GENERATING AND DETECTING MATRICES WITH POSITIVE PRINCIPAL MINORS

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Abstract. A brief but concise review of methods to generate P-matrices (i.e., matrices having positive principal minors) is provided and motivated by open problems on P-matrices and the desire to develop and test efficient methods for the detection of P-matrices. Also discussed are operations that leave the class of P-matrices invariant. Some new results and extensions of results regarding P-matrices are included.

Key words. P-matrix, positive definite matrix, M-matrix, MMA-matrix, H-matrix, B-matrix, totally positive matrix, mime, Schur complement, principal pivot transform, Cayley transform, linear complementarity problem, P-problem.

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1. Introduction. An $n \times n$ complex matrix $A \in \mathcal{M}_n(\mathbb{C})$ is called a *P-matrix* if all its principal minors are positive. Recall that a *principal minor* is simply the determinant of a submatrix obtained from A when the same set of rows and columns are stricken out. The diagonal entries and the determinant of A are thus among its principal minors. We shall denote the class of complex P-matrices by $\mathbb{P}$.

The P-matrices encompass such notable classes as the Hermitian positive definite matrices, the M-matrices, the totally positive matrices and the real diagonally dominant matrices with positive diagonal entries. The first systematic study of P-matrices appears in the work of Fiedler and Ptak [15]. Since then, the class $\mathbb{P}$ and its subclasses have proven a fruitful subject, judged by the attention received in the matrix theory community and the interest generated (or motivated) by their applications in the mathematical and social sciences; see [3, 14, 25]. Of particular concern is the ability to decide as efficiently as possible whether an n-by-n matrix is in $\mathbb{P}$ or not, known as the *P-problem.* It is receiving attention due to its inherent computational complexity (it is known to be NP-hard [5, 6]), and due to the connection of P-matrices to the linear complementarity problem and to self-validating methods for its solution; see, e.g. [4, 26, 42, 43]. A recursive $O(2^n)$ algorithm for the P-problem that is simple to implement and lends itself to computation in parallel is provided in [48] and reviewed here. More recently [44], a strategy is provided for detecting P-matrices, which is not a priori exponential, although it can be exponential in the worst case.

Motivated by the P-problem and other questions about P-matrices, in this article we address the need to generate P-matrices for purposes of experimentation as well as theoretical and algorithmic development. To this end, we provide a review of (i) operations that preserve P-matrices and (ii) techniques that can be used to generate P-matrices. This affords us an opportunity to survey well-known results and to present

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some new and some less known results on P-matrices. An important goal remains a method to generate 'random' or 'generic' P-matrices. This is meant in the sense of having at our disposal a property shared by all P-matrices, other than the defining property, which would then allow us to easily construct them and possibly detect them.

The presentation unfolds as follows: Section 2 contains all the notation and definitions. Section 3 reviews mappings of $\mathbb{P}$ into itself, which can therefore be viewed as methods of generating P-matrices from other P-matrices. It also contains some background material used in subsequent sections. Section 4 contains methods to generate P-matrices, some of which yield P-matrices with additional properties. Section 5 contains a closer examination of mimes, a special subclass of the P-matrices. In section 6, we briefly discuss some related topics such as P-matrix completions and interval P-matrices. In section 7, we provide an accelerated Matlab implementation of the P-matrix test algorithm in [48]; its ability is also extended to complex matrices.

2. Notation, definitions and preliminaries. We group the contents of this section in several categories for convenience.

General notation

The transpose of a complex array X is denoted by X^T and its conjugate transpose by X^*. Entrywise ordering of arrays of the same size is indicated by $\geq$. We write $X > Y$ if X, Y are real and every entry of $X - Y$ is positive. The indication of a matrix interval $[X, Y]$ presumes that $Y \geq X$ and denotes the set of all real matrices each of whose entry lies in the interval of the corresponding entries of X and Y. When $X \geq 0$ (resp., $X > 0$), we refer to X as *nonnegative* (resp., *positive*).

A vector $x \in \mathbb{C}^n$ is called a *unit* vector if $\|x\|_2 = 1$. Let n be a positive integer and $A \in \mathcal{M}_n(\mathbb{C})$. The following notation is also used:

- $\langle n \rangle = \{1, \ldots, n\}$.
- The i-th entry of a vector x is denoted by x_i.
- $\sigma(A)$ denotes the spectrum of A.
- $\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$ is the spectral radius of A.
- $\text{diag}\,(d_1, \ldots, d_n)$ is the diagonal matrix with diagonal entries $d_1, \ldots, d_n$.
- For $\alpha \subseteq \langle n \rangle$, $|\alpha|$ denotes the cardinality of α and $\overline{\alpha} = \langle n \rangle \setminus \alpha$.
- $A[\alpha, \beta]$ is the submatrix of A whose rows and columns are indexed by $\alpha, \beta \subseteq \langle n \rangle$, respectively; the elements of α, β are assumed to be in ascending order. When a row or column index set is empty, the corresponding submatrix is considered vacuous and by convention has determinant equal to 1. We abbreviate $A[\alpha, \alpha]$ by $A[\alpha]$.

Matrix Transforms

Given $A \in \mathcal{M}_n(\mathbb{C})$ and $\alpha \subseteq \langle n \rangle$ such that $A[\alpha]$ is invertible, $A/A[\alpha]$ denotes the *Schur complement* of $A[\alpha]$ in A, that is,

$$A/A[\alpha] \;=\; A[\overline{\alpha}] \;-\; A[\overline{\alpha}, \alpha] A[\alpha]^{-1} A[\alpha, \overline{\alpha}].$$

Definition 2.1. Given a nonempty $\alpha \subseteq \langle n \rangle$ and provided that $A[\alpha]$ is invertible, we define the *principal pivot transform of* $A \in \mathcal{M}_n(\mathbb{C})$ *relative to* α as the matrix $\mathrm{ppt}\,(A,\alpha)$ obtained from A by replacing

$$\begin{array}{llll} A[\alpha] & \text{by } A[\alpha]^{-1}, & A[\alpha,\overline{\alpha}] & \text{by } -A[\alpha]^{-1}A[\alpha,\overline{\alpha}], \\ A[\overline{\alpha},\alpha] & \text{by } A[\overline{\alpha},\alpha]A[\alpha]^{-1} & A[\overline{\alpha}] & \text{by } A/A[\alpha]. \end{array}$$

By convention, $\mathrm{ppt}\,(A,\emptyset) = A$.

To illustrate this definition, when $\alpha = \{1,\dots,k\}$ $(0 < k < n)$,

$$\mathrm{ppt}\,(A,\alpha) = \begin{bmatrix} A[\alpha]^{-1} & -A[\alpha]^{-1}A[\alpha,\overline{\alpha}] \\ A[\overline{\alpha},\alpha]A[\alpha]^{-1} & A/A[\alpha] \end{bmatrix}.$$

Definition 2.2. For $A \in \mathcal{M}_n(\mathbb{C})$ with $-1 \notin \sigma(A)$, consider the fractional linear map $F_A \equiv (I+A)^{-1}(I-A)$. This map is an involution, namely, $A = (I+F_A)^{-1}(I-F_A)$. The matrix F_A is referred to as the *Cayley transform* of A.

Conditions on the entries and stability

We call $A = [a_{ij}] \in \mathcal{M}_n(\mathbb{C})$ *row diagonally dominant* if for all $i \in \langle n \rangle$,

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|.$$

Note that in our terminology the diagonal dominance is strict. Due to the Geršgorin Theorem [24, Theorem 6.1.1], row diagonally dominant matrices with positive diagonal entries are *positive stable*, namely, their eigenvalues lie in the open right half-plane.

Matrix $A = [a_{ij}] \in \mathcal{M}_n(\mathbb{R})$ is called a *B-matrix* if

$$\sum_{k=1}^{n} a_{ik} > 0 \quad \text{and} \quad \frac{1}{n} \sum_{k=1}^{n} a_{ik} > a_{ij} \quad \text{for all } j \neq i.$$

The properties and applications of B-matrices are studied in [39].

Positivity classes

Recall that $\mathbb{P}$ denotes the complex P-matrices (of a given order), that is, matrices all of whose principal minors are positive. We also let $\mathbb{P}_{\mathrm{M}}$ denote the class of matrices all of whose positive integer powers are in $\mathbb{P}$.

We call a matrix A *semipositive* if there exists $x \geq 0$ such that $Ax > 0$. Notice that by continuity of the map $x \to Ax$, semipositivity of A is equivalent to the existence of $u > 0$ such that $Au > 0$

A *positive definite* matrix A is simply an Hermitian (i.e., $A = A^*$) P-matrix. A *totally positive* matrix is a matrix all of whose minors (principal and non-principal) are positive.

A *Z-matrix* is a square matrix all of whose off-diagonal entries are non-positive. An (invertible) *M-matrix* is a positive stable Z-matrix or, equivalently, a semipositive Z-matrix. An *inverse M-matrix* is the inverse of an M-matrix (see [3, 25] for general background on M-matrices). An *MMA-matrix* is a matrix all of whose positive integer powers are irreducible M-matrices (see below for the definition of irreducibility).

It is known that an M-matrix A can be written as $A = sI - B$, where $B \geq 0$ and $s > \rho(B)$. The Perron-Frobenius Theorem applied to B and B^T implies that A possesses right and left nonnegative eigenvectors x, y, respectively, corresponding to the eigenvalue $s - \rho(B)$. We refer to x and y as the *right* and *left Perron* eigenvectors of A, respectively. When B is also irreducible, it is known that $s - \rho(B)$ is a simple eigenvalue of A and that $x > 0$ and $y > 0$.

The *companion matrix* of $A = [a_{ij}]$, denoted by $\mathcal{M}(A) = [b_{ij}]$, is defined by

$$b_{ij} = \begin{cases} -|a_{ij}| & \text{if } i \neq j \\ |a_{ii}| & \text{otherwise.} \end{cases}$$

We call $A \in \mathcal{M}_n(\mathbb{C})$ an *H-matrix* if $\mathcal{M}(A)$ is an M-matrix.

A less known class of matrices is defined next; it was introduced by Pang in [37, 38], extending notions introduced by Mangasarian and Dantzig. Such matrices were called 'hidden Minkowski matrices' in [38]; we adopt a new name for this class of matrices, indicative of their matricial nature and origin:

Definition 2.3. Consider a matrix $A \in \mathcal{M}_n(\mathbb{R})$ of the form

$$A = (s_1 I - P_1)\,(s_2 I - P_2)^{-1},$$

where $s_1,\, s_2 \in \mathbb{R}$, $P_1,\, P_2 \geq 0$, and for some vector $u \geq 0$,

$$P_1 u < s_1 u \quad \text{and} \quad P_2 u < s_2 u.$$

We call A a *mime*, an acronym for '**M**-matrix and **I**nverse **M**-matrix **E**xtension', because the class of mimes contains the M-matrices (by taking $P_2 = 0, s_2 = 1$) and their inverses (by taking $P_1 = 0, s_1 = 1$). We refer to the positive vector u above as a *common semipositivity vector* of A.

We continue with the definition of a type of orthogonal matrix to be used in generating matrices in $\mathbb{P}_{\mathrm{M}}$. An n-by-n *Soules matrix* R is an orthogonal matrix with columns $\{w_1, \ldots, w_n\}$ such that $w_1 > 0$ and $R\Lambda R^T \geq 0$ for every

$$\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$$

with $\lambda_1 \geq \ldots \geq \lambda_n \geq 0$. Soules matrices can be constructed starting with an arbitrary positive vector w_1 such that $\|w_1\|_2 = 1$; for details see [9, 45].

Sign patterns and directed graphs

There are two fundamental methods of generating P-matrices based on the notion of the *sign pattern* of a real matrix A, namely, the corresponding array consisting of $(0, +, -)$ entries according to the sign of each entry of A. In preparation for describing these methods, we need the following definitions.

We call a diagonal matrix S whose diagonal entries equal ± 1 a *signature (matrix).* Note that $S = S^{-1}$; thus we refer to SAS as a *signature similarity* of A.

Matrix $A \in \mathcal{M}_n(\mathbb{R})$ is called *sign nonsingular* if every matrix with the same sign pattern as A is nonsingular.

The *directed graph*, $D(A)$, of $A = [a_{ij}] \in \mathcal{M}_n(\mathbb{R})$ consists of the set of vertices $\{1, \ldots, n\}$ and the set of directed edges (i, j) connecting vertex i to vertex j if and only if $a_{ij} \neq 0$. We say $D(A)$ is *strongly connected* if any two distinct vertices i, j are connected by a path of edges (i, i_1), (i_1, i_2), $\dots$, (i_{k-1}, i_k), (i_k, j); we then refer to A as an *irreducible matrix.* A *cycle of length* k in $D(A)$ consists of edges

$$(i_1, i_2),\ (i_2, i_3),\ \dots,\ (i_{k-1}, i_k),\ (i_k, i_1),$$

where the vertices $i_1, \dots, i_k$ are distinct. The nonzero diagonal entries of A correspond to cycles of length 1 in $D(A)$. The *signed directed graph* of A, $S(A)$, is obtained from $D(A)$ by labeling each edge (i, j) with the sign of a_{ij}. We define the sign of the above cycle on the vertices $\{i_1, \dots, i_k\}$ to be the sign of the product $a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{k-1} i_k} a_{i_k i_1}$.

Denote by $M_{n,k}$ the set of matrices $A \in \mathcal{M}_n(\mathbb{R})$ with nonzero diagonal entries such that the length of the longest cycle in $D(A)$ is no more than k. For matrices in $M_{n,k}$ we adopt the following further notation: $A \in \mathbb{P}_{n,k}$ if A is, in addition, a P-matrix, and $A \in S_{n,k}$ if all the cycles in $S(-A)$ are signed negatively.

3. Mapping $\mathbb{P}$ into $\mathbb{P}$. First, it is worth noting that if $A_1, \dots, A_k \in \mathbb{P}$, then any block triangular matrix with diagonal blocks $A_1, \dots, A_k$ is a P-matrix. In particular, the direct sum of P-matrices is a P-matrix. We proceed with transformations that map $\mathbb{P}$ into itself.

THEOREM 3.1. *Assume that $A \in \mathcal{M}_n(\mathbb{C})$ is a P-matrix. Then the following hold.*

(1) $A^T \in \mathbb{P}$.

(2) $QAQ^T \in \mathbb{P}$ *for every permutation matrix* Q.

(3) $SAS \in \mathbb{P}$ *for every signature matrix* S.

(4) $DAE \in \mathbb{P}$ *for all diagonal matrices* D, E *such that* DE *has positive diagonal entries.*

(5) $A + D \in \mathbb{P}$ *for all diagonal matrices* D *with nonnegative diagonal entries.*

(6) $A[\alpha] \in \mathbb{P}$ *for all nonempty* $\alpha \subseteq \langle n \rangle$.

(7) $A/A[\alpha] \in \mathbb{P}$ *for all* $\alpha \subseteq \langle n \rangle$.

(8) $\operatorname{ppt}(A, \alpha) \in \mathbb{P}$ *for all* $\alpha \subseteq \langle n \rangle$.
In particular, when $a = \langle n \rangle$, *we obtain that* $\operatorname{ppt}(A, \langle n \rangle) = A^{-1} \in \mathbb{P}$.

(9) $I + F_A,\ I - F_A \in \mathbb{P}$.

(10) $T\,I + (I - T)\,A \in \mathbb{P}$ *for all* $T \in [0, I]$.

Proof. Clauses (1)–(4) and (6) are direct consequences of determinantal properties and the definition of a P-matrix.

(5) Notice that if $A = [a_{ij}] \in \mathbb{P}$, then $\dfrac{\partial \det A}{\partial\, a_{ii}} = \det A[\overline{\{i\}}] > 0$, that is, $\det A$ is an increasing function of the diagonal entries. Thus, as the diagonal entries of D are added in succession to the diagonal entries of A, the determinant of A, and similarly every principal minor of A, remains positive.

(7) Since A is a P-matrix, $A[\alpha]$ is invertible and A^{-1}, up to a permutation similarity,

has the block representation (see e.g., [24, (0.7.3) p. 18])

$$\begin{bmatrix} (A/A[\overline{\alpha}])^{-1} & -A[\alpha]^{-1}A[\alpha,\overline{\alpha}](A/A[\alpha])^{-1} \\ -(A/A[\alpha])^{-1}A[\overline{\alpha},\alpha]A[\alpha]^{-1} & (A/A[\alpha])^{-1} \end{bmatrix}.$$

Therefore, every principal submatrix of A^{-1} is of the form $(A/A[\alpha])^{-1}$ for some $\alpha \subseteq \langle n\rangle$ and has determinant $\det A[\alpha]/\det A > 0$. This shows that A^{-1} is a P-matrix and, in turn, that $(A/A[\alpha])^{-1}$ and thus $A/A[\alpha]$ are P-matrices for every $\alpha \subseteq \langle n\rangle$.

(8) Let A be a P-matrix and consider first the case where α is a singleton; without loss of generality assume that $\alpha = \{1\}$. Let $B = \mathrm{ppt}\,(A,\alpha) = [b_{ij}]$. By definition, the principal submatrices of B that do not include entries from the first row coincide with the principal submatrices of $A/A[\alpha]$ and thus, by (7), have positive determinants. The principal submatrices of B that include entries from the first row of B are equal to the corresponding principal submatrices of the matrix B' obtained from B using $b_{11} = (A[\alpha])^{-1} > 0$ as the pivot and eliminating the nonzero entries below it. Notice that

$$B' = \begin{bmatrix} 1 & 0 \\ -A[\overline{\alpha},\alpha] & I \end{bmatrix} \begin{bmatrix} b_{11} & -b_{11}A[\alpha,\overline{\alpha}] \\ A[\overline{\alpha},\alpha]b_{11} & A/A[\alpha] \end{bmatrix} = \begin{bmatrix} b_{11} & -b_{11}A[\alpha,\overline{\alpha}] \\ 0 & A[\overline{\alpha}] \end{bmatrix}.$$

That is, B' is itself a P-matrix for it is block upper triangular and the diagonal blocks are P-matrices. It follows that all the principal minors of B are positive and thus B is a P-matrix. Next, consider the case $\alpha = \{i_1,\ldots,i_k\} \subseteq \langle n\rangle$ with $k \geq 1$. By our arguments so far, the sequence of matrices

$$A_0 = A, \quad A_j = \mathrm{ppt}\,(A_{j-1},\{i_j\}),\; j = 1,\ldots,k$$

is well defined and comprises P-matrices. Moreover, from the uniqueness of $B = \mathrm{ppt}\,(A,\alpha)$ shown in [47, Theorem 3.1] it follows that $A_k = \mathrm{ppt}\,(A,\alpha) = B$ and thus B is a P-matrix.

(9) First, since A is a P-matrix, A is nonsingular and has no negative real eigenvalues. Hence F_A is well-defined. It can also be verified that

$$I + F_A = 2(I+A)^{-1} \quad \text{and} \quad I - F_A = 2(I+A^{-1})^{-1}.$$

As addition of positive diagonal matrices and inversion are operations that preserve P-matrices (see (5) and (8)), it follows that $I - F_A$ and $I + F_A$ are P-matrices.

(10) Let $A \in \mathbb{P}$ and $T = \mathrm{diag}(t_1,\ldots,t_n) \in [0,I]$. Since T and $I - T$ are diagonal,

$$\det(TI + (I-T)A) = \sum_{\alpha\subseteq\langle n\rangle} \prod_{i\notin\alpha} t_i \, \det\left(((I-T)A)\,[\alpha]\right).$$

As $t_i \in [0,1]$ and $A \in \mathbb{P}$, all summands in this determinantal expansion are nonnegative. Unless $T = 0$, in which case $TI + (I-T)A = A$, one of these summands is positive. Hence $\det(A) > 0$. The exact same argument can be applied to any principal submatrix of A, proving that $A \in \mathbb{P}$. □

Remark 3.2. The following comments pertain to the above theorem.

(i) It is quite clear that each one of the clauses (1)–(5), (7) and (10) represents a necessary and sufficient condition that A be a P-matrix.

(ii) Clause (8) is also a necessary and sufficient condition that A be a P-matrix. Sufficiency follows by taking $\alpha = \emptyset$. A stronger result is, however, true: The principal pivot transform of A relative to any one subset α of $\langle n \rangle$ being a P-matrix is necessary and sufficient for **all** principal pivot transforms of A to be in $\mathbb{P}$; see [47].

(iii) A more extensive analysis of the relation between the Cayley transforms implicated in (9), P-matrices and other matrix positivity classes is undertaken in [12].

We conclude this section with a few more basic facts and characterizations of P-matrices with proofs, which are quoted in the remainder.

THEOREM 3.3. *If $A \in \mathcal{M}_n(\mathbb{R})$, then A is a P-matrix if and only if for each nonzero $x \in \mathbb{R}^n$, there exists $j \in \langle n \rangle$ such that $x_j(Ax)_j > 0$.*

Proof. Suppose that $A \in \mathcal{M}_n(\mathbb{R})$ and that there exists $x \in \mathbb{R}^n$ such that for all $j \in \langle n \rangle$, $x_j(Ax)_j \leq 0$. Then there exists positive diagonal matrix D such that $Ax = -Dx$, i.e., $(A + D)x = 0$. By Theorem 3.1 (5), A is not a P-matrix. This proves that when $A \in \mathcal{M}_n(\mathbb{C})$ is a P-matrix, then for every nonzero $x \in \mathbb{R}^n$, there exists $j \in \langle n \rangle$ such that $x_j(Ax)_j > 0$.

Suppose now that $A \in \mathcal{M}_n(\mathbb{R})$ and that for each nonzero $x \in \mathbb{R}^n$, there exists $j \in \langle n \rangle$ such that $x_j(Ax)_j > 0$. Notice that the same holds for every principal submatrix $A[\alpha]$ of A, by simply considering vectors x such that $x[\overline{\alpha}] = 0$. Thus all the real eigenvalues of $A[\alpha]$ are positive, for all nonempty $\alpha \subseteq \langle n \rangle$. As complex eigenvalues come in conjugate pairs, it follows that all the principal minors of A are positive, completing the proof of the theorem. □

THEOREM 3.4. *Let $A \in \mathcal{M}_n(\mathbb{R})$ be a P-matrix. Then A is a semipositive matrix.*

Proof. We will prove the contrapositive. Recall the following Theorem of the Alternative (see [34]), stating that for a real matrix A, exactly one of the following is true:

(i) There is an $x \geq 0$, $x \neq 0$, such that $A^T x \leq 0$.
(ii) There is a $y \geq 0$ such that $Ay > 0$.

Thus, if A is not semipositive, then there is a nonzero $x \geq 0$ such that $A^T x \leq 0$. By Theorem 3.3, A^T is not a P-matrix and so A is not a P-matrix. □

THEOREM 3.5. [31, Theorem 3.1] *Let $A = BC^{-1}\mathcal{M}_n(\mathbb{R})$. Then $A \in \mathbb{P}$ if and only if the matrix $TB + (I - T)C$ is invertible for every $T \in [0, I]$.*

Proof. If $A \in \mathbb{P}$, by Theorem 3.1 (10), $TI + (I - T)BC^{-1} \in \mathbb{P}$ for every $T \in [0, I]$. Thus $TC + (I - T)B$ is invertible for every $T \in [0, I]$.

For the converse, suppose $TI + (I - T)BC^{-1}$ is invertible for every $T \in [0, I]$ and, by way of contradiction, suppose $BC^{-1} \notin \mathbb{P}$. By Theorem 3.3, there exists nonzero $x \in \mathbb{R}^n$ such that $y_j x_j \leq 0$ for every $j \in \langle n \rangle$, where $y = BC^{-1}x$. Consider then $T = \text{diag}(t_1, \ldots, t_n)$, where $t_j \in [0, 1]$ are selected so that $t_j x_j + (1 - t_j)y_j = 0$ for all j. It follows that

$$Tx + (I - T)BC^{-1}x = 0,$$

a contradiction that completes the proof. □

4. Generating matrices in $\mathbb{P}$. Below we originate a list of P-matrix generating methods. We indicate any special properties these P-matrices have and provide or outline proofs as necessary.

Generating method 4.1. *Every row diagonally dominant matrix $A \in \mathcal{M}_n(\mathbb{R})$ with positive diagonal entries is a positive stable P-matrix.*

Proof. Every principal submatrix $A[\alpha]$ is row diagonally dominant with positive diagonal entries and thus positive stable. In particular, every real eigenvalue of $A[\alpha]$ is positive. As the complex eigenvalues come in conjugate pairs, it follows the $\det A[\alpha] > 0$ for all nonempty $\alpha \subset \langle n \rangle$. □

Generating method 4.2. *Let $A \in \mathcal{M}_n(\mathbb{R})$ such that $A + A^T$ is positive definite. Then A is a positive stable P-matrix.*

Proof. As is well-known, every principal submatrix $A[\alpha] + A[\alpha]^T$ of $A + A^T$ is also positive definite. Thus for every $x \in \mathbb{C}^{|\alpha|}$,

$$x^* A[\alpha]\, x = x^* \, \frac{A[\alpha] + A[\alpha]^T}{2} \, x \; + \; x^* \, \frac{A[\alpha] - A[\alpha]^T}{2} \, x$$

has positive real part. It follows that every eigenvalue of $A[\alpha]$ has positive real part and thus every principal submatrix of A is positive stable. As above, this implies that A is a positive stable P-matrix. □

The following is stated and proved in [39].

Generating method 4.3. *Let $A \in \mathcal{M}_n(\mathbb{R})$ be a B-matrix. Then $A \in \mathbb{P}$.*

Generating method 4.4. *For every nonsingular matrix $A \in \mathcal{M}_n(\mathbb{C})$, the matrices A^*A, AA^* are positive definite and thus P-matrices.*

Generating method 4.5. *Given a square matrix $B \geq 0$ and $s > \rho(B)$, $A = sI - \rho(B) \in \mathbb{P}$. In particular, A is an M-matrix.*

Proof. Consider $B = [b_{ij}] \geq 0$ and $s > \rho(B)$. By [24, Theorem 8.1.18], $s > \rho(B[\alpha])$ for every nonempty $\alpha \subseteq \langle n \rangle$. Thus every principal submatrix $A[\alpha]$ of A is positive stable and, as in the proof of Generating Method 4.1, it must have positive determinant. □

Generating method 4.6. *Let $B, C \in \mathcal{M}_n(\mathbb{R})$ be row diagonally dominant matrices with positive diagonal entries. Then $BC^{-1} \in \mathbb{P}$.*

Proof. Let B and C be as prescribed and notice that by the Levy-Desplanques theorem (see [24, Corollary 5.6.17]), $TB + (I - T)C$ is invertible for every $T \in [0, I]$. Thus, by Theorem 3.5, $BC^{-1} \in \mathbb{P}$. □

Remark 4.7. It is easy to construct examples showing that the above result fails to be true when B or C are not real matrices. At the Oberwolfach meeting [51], C.R. Johnson stated the above result and conjectured that every real P-matrix A can be factored into BC^{-1}, where $B, C \in \mathcal{M}_n(\mathbb{R})$ are row diagonally dominant matrices with positive diagonal entries.

We continue with results on mimes. We will restate in our context a result from [37] as well as a remarkable method for constructing entrywise nonnegative mimes. The proofs of these results in [37] rely on (hidden) Leontief matrices and principal pivot transforms. In particular, the proof of the latter construction method is attributed to

Mangasarian [35], who also used ideas from mathematical programming. We include here shorter proofs that are based on what is now considered standard M-matrix and P-matrix theory.

Generating method 4.8. *Let* $A \in \mathcal{M}_n(\mathbb{R})$ *be a mime. Then* $A \in \mathbb{P}$.

Proof. Let A be a mime and s_1, s_2, P_1, P_2 and $u > 0$ be as in Definition 2.3. Then, for every $T \in [0, I]$, the matrix

$$C = T(s_2 I - P_2) + (I - T)(s_1 I - P_1)$$

is a Z-matrix and $Cu > 0$ (i.e., C is a semipositive Z-matrix). This means that C is an M-matrix. In particular, C is invertible for every $T \in [0, I]$. By Theorem 3.5, we can now conclude that $A = (s_1 I - P_1)(s_2 I - P_2)^{-1} \in \mathbb{P}$. □

Generating method 4.9. *Let* $B \geq 0$ *with* $\rho(B) < 1$. *Let* $\{a_k\}_{k=1}^m$ *be a sequence such that* $0 \leq a_{k+1} \leq a_k \leq 1$ *for all* $k = 1, \ldots, m-1$. *Then*

$$A \;=\; I \,+\, \sum_{k=1}^{m} a_k B^k \;\in\; \mathbb{P}.$$

If m *is infinite, under the additional assumption that* $\sum_{k=1}^{\infty} a_k$ *is convergent, we can still conclude that* $A \in \mathbb{P}$. *More specifically,* A *is a mime.*

Proof. Consider the matrix $C \;\equiv\; A(I - B)$ and notice that C can be written as

$$C = I - G, \quad \text{where} \;\; G \;\equiv\; B - \sum_{k=1}^{m} a_k (B^k - B^{k+1}).$$

First we show that G is nonnegative: Indeed, as $0 \leq a_{k+1} \leq a_k \leq 1$, we have

$$\begin{aligned} G \;&\geq\; B - \sum_{k=1}^{m} a_k B^k + \sum_{k=1}^{m-1} a_{k+1} B^{k+1} + a_k B^{k+1} \\ &=\; B - \sum_{k=1}^{m} a_k B^k + \sum_{k=2}^{m} a_k B^k + a_k B^{k+1} \\ &=\; B - a_1 B + a_m B^{m+1} \;=\; (1 - a_1) B + a_m B^{m+1} \;\geq\; 0. \end{aligned}$$

Next we show that $\rho(G) < 1$. For this purpose, consider the function

$$\begin{aligned} f(z) \;&=\; z - \sum_{k=1}^{m} a_k (z^k - z^{k+1}) \\ &=\; z(1 - a_1) + z^2(a_1 - a_2) + \ldots + z^m(a_{m-1} - a_m) + a_m z^{m+1}, \end{aligned}$$

in which expression all the coefficients are by assumption nonnegative. Thus $|f(z)| \leq f(|z|)$. However, for $|z| < 1$, we have

$$f(|z|) = |z| - \sum_{k=1}^{m} a_k (|z|^k - |z|^{k+1}) \leq |z|.$$

That is, for all $|z| < 1$,

$$|f(z)| \leq f(|z|) \leq |z|.$$

We can now conclude that for every $\lambda \in \sigma(B)$,

$$|f(\lambda)| \leq |\lambda| \leq \rho(B) < 1;$$

that is, $\rho(G) < 1$. We have thus shown that

$$A = (I - G)(I - B)^{-1},$$

where B, $G \geq 0$ and $\rho(B)$, $\rho(G) < 1$. Also, as $B \geq 0$, we may consider its Perron vector $u \geq 0$. By construction, u is also an eigenvector of G corresponding to $\rho(G)$. That is, there exists vector $u > 0$ such that

$$Bu \;=\; \rho(B)\, u \;<\; u \quad \text{and} \quad Gu \;=\; \rho(G)\, u \;<\; u\,.$$

Thus A is a mime and so it belongs to $\mathbb{P}$ by Theorem 4.8. □

Generating method 4.10. *Let $B \geq 0$. Then $e^{tB} \in \mathbb{P}$ for all $t \in [0, 1/\rho(B))$.*

Proof. It follows from Generating Method 4.9 by taking $m = \infty$ and $a_k = \frac{1}{k!}$. □

H-matrices having positive diagonal entries are mimes and thus P-matrices by the results in [37]. We include a short direct proof of this fact.

Generating method 4.11. *Let $A \in \mathcal{M}_n(\mathbb{R})$ be an H-matrix with positive diagonal entries. Then $A \in \mathbb{P}$.*

Proof. As $\mathcal{M}(A)$ is an M-matrix, by [3, Theorem 2.3 (M_{35}), Chapter 6], there exists a diagonal matrix $D \geq 0$ such that AD is strictly row diagonally dominant. As AD has positive diagonal entries, the result follows from Theorem 3.1 (4) and Generating Method 4.1. □

We now turn our attention to P-matrices generated via sign patterns.

Generating method 4.12. $S_{n,k} \subset \mathbb{P}_{n,k} \subset \mathbb{P}$.

Proof. Recall that by the definitions in Section 2, $A = [a_{ij}] \in S_{n,k}$ means that the longest cycle in the directed graph of A is $k \geq 1$; the cycles of odd (resp., even) length in the signed directed graph of A are positive (resp., negative). In particular, the diagonal entries of A are **all** positive. In the standard expansion of the determinant of A, the terms that appear are of the form

$$(-1)^{\mathrm{sign}(\sigma)} \;\; a_{1\sigma(1)} \, \ldots \, a_{n\sigma(n)}, \tag{4.1}$$

where σ is a member of the symmetric permutation group on n elements. However, σ can be uniquely partitioned into its permutation cycles, some of odd and some of even length. In fact, $(-1)^{\mathrm{sign}(\sigma)} = (-1)^{n+q}$, where q is the number of odd cycles. As a consequence, the quantity in (4.1) is nonnegative. One such term in $\det A$ consists of the diagonal entries of A, which are positive and thus the sign of that term is $(-1)^{2n}$. It follows that $\det A > 0$. Notice also that a similar argument can be applied to every principal minor of A. Thus A is a P-matrix, in particular a matrix in $\mathbb{P}_{n,k}$. □

Remark 4.13. The argument in the proof above is essentially contained in the proof of [10, Theorem 1.9]. Also, notice that the matrices in $\mathbb{P}_{n,k}$ are sign nonsinular.

Example 4.14. The sign pattern

$$\begin{bmatrix} + & + & + \\ - & + & + \\ 0 & - & + \end{bmatrix}$$

belongs to $S_{3,2}$ and thus every matrix with this sign pattern is a P-matrix.

REMARK 4.15. It has been shown [29] that if $A \in \mathbb{P}_{n,k}$, where $k < n$, then

$$\sigma(A) \subset \{z \in \mathbb{C} \;:\; -\pi + \frac{\pi}{n-1} \;<\; \arg z \;<\; \pi - \frac{\pi}{n-1}\}.$$

This generalizes a result of Kellogg [32], which states that for all $A \in \mathbb{P}$,

$$\sigma(A) \subset \{z \in \mathbb{C} \;:\; -\pi + \frac{\pi}{n} \;<\; \arg z \;<\; \pi - \frac{\pi}{n}\}.$$

GENERATING METHOD 4.16. *Let $A \in \mathcal{M}_n(\mathbb{R})$ be sign nonsingular and $B \in \mathcal{M}_n(\mathbb{R})$ any matrix with the same sign pattern as A. Then BA^{-1}, $B^{-1}A \in \mathbb{P}$.*

Proof. Since A and B are sign-nonsingular having the same sign pattern, we have that $C = TB + (I - T)A$ is also sign-nonsingular for every $T \in [0, I]$. Thus, by Theorem 3.5, $BA^{-1} \in \mathbb{P}$. The conclusion for $B^{-1}A$ follows from a result in [31] dual to the one quoted in Theorem 3.5. ◻

We conclude this section by mentioning that some sufficient conditions for a matrix A to be a P-matrix based on its signed directed graph are provided in [28]. In particular, if $A = [a_{ij}]$ is sign symmetric (i.e., $a_{ij}a_{ji} \geq 0$) and if its undirected graph is a tree (i.e., a connected acyclic graph) then necessary and sufficient conditions that A be a P-matrix are obtained.

5. Generating matrices all of whose powers are in $\mathbb{P}$. As is well-known, if A is positive definite, then so are all of its powers. Thus positive definite matrices belong to $\mathbb{P}_{\mathrm{M}}$. By the Cauchy-Binet formula for the determinant of the product of two matrices (see [24]), it follows that powers of totally positive matrices are totally positive and thus belong to $\mathbb{P}_{\mathrm{M}}$. Below is a constructive characterization of totally positive matrices found in [17] (see also [11]), which therefore allows us to construct matrices in $\mathbb{P}_{\mathrm{M}}$.

THEOREM 5.1. *Matrix $A \in \mathcal{M}_n(\mathbb{R})$ is totally positive if and only if there exist positive diagonal matrix D and positive numbers l_i, u_j $\;(i, j = 1, \ldots, k)$, $k = \binom{n}{2}$, such that $A = FDG$, where*

$$F \;=\; [E_n(l_k)\, E_{n-1}(l_{k-1})\; \ldots\; E_2(l_{k-n+2})]\; [E_n(l_{k-n+1})\; \ldots\; E_3(l_{k-2n+4})]\; \ldots\; [E_n(l_1)]$$

and

$$G \;=\; \left[E_n^T(u_1)\right]\; \left[E_{n-1}^T(u_2)\, E_n^T(u_3)\right]\; \ldots\; \left[E_2^T(u_{k-n+2})\; \ldots\; E_{n-1}^T(u_{k-1}E_n^T(u_k)\;\right]\;;$$

here $E_k(\beta) = I + \beta E_{k,k-1}$, where $E_{k,k-1}$ denotes the $(k, k-1)$ elementary matrix.

Another subclass of $\mathbb{P}_{\mathrm{M}}$ are the MMA-matrices, introduced in [16] (recall that these are matrices all of whose positive integer powers are irreducible M-matrices). By the results in [9] (see also [46]) combined with the notion of a Soules matrix allow us to firstly construct all symmetric (inverse) MMA-matrices:

THEOREM 5.2. *Let $A \in \mathcal{M}_n(\mathbb{R})$ be an invertible and symmetric matrix. Then A is an MMA-matrix if and only if $A^{-1} = R\Lambda R^T$, where R is a Soules matrix and $\Lambda = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_n)$ with $\lambda_1 > \lambda_2 \geq \ldots \geq \lambda_n > 0$.*

It has also been shown in [16] that for every MMA-matrix $B \in \mathcal{M}_n(\mathbb{R})$ there exists a positive diagonal matrix D such that $A = D^{-1}BD$ is a symmetric MMA-matrix. In [21], D is found to be

$$D = \operatorname{diag}(x_1^{1/2}\, y_1^{-1/2}, \ldots, x_n^{1/2}\, y_n^{-1/2}),$$

where x, y are the unit strictly positive right and left Perron eigenvectors of B, respectively. Thus, we have a way of constructing arbitrary MMA-matrices: Determine an orthogonal Soules matrix R starting with an arbitrary unit positive vector w_1 (as described in section 2). Choose a matrix Λ as in Theorem 5.2 and form $R\Lambda R^T$. Then $A = R\Lambda^{-1}R^T$ is a symmetric MMA-matrix with unit left and right Perron eigenvectors equal to w_1. Choose now positive diagonal matrix $\hat{D}$ and let $B = \hat{D}R\Lambda^{-1}R^T\hat{D}^{-1}$. Then B is an MMA-matrix having $\hat{D}w_1$ and $\hat{D}^{-1}w_1$ are right and left Perron vectors, respectively.

Remark 5.3.

(i) Given a matrix A in $\mathbb{P}_{\mathrm{M}}$, the matrices A^T, $D^{-1}AD$ (where D is a positive diagonal matrix), QAQ^T (where Q is a permutation matrix), and SAS (where S is a signature matrix) all belong to $\mathbb{P}_{\mathrm{M}}$.

(ii) As all the classes of matrices in $\mathbb{P}_{\mathrm{M}}$ identified so far comprise positive stable matrices, a natural (yet unanswered) question is posed in [20]: *Are all matrices in* $\mathbb{P}_{\mathrm{M}}$ *positive stable ?*

6. More on mimes. In this section, we further examine mimes. It has been shown in [37] that the definition of a mime A is tantamount to A being 'hidden Z' and a P-matrix at the same time. More relevant to our context is the following result, reproven here using the language and properties of M-matrices.

Proposition 6.1. *Let $A \in \mathcal{M}_n(\mathbb{R})$. Then A is a mime if and only if*
(1) $AX = Y$ for some Z-matrices X and Y, and
(2) A and X are semipositive.

Proof. Clearly, if A is a mime as in Definition 2.3, then (1) holds with the roles of X and Y being played by $(s_2I - P_2)$ and $(s_1I - P_1)$, respectively. That (2) holds follows from the fact that $z \equiv Xu > 0$ (i.e., X is semipositive) and $Yu > 0$, where $u > 0$ is a common semipositivity vector of A. We then have that $Az = YX^{-1}Xu = Yu > 0$; that is, A is also semipositive.

For the converse, suppose (1) and (2) hold. Then X is an M-matrix. As A is assumed semipositive, $Ax > 0$ for some $x > 0$. Let $u = A^{-1}x$. Then, $Yu = Ax > 0$; that is, Y is also semipositive and so an M-matrix as well. In fact, u is a common semipositivity vector of A and thus A is a mime. □

Remark 6.2.

(i) Based on the above result, we can assert that given a mime A and a positive diagonal matrix D, $A + D$ and DA are also mimes.

(ii) Principal pivot transforms, permutation similarities, Schur complementation and extraction of principal submatrices leave the class of mimes invariant; see [37, 47].

Recall that the mimes form a subclass of the P-matrices that includes the M-matrices, the H-matrices with positive diagonal entries, as well as their inverses. The following is another large class of mimes mentioned in [38].

THEOREM 6.3. *Every triangular P-matrix is a mime.*

Proof. We prove the claim by induction on the order k of A. If $k = 1$ the result is obviously true. Assume the claim is true for $k = n - 1$; we will prove it for $k = n$. For this purpose, consider the triangular P-matrix

$$A = \begin{bmatrix} A_{11} & a \\ 0 & a_{22} \end{bmatrix},$$

where A_{11} is an $(n-1)$-by-$(n-1)$ P-matrix, $a \in \mathbb{R}^{n-1}$ and $a_{22} > 0$. By the inductive hypothesis and Theorem 6.1, there exist Z-matrices X_{11}, Y_{11} and positive vector $u_1 \in \mathbb{R}^{n-1}$ such that $A_{11}X_{11} = Y_{11}$, $Y_{11}u_1 > 0$, and $X_{11}u_1 > 0$. Consider then the Z-matrix

$$X \equiv \begin{bmatrix} X_{11} & -X_{11}u_1 \\ 0 & x_{22} \end{bmatrix},$$

where $x_{22} > 0$ is to be chosen. Then let

$$Y \equiv AX = \begin{bmatrix} A_{11}X_{11} & -Y_{11}u_1 + x_{22}a \\ 0 & a_{22}x_{22} \end{bmatrix}.$$

Notice that $x_{22} > 0$ can be chosen so that Y is a Z-matrix. Let also $u^T = [u_1^T \; u_2]$. Choosing $u_2 > 0$ small enough, we have that $Xu > 0$ and $Yu > 0$. Thus A is a mime by Theorem 6.1. □

In relation to the conjecture in Remark 4.7, we can prove the following.

THEOREM 6.4. *Let A be a mime. Then A can be factored into $A = BC^{-1}$, where B, C are row diagonally dominant matrices with positive diagonal entries.*

Proof. Suppose that $A = (s_1I - P_1)(s_2I - P_2)^{-1}$ is a mime with semipositivity vector $u > 0$ and let $D = \text{diag}(u_1, \ldots, u_n)$. Define $B = (s_1I - P_1)D$ and $C = (s_2I - P_2)D$ so that $Be > 0$ and $Ce > 0$, where e is the all ones vector. Notice that as B and C are Z-matrices with positive diagonal entries, they are indeed row diagonally dominant and $A = BC^{-1}$. □

In view of the conjecture in Remark 4.7 and the definition of a mime, it is natural to wonder whether every P-matrix is a mime and whether the sought factorization is indeed provided by Theorem 6.4. Unfortunately, the answer is negative, i.e., not all P-matrices are mimes as shown by the construction of a counterexample in [38].

7. Miscellaneous related facts. We summarize and provide references to some facts on P-matrices that are related to the purpose of this paper: P-matrix completions, P-matrix intervals and linear transformations of $\mathbb{P}$ into $\mathbb{P}$.

In [27], the authors consider matrices some of whose entries are not specified and raise the problem of when do the unspecified entries can be chosen so that the completed matrix is in $\mathbb{P}$. They show that if the specified entries include all of the diagonal entries, if every fully specified principal submatrix is in $\mathbb{P}$, and if the unspecified entries are symmetrically placed (i.e., (i,j) entry is specified if and only if the (j,i) entry is specified), then the matrix can be completed to a P-matrix. In [8], the authors extend the aforementioned class of partially specified matrices that have a P-matrix completion by providing strategies to achieve its conditions. In [8, 27], partially specified matrices that cannot be completed to be in $\mathbb{P}$ are also discussed. General necessary and sufficient conditions for the completion of a partially specified matrix to a P-matrix are not known to date. Finally, completion problems of subclasses of $\mathbb{P}$ are studied in [1, 18, 22, 23, 30].

In the context of mathematical programming and elsewhere, the analysis of matrix intervals and interval equations play an important role and are intimately related to P-matrices. In particular, it is known that $[A, B]$ consists exclusively of n-by-n invertible matrices if and only if for each of the 2^n matrices $T \in [0, I]$ having diagonal entries in $\{0, 1\}$, $TA + (I - T)B$ is invertible [40, 41]. The case of matrix intervals that consist exclusively of P-matrices is considered in [42] (see also [26]): It is shown that $[A, B]$ consists entirely of P-matrices if and only if for every nonzero $x \in \mathbb{R}^n$, there exists $j \in \langle n \rangle$ such that $x_j \, (Cx)_j > 0$ for every $C \in [A, B]$ (cf. Theorem 3.3).

In [2, 19], the authors study *linear* transformations of $\mathbb{P}$ into $\mathbb{P}$. In [2] it is shown that the linear transformations that map $\mathbb{P}$ *onto* itself are necessarily compositions of transposition, permutation similarity, signature similarity, and positive diagonal equivalence (cf. Theorem 3.1 ((1)–(4), respectively). Under the assumption that the kernel of a linear transformation $\mathcal{L}$ intersects trivially the set of matrices with zero diagonal entries, it is shown in [19] that the linear transformations that map $\mathbb{P}$ *into* itself ($n \geq 3$) are compositions of transposition, permutation similarity, signature similarity, positive diagonal equivalence, and the map $A \to A + D$, where D is a diagonal matrix whose diagonal entries are nonnegative linear combinations of the diagonal entries of A.

Finally, we mention the open problem of studying when the Hadamard (i.e., entrywise) product of two P-matrices is a P-matrix. A related problem concerns Hadamard product of inverse M-matrices. In particular, it is conjectured that that Hadamard square of two inverse M-matrices is an inverse M-matrix; see [36, 50].

8. Detection of P-matrices. As mentioned in the introduction, the P-problem is NP-hard. The exhaustive check of all $2^n - 1$ principal minors of $A \in \mathcal{M}_n(\mathbb{R})$ using Gaussian elimination is an $\mathrm{O}(n^3 2^n)$ task; see [48]. Later on we will describe an alternative test for complex P-matrices, which was first presented in [48] and was shown to be $\mathrm{O}(n^3)$ when applied to real P-matrices.

Determining whether a matrix belongs or not to one of the subclasses of $\mathbb{P}$ that we have discussed in this paper is typically an easier task. To be more specific, to detect a positive definite matrix A one needs to check whether $A = A^*$ and whether the eigenvalues of A are positive or not. To detect an M-matrix A, one needs to check whether A is a Z-matrix and whether A is positive stable. An alternative $\mathrm{O}(n^3)$ procedure for testing whether a Z-matrix is an M-matrix is described in [49]. To detect an inverse M-matrix A, one can apply a test for M-matrices to A^{-1}. Similarly, one needs to apply a test for M-matrices to $\mathcal{M}(A)$ to determine whether a matrix A with positive diagonal entries is an H-matrix or not.

Recall that an H-matrix is also characterized by the existence of a positive diagonal matrix D such that AD is row diagonally dominant. This is why H-matrices are also known in the literature as 'generalized diagonally dominant matrices'. Based on this characterization of H-matrices, an iterative method to determine whether A is an H-matrix or not is developed in [33]; this iterative method also determines a diagonal matrix D with the aforementioned scaling property.

To determine a totally positive matrix, Fekete's criterion [13] may be used: Let $S = \{i_1, \ldots, i_k\}$ with $i_j < i_{j+1}$, $(j = 1, \ldots, k-1)$ and define the *dispersion* of S to be

$$d(S) \;=\; \begin{cases} i_k - i_1 - k + 1 & \text{if } \; k > 1 \\ 0 & \text{if } \; k = 1. \end{cases}$$

Then $A \in \mathcal{M}_n(\mathbb{R})$ is totally positive if and only if $\det A[\alpha|\beta] > 0$ for all $\alpha, \beta \subseteq \langle n \rangle$ with $|\alpha| = |\beta|$ and $d(\alpha) = d(\beta) = 0$. That is, one only needs to check for positivity minors whose row and column index sets are contiguous.

Fekete's criterion is improved in [17] as follows:

THEOREM 8.1. *Matrix $A \in \mathcal{M}_n(\mathbb{R})$ is totally positive if and only if for every $k \in \langle n \rangle$*

(a) $\det A[\alpha|\langle k \rangle] > 0$ *for all* $\alpha \subseteq \langle n \rangle$ *with* $|\alpha| = k$ *and* $d(\alpha) = 0$,

(b) $\det A[\langle k \rangle|\beta] > 0$ *for all* $\beta \subseteq \langle n \rangle$ *with* $|\beta| = k$ *and* $d(\beta) = 0$.

The task of detecting mimes is less straightforward and can be based on Proposition 6.1. As argued in [38] and by noting that, without loss of any generality, the semipositivity vector of X in Proposition 6.1 can be taken to be the all ones vector e (otherwise, our considerations apply with X, Y replaced by XD, YD for a suitable positive diagonal matrix D), the following two-step test for mimes is applicable [38]:

Step 1. Determine whether A is semipositive by solving the linear program

`minimize` $e^T x$
`subject to` $x \geq 0$ `and` $Ax \geq e$

If this program is infeasible, then A is **not** semipositive and thus not a mime; stop.

Step 2. Check the consistency of the linear inequality system

$x_{ij} \leq 0 \quad (i, j = 1, \ldots, n,\; i \neq j)$,
$\sum_{k=1}^{n} a_{ik} x_{kj} \leq 0 \quad (i, j = 1, \ldots, n,\; i \neq j)$,
$\sum_{j=1}^{n} x_{ij} > 0 \quad (i = 1, \ldots, n)$.

If this system is inconsistent, A is not a mime; stop. Otherwise, A is mime as it satisfies the conditions of Proposition 6.1.

REMARK 8.2. The task of detecting matrices in $\mathbb{P}_{\mathrm{M}}$ is an open problem, related to the problem of characterizing matrices in $\mathbb{P}_{\mathrm{M}}$ and their spectra.

Now we turn our attention back to the general P-problem. The following theorem (cf. Theorem 3.1 (7) and (8)) is the theoretical basis for the subsequent algorithm. Although the result is stated and proved in [48] for real P-matrices, its proof is valid for complex P-matrices and included here for completeness.

THEOREM 8.3. [48] *Let $\alpha \in \langle n \rangle$ with $|\alpha| = 1$. Then $A \in \mathbb{P}$ if and only if $A[\alpha] > 0$, $A[\overline{\alpha}] \in \mathbb{P}$, and $A/A[\alpha] \in \mathbb{P}$.*

Proof. Without loss of generality, assume that $\alpha = \{1\}$. Otherwise we can consider a permutation similarity of A. If $A = [a_{ij}]$ is a P-matrix, then $A[\alpha]$ and $A(\alpha)$ are also P-matrices. Also $A/A[\alpha]$ is a P-matrix by Theorem 3.1 (7).

For the converse, assume that $A[\alpha] = [a_{11}]$, $A[\overline{\alpha}]$ and $A/A[\alpha]$ are P-matrices. Using $a_{11} > 0$ as the pivot, we can row reduce A to obtain a matrix B with all of its off-diagonal entries in the first column equal to zero. As is well known, $B[\overline{\alpha}] = A/A[\alpha]$. That is, B is a block triangular matrix whose diagonal blocks are P-matrices. It follows readily that B is a P-matrix. The determinant of any principal submatrix of A that includes entries from the first row of A coincides with the determinant of the corresponding submatrix of B and is thus positive. The determinant of any principal submatrix of A with no entries from the first row coincides with a principal minor of $A[\overline{\alpha}]$ and is also positive. Hence A is a P-matrix. □

The following is an algorithmic implementation of the technique suggested by Theorem 8.3 for the P-problem. For a complexity analysis see [48].

Algorithm P(A)

1. Input $A = [a_{ij}] \in \mathcal{M}_n(\mathbb{C})$
2. If $a_{11} \not> 0$ output **'Not a P-matrix'** stop
3. Evaluate $A/A[\alpha]$, where $\alpha = \{1\}$
4. Call P($A(\alpha)$) and P($A/A[\alpha]$)
5. Output **'This is a P-matrix'**

Finally, we provide a Matlab M-file implementing Algorithm P(A).

```
function [r]=ptest(a)
% Return r=1 if 'a' is a P-matrix (r=0 otherwise)
n = length(a);
if ~(a(1,1)>0), r=0;
elseif n==1, r=1;
else
  b=a(2:n,2:n);
  d=a(2:n,1)/a(1,1);
  c=b-d*a(1,2:n);
  % recursively call
  r=ptest(b) & ptest(c);
end
```

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