, 0]^T.$$

We can now observe that $w = v_1 \wedge v_2$ (and that $w \in N(A^{(2)})$), namely, the exterior product of two linearly independent eigenvectors of A that span $N(A)$ is equal to the supplementary exterior product of two linearly independent vectors that span $R(A^T)$. $\square$

In the proof of our main result we shall also make use of the following. For any square matrix $B \in \mathcal{R}^{n\times n}$ and an integer $0 \le \ell < n$ we define the *supplementary ℓ-th compound* of B to be the $\binom{n}{\ell} \times \binom{n}{\ell}$ matrix

$$B^{[\ell]} = \Big((-1)^{s(\alpha)+s(\beta)} det B(\alpha \,|\, \beta) \Big)_{\alpha,\beta \in Q_{\ell,n}}.$$

Observe that the $(n-\ell)$–th compound of B and its ℓ–th supplementary compound are related by

$$B^{[\ell]}_{\alpha,\beta} = (-1)^{s(\alpha)+s(\beta)} B^{(n-\ell)}_{\alpha^c,\beta^c}.$$

Also notice that, by definition, $(B^T)^{(\ell)} = (B^{(\ell)})^T$ and therefore the Laplace Expansion Theorem (see [M], Theorem 4.3.3) can be stated as

$$(B^{(\ell)})^T B^{[\ell]} = (det B) I, \tag{2.9}$$

where I is the identity matrix of order $\binom{n}{\ell}$.

3 Main Result

The observations in the introduction can be generalized to the case where the nullity of $A \in \mathcal{R}^{n\times n}$ is $k \ge 1$, as indicated in Example 2.1. Let $A_1, \ldots, A_n$ denote the rows of A and let $\alpha \in Q_{k,n}$ such that $\{A_i \;;\; i \in \alpha^c\}$ is a basis for the row space of A. Consider then the $\binom{n}{k}$–component vector

$$w = \bigwedge_{i\in\alpha^c}{}' A_i^T \neq 0. \tag{3.1}$$

Let $p = \binom{n}{k}$ and suppose $\beta_1, \ldots, \beta_p$ denote the members of $Q_{k,n}$. We then have from (3.1) and the definition of the supplementary exterior product that

$$w = \left[(-1)^{s(\alpha)+s(\beta_1)} detA(\alpha\,|\,\beta_1), \ldots, (-1)^{s(\alpha)+s(\beta_p)} detA(\alpha\,|\,\beta_p)\right]^T.$$

By the Laplace Expansion Theorem (equation (2.9) applied to A for $\ell = k$) we obtain

$$detA = \sum_{i=1}^{p} (-1)^{s(\alpha)+s(\beta_i)} detA[\alpha\,|\,\beta_i] detA(\alpha\,|\,\beta_i) = 0 \tag{3.2}$$

and also for any $\gamma \in Q_{k,n}$ such that $\gamma \neq \alpha$ we have (alien cofactors) that

$$\sum_{i=1}^{p} (-1)^{s(\gamma)+s(\beta_i)} detA[\alpha\,|\,\beta_i] detA(\gamma\,|\,\beta_i) = 0. \tag{3.3}$$

By (3.2) and (3.3) we get $A^{(k)}w = 0$ and, as $w \neq 0$, we have that w is an eigenvector of $A^{(k)}$ corresponding to 0. From (2.2) and (2.5) it is clear that any decomposable vector with k factors and at least one factor in $N(A)$, belongs to $N(A^{(k)})$. Our aim in the next theorem, our main result, is to show that w in (3.1) is not only an eigenvector of $N(A^{(k)})$ corresponding to 0, but further, that w is decomposable and in fact equal (up to scalar multiples) to the exterior product of any set of k linearly independent eigenvectors of A corresponding to 0.

Theorem 3.1 *Let $A \in \mathcal{R}^{n\times n}$ have nullity $k \geq 1$. Given any basis $\{v_1, \ldots, v_k\}$ of $N(A)$ and any set $\{A_i \;;\; i \in \beta,\ \beta \in Q_{n-k,n}\}$ of linearly independent rows of A, there exists $\mu \in \mathcal{R}$ such that*

$$\bigwedge_{i\in\langle k\rangle} v_i = \mu \bigwedge_{i\in\beta}{}' A_i^T.$$

Proof. Without loss of generality suppose that $\{v_1, \ldots, v_k\}$ is an orthogonal basis for $N(A)$ and that $V = \{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ is a completion to an orthogonal basis of $\mathcal{R}^n$. Define the nonsingular matrix

$$\mathcal{A} = [v_1\,|\,\ldots\,|\,v_n] \in \mathcal{R}^{n\times n}.$$

Let $v = v_1 \wedge \ldots \wedge v_k$ and $w = v_{k+1} \wedge' \ldots \wedge' v_n$. Notice that as $v_{k+1}, \ldots, v_n$ span $(N(A))^{\perp} = R(A^T)$, by the discussion leading up to (2.7), the proof of the theorem will be completed if we show that v is a scalar multiple of w. By definition of the exterior and supplementary exterior products we have that

$$v = (det\mathcal{A}[\alpha \,|\, \langle k\rangle])_{\alpha \in Q_{k,n}} \quad \text{and} \quad w = \Big((-1)^{s(\alpha)+s(\langle k\rangle)} det\mathcal{A}(\alpha \,|\, \langle k\rangle)\Big)_{\alpha \in Q_{k,n}}.$$

As in (2.8), $\{\bigwedge_{i\in\alpha} v_i \ ; \ \alpha \in Q_{k,n}\}$ is an (orthogonal) basis of $\mathcal{R}^{\binom{n}{k}}$ and therefore there exist $\binom{n}{k}$ constants $\mu_\alpha \in \mathcal{R}$ such that

$$w = \sum_{\alpha \in Q_{k,n}} \mu_\alpha (\bigwedge_{i\in\alpha} v_i). \tag{3.4}$$

We will now evaluate the usual inner product $\langle v, w\rangle$. Observe that v is the first column of $\mathcal{A}^{(k)}$ and that w is the last column of $\mathcal{A}^{[k]}$. Thus, by (2.9) applied to $\mathcal{A}$ we have

$$\langle v, w\rangle = det\mathcal{A} \neq 0. \tag{3.5}$$

By the orthogonality of the basis V we see that

$$\mu_{\langle k\rangle} \neq 0, \tag{3.6}$$

namely, w in its representation with respect to the basis of $\mathcal{R}^{\binom{n}{k}}$ obtained from V, has a nonvanishing component corresponding to the basis vector v. Now we claim that this is the only nonvanishing component of w, that is,

$$\alpha \in Q_{k,n},\ \alpha \neq \langle k\rangle \implies \mu_\alpha = \frac{1}{\langle \bigwedge_{i\in\alpha} v_i,\ \bigwedge_{i\in\alpha} v_i\rangle} \langle \bigwedge_{i\in\alpha} v_i, w\rangle = 0. \tag{3.7}$$

Equation (3.7) follows from (2.9) and the fact that $\bigwedge_{i\in\alpha} v_i$ represents the "α–th" column of $A^{(k)}$. Consequently, from (3.4), (3.6) and (3.7),

$$v = \mu w, \ \text{where } \mu = \mu_{\langle k\rangle}^{-1},$$

completing the proof of the theorem. □

Comments

(i) In the proof of Theorem 3.1, on letting $\{\bar{v}_1^T, \ldots, \bar{v}_n^T\}$ be the dual basis of V, so that $\bar{v}_{k+1}, \ldots, \bar{v}_n$ span $R(\bar{A}^T)$, and by considering the inner products in (3.5) and (3.7) with w replaced by its conjugate, $\bar{w}$, the result holds as stated even for matrices and scalars over the complex field.

(ii) Let $*$ denote the Hodge star operator (see Chapter 4 in [M]). Refering to the proof of Theorem 3.1, we have that

$$\bigwedge_{i\in\langle k\rangle} v_i = * \bigwedge_{i\in\langle k\rangle^c} v_i$$

and since $\{v_{k+1},\ldots,v_n\}$ and $\{A_i^T\ ;\ i\in\beta\}$ are bases of $N(A)^\perp$, from Section 2 there exists scalar b such that

$$\bigwedge_{i\in\langle k\rangle^c} v_i = b\bigwedge_{i\in\beta} A_i^T.$$

Therefore,

$$\bigwedge_{i\in\beta}{}' A_i^T = \mu^{-1}\bigwedge_{i\in\langle k\rangle} v_i = *(\mu^{-1}\bigwedge_{\langle k\rangle^c} v_i) = *(\mu^{-1} b\bigwedge_{i\in\beta} A_i^T).$$

This identifies the supplementary exterior product with the Hodge star operator on the exterior product of a set of vectors.

Example 3.2 Let

$$A = \begin{bmatrix} -.5 & 1 & -3 & 5 & 2.5 & -1 \\ -.5 & 1 & -2 & 2 & 1.5 & 0 \\ -.5 & 1 & 0 & 3 & -.5 & -1 \\ 0 & 0 & 1 & 2 & -1 & -1 \\ -.5 & 1 & -1 & 3 & .5 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

The rank of A is 4 and the nullity is $k=2$. Rows 3, 4, 5 and 6 are linearly independent and therefore span the row space of A. For $\beta=\{3,4,5,6\}$ we find that

$$w = \bigwedge_{i\in\beta}{}' A_i = [-.5,-1,0,-1,0,-.5,0,-.5,0,0,0,0,0,0,0]^T.$$

Also $N(A)$ is spanned by

$$v_1 = [1,1,1,0,1,0]^T \text{ and } v_2 = [1,.5839,.1677,0,.1677,0]^T$$

and

$$v_1\wedge v_2 = [-.4163,-.8363,0,-.8363,0,-.4163,0,-.4163,0,0,0,0,0,0,0]^T.$$

Observe that, in accordance with Theorem 3.1, we have (approximately)

$$w = 1.2 v_1 \wedge v_2. \quad \square$$

Our observation in the introduction can now be viewed as a special case of Theorem 3.1, for the geometric multiplicity of 0 being $k = 1$. In that case, the eigenvector corresponding to 0 can be directly computed as explained in Section 1. The case $k > 1$ can also be linked to the case where the geometric multiplicity of 0 is one as follows. Let $A \in \mathcal{R}^{n \times n}$ have rank $n - k,\ k > 1$, and assume there is an $\alpha \subset \langle n \rangle$ with $|\alpha| = n - k + 1$ such that $A[\alpha]$ is of rank $|\alpha| - 1$. We can now apply Theorem 3.1 to $A[\alpha]$ which is of nullity one and obtain a nonzero vector $w_1 \in N(A[\alpha])$. Then, by Theorem 1.4.9 in [HJ], the vector w defined by

$$w[\alpha] = w_1 \quad \text{and} \quad w[\alpha^c] = 0,$$

is indeed an eigenvector of A corresponding to 0. Also note that, for any permutation matrix P, $PAw = 0$ if and only if $Aw = 0$. Thus, if there does not exist a principal submatrix $A[\alpha]$ as required above, then the rows of A can be permuted so that PA has such a principal submatrix, and w may be constructed as above. We illustrate this approach by the following example.

Example 3.3 Let

$$J_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

The rank of J_4 is 1 and the nullity 3. Any principal submatrix

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

of J_4 is of nullity 1 and its cofactors about row 1 yield $[1, -1]^T$ as an eigenvector corresponding to 0. By the discussion preceding this example we have that

$$[1, -1, 0, 0]^T, \ [0, 1, -1, 0]^T, \ \text{and } [0, 0, 1, -1]^T$$

are (linearly independent) eigenvectors of J_4 corresponding to 0. $\square$

4 The Nullspace of the Compound

In this section we obtain further results on the nullspace of the compound. Let $A \in \mathcal{R}^{n\times n}$. If $\ell > rankA$ then $A^{(\ell)} = 0$, while if $\ell \leq rankA$ then A is invertible if and only if $A^{(\ell)}$ is (see [F]). It is clear from (2.5) that the exterior product of ℓ linearly independent eigenvectors of a matrix A forms an eigenvector of $A^{(\ell)}$ (see Theorem 6.22 in [F]). The converse is not in general true. For example, $[1,0,0,0,0,1]^T$ is an eigenvector corresponding to the eigenvalue 0 of $A^{(2)}$ in Example 2.1, but as we saw in Section 2 this is not a decomposable vector. In the following theorem we give a necessary and sufficient condition for a decomposable vector to be in the nullspace of the ℓ–th compound.

Theorem 4.1 *Let $A \in \mathcal{R}^{n\times n}$. Then, $\bigwedge_{i\in\langle\ell\rangle} u_i \in N(A^{(\ell)})$ if and only if either $\bigwedge_{i\in\langle\ell\rangle} u_i = 0$ or $span\{u_1, u_2, \ldots, u_\ell\} \cap N(A) \neq \{0\}$.*

Proof : There is nothing to show if A is invertible so suppose A is singular. If $\ell > rankA$ then $A^{(\ell)} = 0$ and

$$u_1 \wedge \ldots \wedge u_\ell \neq 0 \Rightarrow span\{u_1, \ldots, u_\ell\} \cap N(A) \neq \{0\}.$$

Since any vector is in the nullspace of the 0 matrix, the result is true. Assume then that $\ell \leq rankA$ and that $u_1 \wedge \ldots \wedge u_\ell \in N(A^{(\ell)}) \setminus \{0\}$. By (2.2) and (2.5) we have that $u_1, \ldots, u_\ell$ are linearly independent and that

$$0 = A^{(\ell)}(u_1 \wedge \ldots \wedge u_\ell) = Au_1 \wedge \ldots \wedge Au_\ell,$$

thus $Au_1, \ldots, Au_\ell$ are linearly dependent. Therefore, there exist $a_i,\ i = 1, 2, \ldots, \ell$, not all zero, such that $\sum_{i=1}^{\ell} a_i Au_i = 0$ and hence $\sum_{i=1}^{\ell} a_i u_i \in N(A) \setminus \{0\}$. For the converse of the theorem, there is nothing to show if $u_1 \wedge \ldots \wedge u_\ell = 0$. If $span\{u_1, \ldots, u_\ell\} \cap N(A) \neq \{\emptyset\}$ then, as above, $A^{(\ell)}(u_1 \wedge \ldots \wedge u_\ell) = 0$ □

Corollary 4.2 *Let $A \in \mathcal{R}^{n\times n}$ and $w \in N(A^{(\ell)})$ be decomposable with ℓ factors. Then there exists $v \in N(A)$ and $u_i \in \mathcal{R}^n,\ i = 2, 3, \ldots, \ell$ such that*

$$w = v \wedge u_2 \wedge \ldots \wedge u_\ell.$$

Proof. There is nothing to show if $w = 0$. Suppose $w = \bigwedge_{i\in\langle\ell\rangle} u_i \neq 0$. By Theorem 4.1 there exists $0 \neq u \in span\{u_1, \ldots, u_\ell\} \cap N(A)$. Let $u = \sum_{i=1}^{\ell} a_i u_i$

and suppose that $a_1 \neq 0$ (otherwise use (2.1) to permute). Observe then that by (2.2), (2.3) and (2.4),

$$u \wedge u_2 \wedge \ldots \wedge u_\ell = (a_1 u_1 \wedge u_2 \ldots \wedge u_\ell) + (a_2 u_2 \wedge u_2 \ldots \wedge u_\ell) + \ldots$$

$$+(a_\ell u_\ell \wedge u_2 \ldots \wedge u_\ell) = a_1 u_1 \wedge u_2 \wedge \ldots \wedge u_\ell,$$

so for $v = \frac{1}{a_1} u \in N(A) \setminus \{0\}$ we have

$$w = v \wedge u_2 \wedge \ldots \wedge u_\ell \quad \square$$

Our next result concerns eigenvectors of compound matrices corresponding to 0 that are not necessarily decomposable.

Theorem 4.3 *Let* $A \in \mathcal{R}^{n \times n}$ *have rank* $n - k = k,\ k \geq 1$ *and let* $w \in N(A^{(k)})$. *Then* w *is the sum of decomposable eigenvectors of* $A^{(k)}$ *corresponding to* 0.

Proof. Let $w \in N(A^{(k)})$. There is nothing to show if $w = 0$ or if w is decomposable. Suppose now that $\{v_1, \ldots, v_k\}$ is a basis of $N(A)$ and let $\{v_1, \ldots, v_n\}$ be a completion to a basis of $\mathcal{R}^n$. As we have seen before, there exist $\mu_\alpha \in \mathcal{R}$ such that

$$w = \sum_{\alpha \in Q_{k,n}} \mu_\alpha \Big(\bigwedge_{i \in \alpha} v_i \Big).$$

Since by assumption $n - k = k$, every vector $\bigwedge_{i \in \alpha} v_i$ with $\alpha \in Q_{k,n}$ contains an eigenvector of A corresponding to 0 as a factor, except when $\alpha = \langle k \rangle^c$. Consequently, as $A^{(k)} w = 0$, we have that

$$\mu_{\langle k \rangle^c} A^{(k)} \Big(\bigwedge_{i \in \langle k \rangle^c} v_i \Big) = 0.$$

If $\mu_{\langle k \rangle^c} \neq 0$, by Theorem 4.2, $\bigwedge_{i \in \langle k \rangle^c} v_i \in N(A^{(k)})$ can be written as the exterior product of k vectors one of which is in $N(A)$, a contradiction to the construction of the basis. This proves that $\mu_{\langle k \rangle^c} = 0$, completing the proof of the theorem. $\square$

Refering to Example 2.1, we have seen that $u = [1, 0, 0, 0, 0, 1]^T \in N(A^{(2)})$, while u is not decomposable. According to Theorem 4.3 and since $n = 4$ and $k = 2$, u is the sum of decomposable eigenvectors of $A^{(2)}$ corresponding to 0. Indeed, if $e_i,\ i = 1, 2, 3, 4$ is the standard basis of $\mathcal{R}^4$ it can be verified that, in accordance with Theorem 4.2 and Corollary 4.3,

$$u = e_1 \wedge e_2 + e_3 \wedge e_4 = (v_1 - e_2) \wedge e_2 + (v_2 - e_4) \wedge e_4 = v_1 \wedge e_2 + v_2 \wedge e_4.$$

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