Sign Controllability: Sign Patterns That Require Complete Controllability

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Michael J. Tsatsomeros †

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Abstract

We apply tools from the theory of sign–solvable systems and use the directed graph of a matrix in order to obtain sufficient conditions for a linear control system (A, B) to be completely controllable solely due to the sign patterns of the coefficient matrices A and B. We show that such conditions are necessary and sufficient for a particular class of linear control systems. We also consider an alternative approach to controllability, based on a reformulation of the classical condition (that the controllability matrix is of full rank) and obtain equivalent conditions for the general case.

Keywords: control system, sign pattern, signing, L–matrix, directed graph, aligned vertices, balancing chain.

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†Department of Mathematics and Statistics, University of Regina, Regina, Saskatchewan S4S 0A2. Work supported by an NSERC research grant. E-mail: tsat@math.uregina.ca

1 Introduction

In linear control theory, the basic concepts of controllability (and observability) are intimately related to the image of a matrix of the form

$$\mathcal{C} \;=\; [B \;\; AB \;\; \dots \;\; A^{n-1}B] \in \mathbf{R}^{n\times nm},$$

where $A \in \mathbf{R}^{n\times n}$ and $B \in \mathbf{R}^{n\times m}$. Specifically, a control system of the form

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t)$$

is completely controllable if and only if rank $\mathcal{C} = n$. As the matrices A, B comprise system parameters prone to measurement errors, it is desirable to determine whether rank $\mathcal{C} = n$ based on combinatorial and qualitative information about A and B (e.g., their directed graphs and the signs of their entries). Such qualitative approaches to controllability have been undertaken, for example, in Lin [6], Mayeda and Yamada [9], Murota [8], Johnson, Mehrmann, and Olesky [4], and Olesky, Tsatsomeros, and van den Driessche [10].

In the present work, we shall consider the following question: assume that the sign patterns (namely, the location of the positive, negative, and zero entries) of A and B are known. *When can we conclude that rank $\mathcal{C} = n$, based solely on the sign patterns and regardless of the magnitudes of the nonzero entries of A and B?* The study of this question was initiated in [4], where A was assumed to have nonnegative entries and B was assumed to be a column vector with positive entries.

Our qualitative approach to controllability will be based on extending and combining techniques used in the study of zero/nonzero patterns that *allow* or *require* complete controllability (see [6, 9, 10]) with notions related to the analysis of sign patterns and sign–solvable linear systems. We will find sufficient conditions for complete controllability for general sign patterns (Theorem 3.2), and we will identify a class of linear control systems (Definition 2.2) for which these conditions are necessary and sufficient (Theorem 3.7). In Section 4, we will consider a simple technical recasting of the classical controllability condition that rank $\mathcal{C} = n$ in order to provide an alternative answer to the question posed above.

2 Preliminaries

In this section, we present some of the notation, terminology and basic facts necessary to state and prove the main results in the following sections. In the remainder we let: $\langle k\rangle = \{1, 2, \dots, k\}$ for any positive integer k; $\mid \alpha \mid$ denote the cardinality of a set α; $\mathrm{sgn}(a)$ be 0, 1 or -1, when a is 0, positive or negative, respectively; $\mathrm{Re}(x)$ denote the real part of a complex vector x; $\mathrm{diag}(A_1, A_2, \dots, A_k)$ be a block diagonal matrix whose diagonal blocks are the square matrices $A_1, A_2, \dots, A_k$; $X[\alpha \mid \beta]$ denote the submatrix of $X \in \mathbf{R}^{s,t}$ whose rows and columns are indexed by the sets $\alpha \subseteq \langle s \rangle$ and $\beta \subseteq \langle t \rangle$, respectively; $X[\alpha] = X[\alpha \mid \alpha]$; e denote an all ones column vector of appropriate size.

We let $\Gamma = (V, E)$ denote a *directed graph* with vertex set V and directed edge set E consisting of ordered pairs (i, j) of vertices. A *path* from j to k in Γ is a sequence of vertices $j = r_1, r_2, ..., r_t = k$, with $(r_i, r_{i+1}) \in E$ for $i = 1, ..., t-1$.

The *directed graph of* $X = (x_{ij}) \in \mathbf{R}^{s,t},\ s \leq t$, denoted by $\Gamma = \mathcal{D}(X) = (V, E)$, has $V = \langle t \rangle$ and $E = \{(i, j) \mid x_{ij} \neq 0\}$. Extending the terminology in [6] and [9], when $s < t$ we say that $\mathcal{D}(X)$ is *accessible* if for every $j \in \langle s \rangle$ there exists $k \in \langle t \rangle \setminus \langle s \rangle$, such that there is a path from j to k in Γ. Also, for every $\alpha \subseteq \langle s \rangle$ we denote the *adjacency set* of α by

$$\mathcal{R}(\alpha) = \{j \in \langle t \rangle \mid (i, j) \in E \ \text{ for some } \ i \in \alpha\}.$$

Notice that if there exists an $\alpha \subseteq \langle s \rangle$ such that $\alpha \subseteq \mathcal{R}(\alpha)$ and if Γ is accessible, then $\mathcal{R}(\alpha) \cap (\langle t \rangle \setminus \langle s \rangle) \neq \emptyset$ and hence $\alpha \subset \mathcal{R}(\alpha)$.

In keeping with the notation and terminology of Brualdi and Shader [1] (which is our comprehensive reference on sign patterns), we define the following.

The *sign pattern* of $X \in \mathbf{R}^{s,t}$ is the $(0, 1, -1)$–matrix obtained from X when zero, positive and negative entries are replaced by 0, 1, -1, respectively. The matrix X determines the *qualitative class* $Q(X)$ of all matrices with the same sign pattern as X. We will write $\hat{X} \in Q(X)$ for any matrix $\hat{X}$ having the same sign pattern as X.

A *signing* is a nonzero square diagonal sign pattern. A real vector is called *balanced* if it is the zero vector, or if it has at least one negative and at least one positive entry. A real vector is referred to as *unisigned* if it is not balanced. If a unisigned vector has nonnegative (resp. nonpositive) entries, we refer to it as *of positive (*resp. *negative) type.* We denote the signings S such that all the columns of SX are balanced by $\mathcal{B}(X)$.

The matrix X is called an *L–matrix* provided that every matrix in $Q(X)$ has linearly independent rows. It is well known (see [1, Theorem 2.1.1]) that $X \in \mathbf{R}^{s,t}$ is an L–matrix if and only if $\mathcal{B}(X) = \emptyset$. Next we introduce the notion of aligned vertices in the directed graph of a matrix.

Definition 2.1 Let $X \in \mathbf{R}^{s,t}$, $s \leq t$, and let $\alpha_1 \subseteq \langle s \rangle, \alpha_2 \subseteq \langle t \rangle$ be two nonempty and disjoint sets. We call α_1 *aligned relative to* α_2 if there exists a signing $S \in \mathcal{B}(X[\alpha_1 \mid \alpha_2])$ such that the unisigned columns of $SX[\alpha_1]$ (if any exist) are only of one type (either only positive type or only negative type). When $\mathcal{B}(X[\alpha_1 \mid \alpha_2]) = \emptyset$, α_1 is by definition not aligned relative to α_2.

In other words, $\alpha_1 \subseteq \langle s \rangle$ is aligned relative to a disjoint set $\alpha_2 \subseteq \langle t \rangle$ if there exists a signing of the rows of $X[\alpha_1 \mid \langle t \rangle]$ such that the columns of $X[\alpha_1 \mid \alpha_2]$ become balanced and the columns of $X[\alpha_1]$ become either balanced or unisigned of only one type.

Consider now a linear control system of the form

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t), \quad t \geq 0, \tag{2.1}$$

where $A \in \mathbf{R}^{n,n}$ and $B \in \mathbf{R}^{n,m}$, and where $u(t) \in \mathbf{R}^m$ represents an unconstrained, piecewise continuous control input. We denote the system in (2.1) by (A, B). It is known that the output (viz. solution) $x(t)$ of (2.1) emanating from any initial point in $\mathbf{R}^n$ is controllable (by an appropriate choice of $u(t)$) to any terminal point in $\mathbf{R}^n$ in finite time if and only if

$$\text{rank } \mathcal{C} = n, \tag{2.2}$$

where $\mathcal{C} = [B \;\; AB \;\; \ldots \;\; A^{n-1}B] \in \mathbf{R}^{n,nm}$ is the *controllability matrix* associated with (A, B). When (2.2) holds, we call (A, B) *completely controllable*. It follows easily that if $X \in \mathbf{R}^{n,n}$ is nonsingular, then (A, B) is completely controllable if and only if (XAX^{-1}, XB) is completely controllable, or if and only if $(-A, B)$ is completely controllable.

As with many questions arising in the study of sign patterns, the presence of implicit relations among the entries of the matrix in question can complicate the qualitative analysis significantly. In the case of the controllability matrix $\mathcal{C}$ this difficulty is evident because of the presence of the products of powers of A with B. For this reason, it is useful to consider a condition known to be equivalent to rank $\mathcal{C} = n$ (see e.g., Theorem 4.3.3 in Lancaster and Rodman [7]), namely,

$$\text{rank}[A - \lambda I \;\;\; B] = n \quad \text{for all} \quad \lambda \in \mathbf{C}. \tag{2.3}$$

The compromise in dealing with the latter condition, rather than $\mathcal{C}$, is the introduction of the complex parameter λ.

Given a linear control system (A, B), we consider the qualitative class consisting of all linear control systems $(\hat{A}, \hat{B})$ such that $\hat{A} \in Q(A)$, $\hat{B} \in Q(B)$. In this paper, we say that (A, B) is *sign controllable* [1] if $(\hat{A}, \hat{B})$ is completely controllable for all $\hat{A} \in Q(A)$ and all $\hat{B} \in Q(B)$.

Next we introduce a classification of control systems (A, B) based on the directed graph of $[A \; B]$ and the signs of the diagonal entries of A.

Definition 2.2 Let $A \in \mathbf{R}^{n,n}$, $B \in \mathbf{R}^{n,m}$, $T = [A \; B]$, $\Gamma = \mathcal{D}(T)$. We call (A, B) a *strict linear control system* if

(a) the diagonal entries of A are nonzero and have the same sign, and

(b) for all $\alpha \subseteq \langle n \rangle$ such that $\alpha \subset \mathcal{R}(\alpha)$ in Γ, either $T[\alpha \mid \mathcal{R}(\alpha) \setminus \alpha]$ is an L–matrix or $\mathcal{B}(T[\alpha \mid \mathcal{R}(\alpha) \setminus \alpha])$ contains a nonsingular signing.

In the next section, we will find sufficient conditions for sign controllability and we will show that these conditions are necessary and sufficient for sign controllability of a strict linear control system.

[1] We caution the reader that the term sign controllability has also been used in the literature to describe a different property of the controllability matrix (see [3, 7]).

3 Conditions for Sign Controllability

First we mention a necessary condition for complete controllability (that is observed in [6] as a necessary condition for *structural controllability*).

Lemma 3.1 *Let $A \in \mathbf{R}^{n,n}$, $B \in \mathbf{R}^{n,m}$ and suppose that (A, B) is completely controllable. Then $\Gamma = \mathcal{D}([A\ B])$ is accessible.*

Proof. Suppose that Γ is not accessible. Then there exists $j \in \langle n \rangle$ such that there is no path from j to k in Γ for every $k \in \langle n+m \rangle \setminus \langle n \rangle$. Let $\alpha \subseteq \langle n \rangle$ be the set consisting of j and all vertices of $\mathcal{D}(A)$ that lie on a path emanating from j. It follows that there is no path from ℓ to k in Γ for every $\ell \in \alpha$ and every $k \in \langle n+m \rangle \setminus \langle n \rangle$. Moreover, letting α^c be the complement of α in $\langle n \rangle$, there exists a permutation matrix P such that

$$PAP^T = \begin{bmatrix} A[\alpha] & 0 \\ A[\alpha^c \mid \alpha] & A[\alpha^c] \end{bmatrix} \quad \text{and} \quad PB = \begin{bmatrix} 0 \\ B[\alpha^c \mid \langle m \rangle] \end{bmatrix}.$$

So, if $x = [\hat{x}^T\ 0]^T \in \mathbf{R}^n$, where $\hat{x}$ is a left eigenvector of $A[\alpha]$ corresponding to an eigenvalue λ, then $x^T[PAP^T - \lambda I \quad PB] = 0$, showing that (PAP^T, PB) (and hence (A, B)) is not completely controllable. ■

From condition (2.3) for $\lambda = 0$ and the above lemma, we have that $[A\ B]$ being an L–matrix and the directed graph of $[A\ B]$ being accessible are two necessary conditions for sign controllability of (A, B). We continue by showing that these two conditions, together with some additional conditions on the directed graph of $[A\ B]$, are also sufficient for sign controllability of (A, B).

Theorem 3.2 *Let $A \in \mathbf{R}^{n,n}$, $B \in \mathbf{R}^{n,m}$ and $\Gamma = \mathcal{D}([A\ B]) = (V, E)$. Suppose that*

(1) Γ is accessible,

(2) $[A\ B]$ is an L–matrix, and

(3) for all $\alpha \subseteq \langle n \rangle$ satisfying $\alpha \subset \mathcal{R}(\alpha)$ in Γ, either there exists $j \in \mathcal{R}(\alpha) \setminus \alpha$ and exactly one $i \in \alpha$ such that $(i, j) \in E$, or α is not aligned relative to $\mathcal{R}(\alpha) \setminus \alpha$.

Then the linear control system (A, B) is sign controllable.

Proof. Suppose (A, B) is not completely controllable and that (1) and (2) hold. It is enough to show that (3) is not true. By condition (2.3) and because $[A\ B]$ is an L–matrix, there exists $\lambda \in \mathbf{C} \setminus \{0\}$ and $x \in \mathbf{C}^n \setminus \{0\}$ such that

$$x^T[A\ B] = [\lambda x^T\ 0]. \tag{3.1}$$

Without loss of generality, assume that $x = (x_1, x_2, \ldots, x_k, 0, \ldots 0)^T$, $x_i \neq 0$ for $i = 1, 2, \ldots, k$ (otherwise we can work with (PAP^T, PB) for some permutation matrix P). Also without loss of generality assume that $\mathrm{Re}(x) \neq 0$ (otherwise we can replace x in our arguments by $\sqrt{-1}x$). Observe that $\mathrm{Re}(\lambda x^T) \neq 0$ or else, by (3.1), $\mathrm{Re}(x^T)[A\ B] = 0$ and (2) is contradicted. Consider an invertible signing $S = \mathrm{diag}(s_1, s_2, \ldots, s_n)$ so that $\mathrm{Re}(\lambda x^T S) \geq 0$ (entrywise). On letting $T = [SAS\ SB] = (t_{ij})$, we have from (3.1) that

$$x^T S[SAS\ SB] = [\lambda x^T S\ 0], \tag{3.2}$$

namely,

$$\sum_{i=1}^{k} x_i s_i t_{ij} = \lambda x_j s_j \neq 0 \quad (j = 1, 2, \ldots, k). \tag{3.3}$$

Take now $\alpha = \langle k \rangle \subseteq \langle n \rangle$ and let α^c be its complement in $\langle n + m \rangle$. From (3.3), we can conclude that every column of $T[\alpha]$ contains at least one nonzero entry. Hence for every $j \in \alpha$ there exists $i \in \alpha$ such that $(i, j) \in E$. This means that $\alpha \subseteq \mathcal{R}(\alpha)$. Since Γ is assumed accessible, we have that $\alpha \subset \mathcal{R}(\alpha)$. We also have from (3.2) that

$$\sum_{i=1}^{k} x_i s_i t_{ij} = 0 \quad (j = k + 1, k + 2, \ldots, n + m), \tag{3.4}$$

which implies that every column of $T[\alpha \mid \alpha^c]$ has either no or at least two nonzero entries. Hence for every $j \in \mathcal{R}(\alpha) \setminus \alpha \subseteq \alpha^c$ there are at least two vertices $i \in \alpha$ such that $(i, j) \in E$. As a consequence, to show that condition (3) is violated it remains to argue that α is aligned relative to $\mathcal{R}(\alpha) \setminus \alpha$. From (3.3) and (3.4) we get, respectively, that

$$\sum_{i=1}^{k} \mathrm{Re}(x_i) s_i t_{ij} = \mathrm{Re}(\lambda x_j s_j) \geq 0 \quad (j = 1, 2, \ldots, k) \tag{3.5}$$

and

$$\sum_{i=1}^{k} \mathrm{Re}(x_i) s_i t_{ij} = 0 \quad (j = k + 1, k + 2, \ldots, n + m). \tag{3.6}$$

Equations (3.5) and (3.6) have the following interpretation: if we consider the signing

$$\hat{S} = \mathrm{diag}(\mathrm{sgn}(\mathrm{Re}(x_1))s_1, \mathrm{sgn}(\mathrm{Re}(x_2))s_2, \ldots, \mathrm{sgn}(\mathrm{Re}(x_k))s_k),$$

then the columns of $\hat{S}T[\alpha \mid \alpha^c]$, and in particular the columns of $\hat{S}T[\alpha \mid \mathcal{R}(\alpha) \setminus \alpha]$, are balanced, while all unisigned columns of $\hat{S}T[\alpha]$ are of positive type. Hence α is aligned relative to $\mathcal{R}(\alpha) \setminus \alpha$ in Γ. ■

We continue with some examples in order to illustrate the use of Theorem 3.2 and various situations that arise.

Example 3.3 Let

$$A = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Notice that the directed graph of $[A\ B]$ is accessible and that $[A\ B]$ is an L–matrix because $\det\hat{A} < 0$ for all $\hat{A} \in Q(A)$ (i.e., A is a sign nonsingular matrix, see [1]). Regarding condition (3) of Theorem 3.2, we find that $\alpha_i \subset \mathcal{R}(\alpha_i)$, $i = 1, 2, 3, 4$, where $\alpha_1 = \{3\}$, $\alpha_2 = \{1, 2\}$, $\alpha_3 = \{1, 3\}$, and $\alpha_4 = \langle 3 \rangle$. We also have that

$$\mathcal{R}(\alpha_1) \setminus \alpha_1 = \{1\}, \quad \mathcal{R}(\alpha_2) \setminus \alpha_2 = \{3, 4\}, \quad \mathcal{R}(\alpha_3) \setminus \alpha_3 = \{2, 4\}, \quad \text{and} \quad \mathcal{R}(\alpha_4) \setminus \alpha_4 = \{4\}.$$

In all four cases, the first part of condition (3) is satisfied with the edge from α_i to $\mathcal{R}(\alpha_i) \setminus \alpha_i$, $i = 1, 2, 3, 4$, being $(3, 1)$, $(1, 4)$, $(1, 4)$, and $(1, 4)$, respectively. So by Theorem 3.2, (A, B) is sign controllable. We comment that, in the language of [10], the pair of zero/nonzero patterns $(\mathbf{A}, \mathbf{B})$ associated with (A, B) is not *qualitatively controllable* (see ([10, Theorem 2.2]).

Example 3.4 Let

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

The directed graph of $T = [A\ B]$ is accessible and T is an L–matrix. Regarding condition (3) of Theorem 3.2, we find that with one exception, for all $\alpha \subseteq \langle 3 \rangle$ for which $\alpha \subset \mathcal{R}(\alpha)$, there is exactly one edge from α to some $j \in \mathcal{R}(\alpha) \setminus \alpha$. The only exception is $\hat{\alpha} = \langle 3 \rangle$ for which $\mathcal{R}(\hat{\alpha}) \setminus \hat{\alpha} = \{4, 5\}$. Notice that every $S \in \mathcal{B}(T[\hat{\alpha} \mid \mathcal{R}(\hat{\alpha}) \setminus \hat{\alpha}])$ has its first two diagonal entries zero and the third diagonal entry nonzero. But then the last two columns of $ST[\hat{\alpha}]$ are unisigned of opposite type. Hence, by Theorem 3.2, (A, B) is sign controllable.

We continue with a result on sign controllability, which will lead to a characterization of strict sign controllable systems.

Proposition 3.5 *Let $A \in \mathbf{R}^{n,n}$, $B \in \mathbf{R}^{n,m}$, $T = [A\ B]$, and $\Gamma = \mathcal{D}(T)$. Assume that there exists $\alpha \subseteq \langle n \rangle$ with $\alpha \subset \mathcal{R}(\alpha)$ in Γ such that $\mathcal{B}(T[\alpha \mid \mathcal{R}(\alpha) \setminus \alpha])$ contains a nonsingular signing S. Also assume that the unisigned columns of $ST[\alpha]S$ (if any exist) are only of one type. Then (A, B) is not sign controllable.*

Proof. Let S be as prescribed above and $\Gamma = (V, E)$. Since (A, B) is completely controllable if and only if $(-A, B)$ is, we will assume, without loss of generality, that the unisigned columns of $ST[\alpha]S$ (if any exist) are all of positive type. Since there is no $i \in \alpha$ and $j \notin \mathcal{R}(\alpha)$ such that $(i, j) \in E$, $T[\alpha \mid \langle n + m \rangle \setminus \mathcal{R}(\alpha)] = 0$. Also, since $\alpha \subset \mathcal{R}(\alpha)$, every column of $T[\alpha]$ contains a nonzero entry. Letting $\hat{S} \in \mathbf{R}^{n,n}$ be a nonsingular signing such that $\hat{S}[\alpha] = S$ and considering $\hat{T} = [\hat{S}A\hat{S}\ \hat{S}B]$, we have that:

(1) every column of $\hat{T}[\alpha]$ contains a positive entry,

(2) every column of $\hat{T}[\alpha \mid \mathcal{R}(\alpha) \setminus \alpha]$ is balanced, and

(3) every column of $\hat{T}[\alpha \mid \langle n+m \rangle \setminus \mathcal{R}(\alpha)]$ is zero.

Therefore, by (1)–(3) above, we can assume that the nonzero entries of A and B have been chosen so that the entries of each column of $\hat{T}[\alpha]$ add up to 1, and the entries of each column of $\hat{T}[\alpha \mid \langle n+m \rangle \setminus \alpha]$ add up to 0. That is, if we let

$$x = (x_1, x_2, \ldots x_n)^T, \quad x_i = \begin{cases} 1 & \text{if } i \in \alpha \\ 0 & \text{otherwise,} \end{cases}$$

we have shown that

$$x^T[\hat{S}A\hat{S}\ \hat{S}B] = [x^T\ 0],$$

for an invertible signing $\hat{S}$. Hence, using $\lambda = 1$ in condition (2.3), it follows that $(\hat{S}A\hat{S}, \hat{S}B)$ and thus (A, B) is not sign controllable. ■

Corollary 3.6 *Let $A \in \mathbf{R}^{n,n}$, $B \in \mathbf{R}^{n,m}$, $T = [A\ B]$, $\Gamma = \mathcal{D}(T)$, and suppose that the diagonal entries of A are nonzero and have the same sign. Let $\alpha \subseteq \langle n \rangle$ with $\alpha \subset \mathcal{R}(\alpha)$ such that $\mathcal{B}(T[\alpha \mid \mathcal{R}(\alpha) \setminus \alpha])$ contains a nonsingular signing. Then (A, B) is not sign controllable.*

Proof. Let α be as prescribed and $S \in \mathcal{B}(T[\alpha \mid \mathcal{R}(\alpha) \setminus \alpha])$ be nonsingular. Since all diagonal entries of $ST[\alpha]S$ are nonzero and have the same sign, all the assumptions of Proposition 3.5 are satisfied and the corollary follows. ■

Theorem 3.7 *Let $A \in \mathbf{R}^{n,n}$, $B \in \mathbf{R}^{n,m}$, $T = [A\ B]$ and $\Gamma = \mathcal{D}(T)$. Suppose that (A, B) is a strict linear control system. Then (A, B) is sign controllable if and only if the following conditions hold:*

(1) Γ is accessible,

(2) $[A\ B]$ is an L–matrix, and

(3) for all $\alpha \subseteq \langle n \rangle$ satisfying $\alpha \subset \mathcal{R}(\alpha)$ in Γ, α is not aligned relative to $\mathcal{R}(\alpha) \setminus \alpha$.

Proof. The sufficiency of conditions (1)–(3) follows from Theorem 3.2. We have discussed the necessity of conditions (1) and (2) after Lemma 3.1. To prove the necessity of condition (3), assume that (1) and (2) hold and that (3) does not hold. Since Γ is accessible and the diagonal entries of A are nonzero, we have that for all $\alpha \subseteq \langle n \rangle$, $\alpha \subset \mathcal{R}(\alpha)$ and hence

$\mathcal{R}(\alpha)\backslash\alpha \neq \emptyset$. So since (3) is not true, there exists $\alpha \subset \mathcal{R}(\alpha)$ such that $\mathcal{B}(T[\alpha \mid \mathcal{R}(\alpha)\backslash\alpha]) \neq \emptyset$. Because (A, B) is strict, $\mathcal{B}(T[\alpha \mid \mathcal{R}(\alpha)\backslash\alpha])$ must contain a nonsingular signing. By Corollary 3.6 it follows that (A, B) is not sign controllable. ∎

Example 3.8 Let

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

The directed graph of $T = [A\ B]$ is accessible and T is an L–matrix. For all $\alpha \subseteq \langle n \rangle$ we have that $\alpha \subset \mathcal{R}(\alpha)$ in Γ. In fact, for all $\alpha \subseteq \langle n \rangle$, except $\hat{\alpha} = \{1, 2, 3\}$, $T[\alpha \mid \mathcal{R}(\alpha) \setminus \alpha]$ is an L–matrix. We have that $T[\hat{\alpha} \mid \mathcal{R}(\hat{\alpha}) \setminus \hat{\alpha}] = B$, and $S = \mathrm{diag}(-1, 1, 1) \in \mathcal{B}(B)$. So (A, B) is a strict linear control system. Since $ST[\hat{\alpha}]$ has unisigned columns of only positive type, $\hat{\alpha}$ is aligned relative to $\mathcal{R}(\hat{\alpha}) \setminus \hat{\alpha}$. By Theorem 3.7, (A, B) is not sign controllable.

4 The Extended Controllability Matrix

We will now introduce some additional concepts and terminology pertaining to an alternative analysis of sign controllability.

For the purposes of this section, we append to the set of signings the zero (square) matrix and refer to them as *weak signings*. We let $\mathcal{B}_0(X)$ denote $\mathcal{B}(X) \cup \{0\}$ for any $X \in \mathbf{R}^{s,t}$.

It is clear that for every $S \in \mathcal{B}_0(X)$, there exists $\hat{X} \in Q(X)$ such that the column sums of $S\hat{X}$ equal to 0 (and hence equal to the column sums of the 0 matrix). Based on this observation, we extend the notion of $\mathcal{B}_0(X)$ as follows. Given a matrix X and a weak signing S', we denote by $\mathcal{B}_0(X, S')$ the set of all weak signings S such that there exists $\hat{X} \in Q(X)$ with the column sums of $S\hat{X}$ equal to the column sums of S'. Notice that $\mathcal{B}_0(X) = \mathcal{B}_0(X, 0)$. To illustrate the definition of $\mathcal{B}_0(X, S')$, let

$$X = \begin{bmatrix} 1 & -3 & 2 \\ -1 & 0 & 1 \\ 2 & -1 & -1 \end{bmatrix} \quad \text{and} \quad S' = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then $\mathcal{B}_0(X, S')$ consists of all signings S such that SX has a negative entry in the first column, a positive entry in the second column, and a balanced third column. For example, $S = \mathrm{diag}(-1, 1, 1) \in \mathcal{B}_0(X, S')$ and $I \notin \mathcal{B}_0(X, S')$.

Definition 4.1 Let $A \in \mathbf{R}^{n,n}$ and $B \in \mathbf{R}^{n,m}$ be given. A nonzero ordered n–tuple $(S_1, S_2, \ldots, S_n)$ of weak signings is called an (A, B)–*balancing chain* if

$$S_i \in \mathcal{B}_0(B) \quad (i = 1, 2, \ldots, n)$$

and

$$S_{i+1} \in \mathcal{B}_0(A, S_i) \quad (i = 1, 2, \ldots, n-1).$$

If in addition there exist $\hat{A} \in Q(A)$, $\hat{B} \in Q(B)$, and entrywise positive vectors $x_i \in \mathbf{R}^n$, such that

$$x_i^T S_i \hat{B} = 0 \quad (i = 1, 2, \ldots, n)$$

and

$$x_i^T S_i = x_{i+1}^T S_{i+1} \hat{A} \quad (i = 1, 2, \ldots, n-1),$$

we call $(S_1, S_2, \ldots, S_n)$ a *compatible* (A, B)–balancing chain.

The notion of an (A, B)–balancing chain depends only on the sign patterns of A and B. If $(S_1, S_2, \ldots, S_n)$ is an (A, B)–balancing chain, then there always exist $A_i \in Q(A)$, $B_i \in Q(B)$, and x_i with positive entries such that $x_i^T S_i B_i = 0$ for $i = 1, 2, \ldots, n$ and $x_i^T S_i = x_{i+1}^T S_{i+1} A_{i+1}$ for $i = 1, 2, \ldots, n-1$. In fact, we can take $x_i = e$ for all i. In the definition of a compatible (A, B)–balancing chain we require, in addition, that there are common matrices $\hat{A} \in Q(A)$ and $\hat{B} \in Q(B)$ that satisfy the above conditions.

Observe that an (A, B)–balancing chain $(S_1, S_2, \ldots, S_n)$ may contain some zero weak signings, which could appear only as the leading part of the chain. Indeed, if $S_i \neq 0$, then since $S_{i+1}\hat{A}$ must have the same column sums as S_i for some $\hat{A} \in Q(A)$, S_{i+1} must be nonzero.

Definition 4.2 With the linear control system (A, B) we will associate (in Lemma 4.3) the *extended controllability matrix*, $\mathcal{G}$, defined as follows:

$$\mathcal{G} = \begin{bmatrix} I & 0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 0 & B \\ -A & I & 0 & \ldots & \ldots & \ldots & \ldots & \ldots & 0 & B & 0 \\ 0 & -A & I & \ldots & \ldots & \ldots & \ldots & 0 & B & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & ⋰ & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & -A & I & 0 & B & 0 & \ldots & \ldots & 0 \\ 0 & 0 & \ldots & 0 & -A & B & 0 & \ldots & \ldots & \ldots & 0 \end{bmatrix} \in \mathbf{R}^{n^2,\ n(n+m-1)}.$$

The following result is a recasting of the classical condition for controllability in (2.2); its proof can be found in Casti [2].

Lemma 4.3 ([2, Corollary 5, §3.5]) *Let $A \in \mathbf{R}^{n,n}$, $B \in \mathbf{R}^{n,m}$. The control system (A, B) is completely controllable if and only if rank $\mathcal{G} = n^2$.*

We now have the following equivalent condition for sign controllability.

Theorem 4.4 *Let $A \in \mathbf{R}^{n,n}$, $B \in \mathbf{R}^{n,m}$. The linear control system (A, B) is sign controllable if and only if there is no compatible (A, B)–balancing chain.*

Proof. Let $A \in \mathbf{R}^{n,n}$, $B \in \mathbf{R}^{n,m}$ and suppose that (A, B) is not sign controllable. Then, by Lemma 4.3, there are matrices in $\hat{A} \in Q(A)$ and $\hat{B} \in Q(B)$ so that the corresponding extended controllability matrix $\hat{\mathcal{G}}$ is of deficient rank, i.e., $w^T \hat{\mathcal{G}} = 0$ for some $w \in \mathbf{R}^{n^2} \setminus \{0\}$. Let now $S \in \mathbf{R}^{n^2,n^2}$ be a signing such that $w = Sx$, where x has positive entries. It follows that $x^T S \hat{\mathcal{G}} = 0$. Hence, if S is partitioned into n diagonal blocks $S_1, S_2, \ldots, S_n$ of size n by n, then $(S_1, S_2, \ldots, S_n)$ is a compatible (A, B)–balancing chain.

Conversely, if $(S_1, S_2, \ldots, S_n)$ is a compatible (A, B)–balancing chain, then there exist $\hat{A} \in Q(A)$, $\hat{B} \in Q(B)$, and vectors x_i with positive entries such that

$$x_i^T S_i \hat{B} = 0 \quad (i = 1, 2, \ldots, n)$$

and

$$x_i^T S_i = x_{i+1}^T S_{i+1} \hat{A} \quad (i = 1, 2, \ldots, n-1).$$

It follows that for $S = \operatorname{diag}(S_1, S_2, \ldots, S_n)$ and for $x = [x_1^T, x_2^T, \ldots, x_n^T]^T$, $w = Sx$ is a nonzero left nullvector of the extended controllability matrix of $(\hat{A}, \hat{B})$; that is, by Lemma 4.3, (A, B) is not sign controllable. ∎

The existence or not of a compatible (A, B)–balancing chain can be a hard condition to check, but in some instances the clauses in the definition of a compatible (A, B)–balancing chain can serve as useful necessary or sufficient conditions. This is illustrated in the following examples.

Example 4.5 This example is mentioned in [4]. Let

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Notice that $\mathcal{B}_0(B)$ consists of the weak signings $S_1, S_2, \ldots, S_9$ having their (1,1) entry equal to 0. It is easy to check the sign patterns of $S_i A$ for $i = 1, 2, \ldots, 9$ and discover that there is no (A, B)–balancing chain and hence, by Theorem 4.4, (A, B) is sign controllable.

Example 4.6 Let

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

It can be checked that

$$R = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad S = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

are in $\mathcal{B}(B)$, and that (R, T, R), (T, R, T), (T, R, S) are some of the (A, B)–balancing chains. In this case the knowledge of a balancing chain leads to a straightforward search for the vectors x_i and the matrices $\hat{A}$ and $\hat{B}$ in the definition of a compatible balancing chain. One finds that with $x_1 = x_2 = x_3 = e$ and

$$\hat{A} = \begin{bmatrix} -1/3 & 0 & 1/3 \\ 1/3 & 0 & -1/3 \\ 1/3 & -1 & -1/3 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$$

(R, T, R) is a compatible (A, B)–balancing chain and thus, by Theorem 4.4, (A, B) is not sign controllable.

In conclusion, we have presented an alternative approach to sign controllability of a linear control system (A, B) based on the existence of a balancing chain of signings. We do not know if there exists an algorithm to verify the (non–)existence of a compatible balancing chain, regardless of the complexity. We have also found sufficient conditions for sign controllability, based on the sign pattern and the directed graph of $[A\ B]$, which are necessary and sufficient when the linear control system is strict. We have not addressed computational matters regarding these conditions. However, we remark that the recognition of one of these conditions, namely that the rectangular matrix $[A\ B]$ be an L–matrix, has been shown to be an NP–complete problem (see Klee, Ladner, and Manber [5]).

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