QUALITATIVE CONTROLLABILITY AND UNCONTROLLABILITY BY A SINGLE ENTRY

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Abstract

We characterize matrices $A \in \mathcal{C}^{n\times n}$ whose zero/nonzero pattern requires that the controllability matrix $[b \quad Ab \quad A^2b \quad \dots \quad A^{n-1}b] \in \mathcal{C}^{n\times n}$ is of full rank, where $b \in \mathcal{C}^{n\times 1}$ has exactly one nonzero entry. When all the diagonal entries of A are nonzero we show that this occurs if and only if QAQ^T is unreduced upper Hessenberg, with Q being a permutation matrix for which $Qb = [b_1, 0, \dots, 0]^T$. We also characterize matrices A whose zero/nonzero pattern requires that the controllability matrix is of deficient rank.

1 Introduction

Consider a linear control system of the form

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t), \quad t \geq 0, \tag{1.1}$$

where $x(t) \in \mathcal{C}^n$ represents the state vector with entries from the complex field $\mathcal{C}$, $u(t) \in \mathcal{C}^m$ represents the (unconstrained, piecewise continuous) control input, and $A \in \mathcal{C}^{n\times n}$, $B \in \mathcal{C}^{n\times m}$ are constant matrices. This system is called *completely controllable* provided that any initial point in $\mathcal{C}^n$ is controllable, via the dynamics of (1.1), to any other point in $\mathcal{C}^n$ in finite time.

We denote the control system in (1.1) by (A, B). The *controllability matrix* associated with (A, B) is defined to be

$$[B : A] \;\equiv\; [B \;\; AB \;\; A^2B \;\; \ldots \;\; A^{n-1}B] \;\in\; \mathcal{C}^{n\times nm}.$$

It is well known (see [P], §34, Theorem 1) that the following are equivalent:

$$(A, B) \text{ is completely controllable.} \tag{1.2}$$

$$(RAR^{-1}, RB) \text{ is completely controllable, for any nonsingular } R \in \mathcal{C}^{n\times n}. \tag{1.3}$$

$$\text{The } n \times nm \text{ matrix } [B : A] \text{ is of full rank.} \tag{1.4}$$

$$\text{There exists no left eigenvector } x \text{ of } A \text{ such that } x^*B = 0. \tag{1.5}$$

It follows from (1.5) that the following condition can be added to the list:

$$\text{The } n \times (n + m) \text{ matrix } [A - \lambda I \;\; B] \text{ is of full rank for all } \lambda \in \mathcal{C}. \tag{1.6}$$

One of the most difficult problems in determining whether or not a control system is completely controllable stems from the fact that the matrices A and B comprise system parameters that, as a result of noise or measuring errors, are not known precisely. However, in many instances, due to the particular design characteristics of the control system, those entries of A and B that are necessarily zero or nonzero are known. This situation has led to various approaches in studying controllability from a qualitative point of view, by means of combinatorial and graph theory. An example of this is the notion of structural controllability of linear control systems (introduced in [L], see also [C] and [M]), which is equivalent to $[B : A]$ having generic rank n. This means that for all choices of the nonzero entries of A and B, except for those from a proper algebraic variety (see e.g., [C] or §16.1 in [VDW]), $\operatorname{rank}[B : A] = n$. Another qualitative approach to controllability can be found in the recent paper [JMO] where the authors introduce the notion of sign controllability; in this case the control system is completely controllable solely due to the sign pattern of the real matrices A and B and regardless of the magnitude of their nonzero entries. Finally, the reader is refered to [M], where some aspects of linear dynamical systems are studied under the assumption that some of the coefficients in (1.1) are fixed constants, while others are treated as algebraically independent parameters.

We will discuss now the notion of qualitative controllability. By a *zero/nonzero pattern* we mean a rectangular array $\mathbf{A}$ of zeros and nonzeros, the latter denoted by the symbol $*$. We write $A \in \mathbf{A}(\mathcal{C})$ when A is a matrix over $\mathcal{C}$ and when $\mathbf{A}$ is obtained from A by replacing its nonzero entries by $*$. We assume that there is no algebraic dependence between the nonzero entries of $\mathbf{A}$.

Given $n \times n$ and $n \times m$ zero/nonzero patterns $\mathbf{A}$ and $\mathbf{B}$, respectively, we say that $(\mathbf{A}, \mathbf{B})$ is *qualitatively controllable (uncontrollable)* if for all $A \in \mathbf{A}(\mathcal{C})$ and all $B \in \mathbf{B}(\mathcal{C})$, $\operatorname{rank}[B : A] = n$ $(\operatorname{rank}[B : A] < n)$; in such a case we say that $\mathbf{A}$ and $\mathbf{B}$ *require (do not allow)* complete controllability.

It should be noted that structural controllability pertains to zero/nonzero patterns $\mathbf{A}$ and $\mathbf{B}$ that may allow a system that is not completely controllable; that is, the set of qualitatively controllable systems is a proper subset of the structurally controllable systems. Also, qualitatively controllable systems give rise to sign controllable systems, but the converse is not true (see Example 2.7).

Qualitative controllability for scalar input (i.e., $m = 1$) was studied in [MY] from a graph theoretic point of view, where it was called strong structural controllability. In the present work we consider the following scalar input qualitative problems. Given an $n \times 1$ zero/nonzero pattern $\mathbf{b}$ with exactly one nonzero entry, we characterize the $n \times n$ zero/nonzero patterns $\mathbf{A}$ such that $(\mathbf{A}, \mathbf{b})$ is qualitatively controllable or qualitatively uncontrollable. In other words, we are concerned with control systems which, by virtue of their zero/nonzero structure alone, either require or do not allow the ability to exercise full control over the entries of the state vector by means of controlling a single entry. Our approach is matrix theoretic, associated with the bipartite graph of $\mathbf{A}$.

In Section 2, after we establish some basic concepts and the relevant notation, we state a key result found in [HS], characterizing zero/nonzero patterns that require a particular rank. This result, along with ideas from its proof, is used to obtain a necessary and sufficient condition for qualitative controllability of $(\mathbf{A}, \mathbf{b})$ in general (Theorem 2.2) and then under the additional assumption that the diagonal entries of $\mathbf{A}$ are nonzero (see Theorems 2.4 and 2.6 and Proposition 2.5). Section 3 contains our results on qualitative uncontrollability.

2 Qualitatively Controllable Patterns

We begin with notation, definitions and some concepts needed to prove our results.

Let $\mathbf{A}$ be an $l \times k$ zero/nonzero pattern and $A = (a_{ij}) \in \mathbf{A}$. We denote by $A(i_1, i_2, \ldots, i_s \,|\, j_1, j_2, \ldots, j_t)$ the submatrix of A obtained when rows $i_1, i_2, \ldots, i_s$ and columns $j_1, j_2, \ldots, j_t$ are deleted, and by $A(i_1, i_2, \ldots, i_s \,|\, \cdot)$ the submatrix of A obtained when rows $i_1, i_2, \ldots, i_s$ (but no columns) are deleted. Similar notation is used for the zero/nonzero pattern $\mathbf{A}$. Finally, $\sigma(A)$ denotes the spectrum of A, and A_λ denotes the matrix $A - \lambda I$.

Let now $I = \{i_1, i_2, \ldots, i_t\}$ and $J = \{j_1, j_2, \ldots, j_t\}$, where $t \leq min(l, k)$, be two subsets of $\{1, 2, \ldots, l\}$ and $\{1, 2, \ldots, k\}$, respectively, both consisting of distinct integers. If $T = \{a_{i_1,j_1}, a_{i_2,j_2}, \ldots, a_{i_t,j_t}\}$ is a set of nonzero entries of A, then T is called a *t–transversal of A* (or $\mathbf{A}$) *on I, J*. If T is the only t–transversal on the index sets I and J, then T is called a *constrained t–transversal of A* (or $\mathbf{A}$). Note that a matrix may have more than one constrained t–transversal, provided that they are defined on different pairs of index sets.

The following theorem characterizes the zero/nonzero patterns that require rank r.

Theorem 2.1 ([HS], Theorem 3.9) *Let $\mathbf{A}$ be an $l \times k$ zero/nonzero pattern. Then every $A \in \mathbf{A}(\mathcal{C})$ has rank r if and only if $\mathbf{A}$ has no t–transversal with $t > r$, and there exists at least one constrained r–transversal.*

The concepts of t–transversal and constrained t–transversal correspond to those of t–matching and constrained t–matching, respectively, in a bipartite graph (see [HS], where this terminology is used). We also comment that the above theorem holds for matrices over any field of at least three elements.

We can state now the following necessary and sufficient condition for qualitative controllability.

Theorem 2.2 *Let* $\mathbf{A}$ *and* $\mathbf{b} = [*, 0, \ldots, 0]^T$ *be* $n \times n$ *and* $n \times 1$ *zero/nonzero patterns, respectively. Then* $(\mathbf{A}, \mathbf{b})$ *is qualitatively controllable if and only if for all* $A \in \mathbf{A}(\mathcal{C})$ *and all* $\lambda \in \mathcal{C}$*,* $A_\lambda(1\,|\,\cdot)$ *has a constrained* $(n-1)$*–transversal.*

Proof.
(i) (sufficiency) Suppose that for all $A \in \mathbf{A}(\mathcal{C})$ and all $\lambda \in \mathcal{C}$, $A_\lambda(1\,|\,\cdot)$ has a constrained $(n-1)$–transversal. By Theorem 2.1 any matrix with the zero/nonzero pattern of $A_\lambda(1\,|\,\cdot)$ must have rank $n-1$; hence $[A - \lambda I \;\; b]$ has rank n for all $b \in \mathbf{b}$ and, by condition (1.6), $(\mathbf{A}, \mathbf{b})$ is qualitatively controllable.
(ii) (necessity) Assume that $(\mathbf{A}, \mathbf{b})$ is qualitatively controllable. From the form of $\mathbf{b}$ and condition (1.6) it follows that for all $A \in \mathbf{A}(\mathcal{C})$ and all $\lambda \in \mathcal{C}$, $A_\lambda(1\,|\,\cdot)$ has rank $n-1$. We will show that $A_\lambda(1\,|\,\cdot)$ has a constrained $(n-1)$–transversal.

For any fixed (but arbitrary) $\lambda \in \mathcal{C}$ and any fixed (but arbitrary) $A \in \mathbf{A}(\mathcal{C})$, if no column of $A_\lambda(1\,|\,\cdot)$ contains exactly one nonzero entry, then the nonzero entries $a_{ij},\ i \neq j$, can be varied to give a matrix $\hat{A} \in \mathbf{A}(\mathcal{C})$ such that every column sum of $\hat{A}_\lambda(1\,|\,\cdot)$ is zero. Therefore $\mathrm{rank}\hat{A}_\lambda(1\,|\,\cdot) < n-1$, a contradiction, so there must exist some column of $A_\lambda(1\,|\,\cdot)$ with exactly one nonzero entry. Suppose this entry is in column c_1 and row r_1 of A_λ. Consider now $A_\lambda(1, r_1\,|\,c_1)$, which must have rank $n-2$. The above argument can be repeated to show that there must exist some column of $A_\lambda(1, r_1\,|\,c_1)$ with exactly one nonzero entry in, say, column c_2 and row r_2 of A_λ. Then $A_\lambda(1, r_1, r_2\,|\,c_1, c_2)$ must have rank $n-3$. Continuing in this manner, it follows that there exist sets of distinct indices $\{r_i\,|\,1 \leq i \leq n-2\}$ and $\{c_i\,|\,1 \leq i \leq n-2\}$ with $2 \leq r_i \leq n$ and $1 \leq c_i \leq n$ such that the 1×2 matrix $A_\lambda(1, r_1, r_2, \ldots, r_{n-2}\,|\,c_1, c_2, \ldots c_{n-2})$ has rank 1. Let the (r_{n-1}, c_{n-1}) entry of A_λ be a nonzero entry of this 1×2 matrix. Then, the submatrix of $A_\lambda(1\,|\,\cdot)$ on rows $r_1, r_2, \ldots, r_{n-1}$ and columns $c_1, c_2, \ldots, c_{n-1}$ (of A_λ) is permutation equivalent to an upper triangular matrix with the (r_k, c_k) entries, $1 \leq k \leq n-1$, on its main diagonal. These entries constitute a constrained $(n-1)$–transversal of $A_\lambda(1\,|\,\cdot)$. $\square$

Before we proceed we must clarify an ambiguity arising in Theorem 2.2, regarding the qualitative nature of our problem and the involvement of an arbitrary complex variable λ. It is apparent that the condition for qualitative controllability stated in Theorem 2.2 forces quantitative relations between the diagonal entries of $A_\lambda = A - \lambda I$. For example, depending on the choice of λ, A_λ may not belong to $\mathbf{A}$ and furthermore, its diagonal entries corresponding to any zero diagonal entries of $\mathbf{A}$ are all λ, and hence equal. As a consequence, there are zero/nonzero patterns $\mathbf{A}$ which satisfy the condition for qualitative controllability of Theorem 2.2, however, the required constrained transversal differs for different choices of the parameter

λ. This situation is illustrated by the following example.

Example 2.3 Consider $A = (a_{ij}) \in \mathbf{A}(\mathcal{C})$ and $b \in \mathbf{b}(\mathcal{C})$, where

$$\mathbf{A} = \begin{bmatrix} * & 0 & 0 \\ * & * & 0 \\ * & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} * \\ 0 \\ 0 \end{bmatrix}.$$

When the first row of A_λ is deleted we obtain

$$A_\lambda(1\,|\,\cdot) = \begin{bmatrix} a_{21} & a_{22} - \lambda & 0 \\ a_{31} & 0 & -\lambda \end{bmatrix}.$$

Notice now that if $\lambda = 0$ then $\{a_{22}, a_{31}\}$ is a constrained 2–transversal of $A_\lambda(1\,|\,\cdot)$. If $\lambda = a_{22}$ then the constrained transversal becomes $\{a_{21}, -\lambda\}$. Finally, if λ is different from 0 and a_{22}, then $A_\lambda(1\,|\,\cdot)$ has three constrained 2–transversals. By Theorem 2.2, $(\mathbf{A}, \mathbf{b})$ is qualitatively controllable. This can be verified independently by computing $[b : A]$ for any $b \in \mathbf{b}(\mathcal{C})$ and noting the exact cancellation. □

We will now consider the situation in Theorem 2.2 under the additional assumption that all the diagonal entries of $\mathbf{A}$ are nonzero. It will be shown that in this case the constrained transversal required in Theorem 2.2 does not contain any diagonal entries and that controllability is required if and only if $\mathbf{A}$, under a permutation similarity $Q\mathbf{A}Q^T$ such that $Q\mathbf{b} = \mathbf{b}$, is in unreduced upper Hessenberg form. Recall that a matrix is of this form if all entries on the first subdiagonal are nonzero and all entries below the first subdiagonal are zero. The necessity part of the proof of Theorem 2.4 requires the assumption that the diagonal entries of $\mathbf{A}$ are nonzero. This is evident from Example 2.3, where $(\mathbf{A}, \mathbf{b})$ is qualitatively controllable but there is no permutation similarity QAQ^T that puts $\mathbf{A}$ into unreduced upper Hessenberg form and is such that $Q\mathbf{b} = \mathbf{b}$.

Theorem 2.4 *For $n \geq 2$, let $\mathbf{A}$ be an $n \times n$ zero/nonzero pattern with all diagonal entries nonzero and $\mathbf{b}$ an $n \times 1$ zero/nonzero pattern with exactly one nonzero entry. Then $(\mathbf{A}, \mathbf{b})$ is qualitatively controllable if and only if there exists a permutation matrix Q such that $Q\mathbf{A}Q^T$ is unreduced upper Hessenberg and $Q\mathbf{b} = [*, 0, \ldots, 0]^T$.*

Proof.
(i) (sufficiency) If $Q\mathbf{A}Q^T$ is in unreduced upper Hessenberg form and $Q\mathbf{b} =$

$[*, 0, \ldots, 0]^T$, then for all $A \in \mathbf{A}(\mathcal{C})$ and $b \in \mathbf{b}(\mathcal{C})$ the controllability matrix $[Qb : QAQ^T]$ is upper triangular with nonzero diagonal entries and therefore has rank n. Hence, by conditions (1.3) and (1.4), $(\mathbf{A}, \mathbf{b})$ is qualitatively controllable.
(ii) (necessity) Assume that $(\mathbf{A}, \mathbf{b})$ is qualitatively controllable. Without loss of generality let $\mathbf{b} = [*, 0, \ldots, 0]^T$, otherwise we can apply our considerations to a permutation similarity of $\mathbf{A}$ and use condition (1.3). By condition (1.6) and the form of $\mathbf{b}$ we have that for all $A = (a_{ij}) \in \mathbf{A}(\mathcal{C})$ and all $\lambda \in \mathcal{C}$, $A_\lambda(1 \,|\, \cdot)$ has rank $n-1$. We will use this fact and induction on n to show that, up to a permutation similarity of rows and columns $2, 3, \ldots, n$, $\mathbf{A}(1 \,|\, \cdot)$ is upper trapezoidal with all diagonal entries nonzero (or equivalently that $\mathbf{A}$ is in unreduced upper Hessenberg form).

For $n = 2$ the claim is clear because if the (2,1) entry of A is zero then for $\lambda = a_{22}$, $A_\lambda(1 \,|\, \cdot)$ is a zero row vector. Assume the claim is true for $n = k-1$; we will show it for $n = k$.

Notice that, as the diagonal entries of $\mathbf{A}$ are nonzero, for an appropriate choice of $A \in \mathbf{A}(\mathcal{C})$ and $\lambda \in \mathcal{C}$, the diagonal entries of A_λ can assume any value in $\mathcal{C}$, including zero, *independently* of one another and of any off-diagonal entry in $\mathbf{A}$. Therefore A and λ can be chosen so that the column sums of $A_\lambda(1 \,|\, 1)$ are zero. As a consequence, the first column of $A_\lambda(1 \,|\, \cdot)$ must have exactly one nonzero entry or else $A \in \mathbf{A}(\mathcal{C})$ and $\lambda \in \mathcal{C}$ can be chosen so that every column sum of $A_\lambda(1 \,|\, \cdot)$ is zero, in which case the sum of its rows is the zero vector, a contradiction to $\text{rank} A_\lambda(1 \,|\, \cdot) = n-1$. Without loss of generality we can assume that the nonzero entry of the first column of $A_\lambda(1 \,|\, \cdot)$ is in the $(2,1)$ position of A_λ. Otherwise there exists a permutation similarity $PA_\lambda P^T$ that puts this entry into the $(2,1)$ position, and clearly $P\mathbf{b} = \mathbf{b}$. Consider now $A_\lambda(1,2 \,|\, 1)$ and notice that its rank must be $n-2$ for all $A \in \mathbf{A}$ and all $\lambda \in \mathcal{C}$. By our inductive assumption applied to the zero/nonzero pattern $\mathbf{A}(1 \,|\, 1)$, $\mathbf{A}(1,2 \,|\, 1)$ must be (up to a permutation similarity of rows and columns $3, 4, \ldots, n$ of $\mathbf{A}$) upper trapezoidal with all diagonal entries nonzero. Thus, we have proved that up to a permutation similarity $Q\mathbf{A}Q^T$ such that $Q\mathbf{b} = \mathbf{b}$, $\mathbf{A}$ is in unreduced upper Hessenberg form. $\square$

With $\mathbf{A}$ and $\mathbf{b}$ as in Theorem 2.4, we have shown that there exists a constrained $(n-1)$–transversal of $A_\lambda(1 \,|\, \cdot)$ that is independent of the choice of $A \in \mathbf{A}(\mathcal{C})$ and of $\lambda \in \mathcal{C}$. It can also be seen from the proof of Theorem 2.4 that the $(1,1)$ entry of $Q\mathbf{A}Q^T$ need not be assumed to be nonzero. Finally, we note that the sufficiency part of the proof of Theorem 2.4 does not require that the diagonal entries of $\mathbf{A}$ are nonzero.

The next proposition shows that if all the diagonal entries of $\mathbf{A}$ are nonzero and $(\mathbf{A}, \mathbf{b})$ is qualitatively controllable, then $\mathbf{b}$ cannot have more than one nonzero entry.

Proposition 2.5 *Let* $\mathbf{A}$ *be an* $n \times n$ *zero/nonzero pattern with all diagonal entries nonzero and* $\mathbf{b}$ *an* $n \times 1$ *zero/nonzero pattern. If* $(\mathbf{A}, \mathbf{b})$ *is qualitatively controllable then* $\mathbf{b}$ *has exactly one nonzero entry.*

Proof.
Suppose $\mathbf{b}$ has two or more nonzero entries. As in the proof of Theorem 2.4, since the diagonal entries of $\mathbf{A}$ are nonzero, for an appropriate choice of $A \in \mathbf{A}(\mathcal{C})$ and $\lambda \in \mathcal{C}$ the diagonal entries of A_λ can independently assume any value in $\mathcal{C}$. Consequently, $b \in \mathbf{b}(\mathcal{C})$, $A \in \mathbf{A}(\mathcal{C})$ and $\lambda \in \mathcal{C}$ can be chosen so that the column sums of $[A - \lambda I \; b]$ are zero and hence its rank less than n. By condition (1.6), this contradicts the qualitative controllability of $(\mathbf{A}, \mathbf{b})$ and completes the proof of the proposition. $\Box$

In view of Proposition 2.5 we can strengthen Theorem 2.4 as follows.

Theorem 2.6 *For* $n \geq 2$*, let* $\mathbf{A}$ *be an* $n \times n$ *zero/nonzero pattern with all diagonal entries nonzero and* $\mathbf{b}$ *an* $n \times 1$ *zero/nonzero pattern. Then* $(\mathbf{A}, \mathbf{b})$ *is qualitatively controllable if and only if there exists a permutation matrix* Q *such that* $Q\mathbf{A}Q^T$ *is unreduced upper Hessenberg and* $Q\mathbf{b} = [*, 0, \ldots, 0]^T$.

Theorem 2.6 is not true when the zero/nonzero pattern $\mathbf{A}$ is replaced by a sign pattern (i.e., when the $*$ entries of $\mathbf{A}$ can only be real numbers of a prescribed sign, denoted by $+$ or $-$). This is illustrated by the following example.

Example 2.7 Let A be any real matrix with sign pattern

$$\begin{bmatrix} + & + & + \\ + & + & 0 \\ + & 0 & - \end{bmatrix}.$$

Notice that for all $\lambda \in \mathcal{C}$ the matrix

$$A_\lambda(1\,|\,\cdot) = \begin{bmatrix} a_{21} & a_{22} - \lambda & 0 \\ a_{31} & 0 & a_{33} - \lambda \end{bmatrix}$$

has rank 2, because $a_{22} - \lambda$ and $a_{33} - \lambda$ cannot be simultaneously zero. Therefore, as in the proof of Theorem 2.2, for any $b \in [*, 0, 0]^T$, (A, b) is completely controllable (and thus the sign patterns of A and b are a sign controllable pair). However, there does *not* exist a permutation Q such that QAQ^T is unreduced upper Hessenberg and $Qb \in [*, 0, 0]^T$. Furthermore, if $\mathbf{A}$ is the zero/nonzero pattern of A and $\mathbf{b} = [*, 0, 0]^T$, by Theorem 2.6, $(\mathbf{A}, \mathbf{b})$ is not qualitatively controllable, which shows that sign controllability does not imply qualitative controllability of the corresponding zero/nonzero patterns. It is however clear that, in general, if ($\mathbf{A}$, $\mathbf{B}$) is a qualitatively controllable pair, any sign patterns obtained from $\mathbf{A}$ and $\mathbf{B}$ are a sign controllable pair. □

We conclude this section by noting that if $A \in \mathcal{C}^{n\times n}$, $b \in \mathcal{C}^n$ and $\text{rank}[b : A] = n$, then there is only one eigenvector (up to scalar multiples) corresponding to any eigenvalue of A (see §2.6 in [GLR]). Therefore the zero/nonzero patterns characterized in Theorems 2.2 and 2.6 have the property of requiring eigenvalues of geometric multiplicity one. This is a known fact for unreduced upper Hessenberg matrices (see e.g., Exercise 22, page 274 in [S]).

3 Qualitatively Uncontrollable Patterns

In this section we consider the problem of characterizing the $n\times n$ zero/nonzero patterns $\mathbf{A}$ such that $(\mathbf{A}, \mathbf{b})$ is qualitatively uncontrollable, where $\mathbf{b}$ is an $n \times 1$ zero/nonzero pattern with exactly one nonzero entry. First we need to discuss the notion of term rank. For more details the reader is refered to [BR]. The *term rank* of an $l \times k$ matrix A (or of its zero/nonzero pattern $\mathbf{A}$) is the minimum number of lines (rows or columns) that cover its nonzero entries. It is well known that the term rank is equal to the maximal length of a transversal of A. Also, if the term rank of $\mathbf{A}$ is t, then $\text{rank}A \leq t$ for all $A \in \mathbf{A}(\mathcal{C})$. Furthermore, the entries in a t–transversal can be chosen so that $\text{rank}A = t$. In fact, except for the nonzero entries of A chosen from an algebraic variety, the rank of A is always t.

Theorem 3.1 *Let* $\mathbf{A}$ *and* $\mathbf{b} = [*, 0, \ldots, 0]^T$ *be* $n \times n$ *and* $n \times 1$ *zero/nonzero patterns, respectively. Then* $(\mathbf{A}, \mathbf{b})$ *is qualitatively uncontrollable if and only if for all* $A \in \mathbf{A}(\mathcal{C})$ *there exists* $\lambda \in \mathcal{C}$ *such that* $A_\lambda(1\,|\,\cdot)$ *has no* $(n-1)$*–transversal with an entry from its first column.*

Proof.
As before, by condition (1.6) and the form of $\mathbf{b}$, $(\mathbf{A}, \mathbf{b})$ is qualitatively uncontrollable if and only if for all $A \in \mathbf{A}(\mathcal{C})$ there exists $\lambda \in \mathcal{C}$ such that $\text{rank}A_\lambda(1\,|\,\cdot) <$

$n-1$.

(i) (sufficiency) If the first column of $\mathbf{A}(1\,|\,\cdot)$ is zero, then for any $A \in \mathbf{A}(\mathcal{C})$ and for $\lambda \in \sigma(A(1\,|\,1))$, $\text{rank}A_\lambda(1\,|\,\cdot) < n-1$ and we are done. Suppose now that the first column of $\mathbf{A}(1\,|\,\cdot)$ contains a nonzero entry and that for all $A \in \mathbf{A}(\mathcal{C})$ there exists $\lambda = \lambda(A)$ such that $A_\lambda(1\,|\,\cdot)$ has no $(n-1)$–transversal with an entry from the first column. Notice then that $A_\lambda(1\,|\,1)$ cannot have an $(n-1)$–transversal or else $A_\lambda(1\,|\,\cdot)$ would have one, with an entry from the first column. Therefore $A_\lambda(1\,|\,\cdot)$ has no $(n-1)$–transversal at all, i.e., its term rank and hence its rank is less than $n-1$, proving that $(\mathbf{A}, \mathbf{b})$ is qualitatively uncontrollable.

(ii) (necessity) We will show the contrapositive. Suppose that for some $A \in \mathbf{A}(\mathcal{C})$ and for all $\lambda \in \mathcal{C}$, $A_\lambda(1\,|\,\cdot)$ has an $(n-1)$–transversal T_λ with an entry from the first column. We will show that, except in the case where the nonzero entries of the first column of A are chosen from an algebraic variety, $\text{rank}A_\lambda(1\,|\,\cdot) = n-1$ for all $\lambda \in \mathcal{C}$ and moreover that the nonzero entries in the first column of A can be varied (if necessary) giving a matrix $\hat{A} \in \mathbf{A}(\mathcal{C})$ so that $\text{rank}\hat{A}_\lambda(1\,|\,\cdot) = n-1$, for all $\lambda \in \mathcal{C}$.

Let $\lambda \in \sigma(A(1\,|\,1))$. Note that $\sigma(A(1\,|\,1))$ and the existence of T_λ are independent of the value of the nonzero entry of T_λ in the first column of $A(1\,|\,\cdot)$. Thus this nonzero entry can be varied giving a matrix $\hat{A} \in \mathbf{A}(\mathcal{C})$ such that the submatrix of $\hat{A}_\lambda(1\,|\,\cdot)$ indexed by the index sets of T_λ has rank $n-1$ and hence

$$\text{rank}\hat{A}_\lambda(1\,|\,\cdot) = n-1. \tag{3.1}$$

In fact, the values of this nonzero entry for which (3.1) does not hold constitute an algebraic variety. Consequently, this argument can be applied simultaneously for all $\lambda \in \sigma(A(1\,|\,1))$ to conclude that the entries in the first column of $A(1\,|\,\cdot)$ can be varied giving a matrix $\hat{A} \in \mathbf{A}(\mathcal{C})$ such that

$$\text{rank}\hat{A}_\lambda(1\,|\,\cdot) = n-1, \quad \text{for all} \ \ \lambda \in \sigma(A(1\,|\,1)) = \sigma(\hat{A}(1\,|\,1)).$$

Also, for $\lambda \notin \sigma(\hat{A}(1\,|\,1))$ the rank of $\hat{A}_\lambda(1\,|\,1))$ and thus the rank of $\hat{A}_\lambda(1\,|\,\cdot)$ is $n-1$. Hence

$$\text{rank}\hat{A}_\lambda(1\,|\,\cdot) = n-1, \quad \text{for all} \ \ \lambda \in \mathcal{C};$$

that is, $(\hat{A}, b)$ is completely controllable for all $b \in \mathbf{b}$, completing the proof of the theorem. $\square$

Example 3.2 In this example we apply Theorem 3.1 in order to identify 4×4 zero/nonzero patterns that do not allow complete controllability by controlling the first entry of the system output. The entries of the patterns below denoted by $\cdot$ can be either $*$ or 0.

$$\mathbf{A}_1 = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & 0 & 0 & \cdot \\ \cdot & 0 & 0 & \cdot \\ \cdot & 0 & 0 & \cdot \end{bmatrix}, \quad \mathbf{A}_3 = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix},$$

$$\mathbf{A}_4 = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & 0 \end{bmatrix}, \quad \mathbf{A}_5 = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & 0 & 0 & 0 \\ \cdot & 0 & 0 & 0 \end{bmatrix}.$$

The fact that these patterns have the desired property can alternatively be justified as follows: For every $A \in \mathbf{A}_1$ and for $\lambda \in \sigma(A(1\,|\,1))$, the rank of $A_\lambda(1\,|\,1)$ and hence that of $A_\lambda(1\,|\,\cdot)$ is less than 3. Also, for all $A \in \mathbf{A}_2$, the term rank of $A(1\,|\,1)$ is less than or equal to 1 and hence (the term) rank of $A(1\,|\,\cdot)$ is less than 3. For any $A = (a_{ij}) \in \mathbf{A}_3$ and for $\lambda = a_{22}$, $A_\lambda(1\,|\,\cdot)$ has a zero row, thus rank$A_\lambda(1\,|\,\cdot) < 3$. Similar comments can be made for the rest of these patterns. Notice that, except for $\mathbf{A}_1$, all these patterns satisfy the property that there exists $\lambda \in \mathcal{C}$ such that $A_\lambda(1\,|\,\cdot)$ has term rank less than 3. $\Box$

We conclude with a remark on the qualitative notions we have considered. For $n = 2$ and $\mathbf{b} = [*, 0]^T$ it can be verified, from the conditions we have obtained, that $(\mathbf{A}, \mathbf{b})$ is qualitatively controllable if and only if the (2,1) entry of $\mathbf{A}$ is $*$, and is qualitatively uncontrollable if and only if the (2,1) entry of $\mathbf{A}$ is zero. That is, for $n = 2$ qualitative and structural controllability coincide in this particular case. For $n = 3$, $\mathbf{b} = [*, 0, 0]^T$ and $\mathbf{A}$ given by

$$\begin{bmatrix} * & 0 & 0 \\ * & * & * \\ * & 0 & 0 \end{bmatrix},$$

the pair $(\mathbf{A}, \mathbf{b})$ is neither qualitatively controllable nor qualitatively uncontrollable, but is structurally controllable.

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