SYMBIOSIS POINTS FOR LINEAR DIFFERENTIAL SYSTEMS

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Abstract

Let A be an $n \times n$ essentially nonnegative matrix and consider the linear differential system $\dot{x} = Ax$. An initial point v lies in the reachability cone $X_A(R^n_+)$ for the nonnegative orthant if the trajectory $x(t)$ emanating from v reaches the nonnegative orthant at some time t_0. Due to the essential nonnegativity of A, once $x(t)$ enters the nonnegative orthant it remains in it thereafter.In this paper we introduce the notion of a *symbiosis point* for the system. This is a point in $X_A(R^n_+)$ such that also the velocity vector at the point is in the reachability cone. This means that not only does the trajectory become and remain nonnegative, but there comes a time such that from then on the components of the trajectory become and remain nondecreasing.

We characterize all symbiosis points for the system. We also show that if $0 < h < h(A) = \sup\{h' > 0 | I + h'A \geq 0\}$ and $\det(I + hA) \neq 0$, then any sequence of finite differences approximation which initiates at a symbiosis point becomes nondecreasing in the ordering of the nonnegative orthant and vice versa. In the case that A is weakly stable, we use a result of Hans Schneider to show that symbiosis points can be described from a matrix–combinatorial point of view. For weakly stable systems we also characterize trajectories whose higher order derivatives are required to lie in the reachability cone.

1 INTRODUCTION

Let $A = (a_{i,j})$ be an $n \times n$ matrix and consider the linear differential system

$$\dot{x} \;=\; Ax. \tag{1.1}$$

We say that an initial vector $v \in R^n$ is a *symbiosis point* if there exists a time $t_0 = t(v)$ such that the trajectory $x(t) = e^{tA}v$ emanating from v satisfies that

$$\dot{x}(t) \;\geq\; 0 \quad \forall t \geq t_0. \tag{1.2}$$

The motivation for this concept is as follows: If $x(t)$ represents the state of a physical or a biological system which is governed by (1.1) and which has evolved from the initial state v, then condition (1.2) means that at some point in time the system will reach a state such that from this time onwards each constituent population in the system is nondecreasing. In this paper for differential systems (1.1) with A essentially nonnegative we shall characterize all initial points in R^n which give rise to trajectories which eventually reach and remain in the nonnegative orthant R^n_+ and which are symbiosis points.

Some background to our problem is now in order. An $n \times n$ matrix $A = (a_{i,j})$ is called *essentially nonnegative*, in notation $A \overset{e}{\geq} 0$, if $a_{i,j} \geq 0, \ \forall i \neq j$. It is well known (cf. Birkhoff and Varga [**?**] and Berman and Plemmons [**?**]) that the essential nonnegativity of A is equivalent to the condition that $e^{tA} \geq 0, \ \forall t \geq 0$. Suppose now that A is an $n \times n$ essentially nonnegative matrix and consider the trajectory $x(t) = e^{tA}v$ emanating from v. If at some time $t_0 = t(v)$, $x(t_0) \geq 0$, then due to the essential nonnegativity of A, $x(t) \geq 0, \ \forall t \geq t_0$, that is, the trajectory remains in the nonnegative orthant for all time after t_0. Thus on letting $X_A(R^n_+)$ denote the set of all initial points in R^n such that the trajectories emanating from these points reach R^n_+ (and remain in it thereafter) we see that

$$X_A(R^n_+) \;=\; \bigcup_{t \geq 0} e^{-tA}(R^n_+). \tag{1.3}$$

In Neumann and Stern [**?**] it was shown that $X_A(R^n_+)$ is a convex cone, which contains R^n_+, but which is not necessarily pointed or closed. $X_A(R^n_+)$ was therefore termed *the reachability cone (for R^n_+ under A).* In that paper and in the subsequent paper Berman Neumann and Stern [**?**], formulas were given, mainly in terms of R^n_+ and the eigenspaces of A, for the closure

$\overline{X}_A(R^n_+)$ under additional spectral assumptions on A. The difficulties in applying these formulas were twofold. First, they appear to be too complicated. Second, as $\overline{X}_A(R^n_+)$ can contain points which are not in $X_A(R^n_+)$ itself, it was necessary to find an algorithm to determine $X_A(R^n_+)$ from its closure. To overcome these difficulties Neumann and Stern [**?**] and subsequently Neumann, Stern, and Tsatsomeros [**?**] have suggested to use finite differences approximations to determine whether a point v is in $X_A(R^n_+)$. Because of properties inherent to essential nonnegativity, the application of the finite differences method to this problem does not require that the time–step become very small regardless of the extent to which the continuous trajectory and the sequence of its discrete approximations can diverge from each other. The sole constraint on the time–step is that it be bounded above by a constant which is only dependent on A. As an incidental result to this paper we shall show that if the time-step used in the generation of the finite differences approximation satisfies the aforementioned bound, then the sequence of finite differences approximations emanating from a point $v \in X_A(R^n_+)$ which is also a symbiosis point eventually becomes nonnegative and nondecreasing (in the ordering of R^n_+).

From now on A will always be an $n \times n$ essentially nonnegative matrix. The Perron–Frobenius theory tells us that the *spectral abscissa* of A which is given by

$$\lambda_1 \; := \; max\{\Re(\lambda) \mid \lambda \in \sigma(A)\},$$

where $\Re(\cdot)$ denotes the real part of a complex number and where $\sigma(\cdot)$ denotes the spectrum of a matrix, is an eigenvalue of A. If $\lambda_1 < 0$, then A is a *stability* matrix, meaning $e^{tA}x \to 0$ for all $x \in R^n$ and showing that the origin is the only symbiosis point in $X_A(R^n_+)$ as then $e^{tA}v \to 0$, $\forall v \in R^n$. Thus the case of interest to us is when $\lambda_1 \geq 0$. Assume the latter. Given a vector $v \in R^n$ decompose v into

$$v \; = \; v_+ - v_-, \tag{1.4}$$

where $v_+ \in \mathcal{N}$, the *join of all eigenspaces of A corresponding to eigenvalues with a nonnegative real part* and where $v_- \in \mathcal{R}$, the *join of all eigenspace of A corresponding to eigenvalues with a negative real part.* Of course, the fact that $v_- \in \mathcal{R}$, the *stability* part of the space, means that $e^{tA}v_- \to 0$ as $t \to \infty$. In Section 2 we shall first show that a necessary condition for $v \in X_A(R^n_+)$ is that $v_+ \in X_A(R^n_+)$.

For a vector $v \in R^n$ let

$$\mathcal{I}(v) \ = \ \{i \in \langle n \rangle \mid (e^{tA} v_+)_i = (v_+)_i, \ \forall t \geq 0\}. \tag{1.5}$$

In the main result of this paper we shall show that for a vector $v \in X_A(R^n_+)$, the *velocity vector* $Av \in X_A(R^n_+)$ if and only if for each $i \in \mathcal{I}(v)$ there exists a time t_i such that

$$0 \leq \ (e^{tA} v_-)_i \ \downarrow 0, \quad \forall t \geq t_i. \tag{1.6}$$

In particular $(v_+)_i = 0$ for $i \in \mathcal{I}(v)$ if and only if $(e^{tA} v_-)_i = 0, \ \forall t \geq 0$. In this case the trajectory emanating form v cannot reach the interior of $R^n_+$and so, according to a result of Neumann and Stern [**?**], v must lie in the *effective part of the boundary* of $X_A(R^n_+)$ which is the set $X_A(R^n_+) \cap \partial X_A(R^n_+)$.

In the special case when the differential system (1.1) is *weakly stable*, that is, $\lambda_1 = 0$, symbiosis points admit a matrix–combinatorial description. Recall that now $-A$ is a singular M–matrix. For a point $v = v_+$ Hans Schneider, [**?**, Lemma 6.4.32], has characterized when $v_+ \in X_A(R^n_+)$. His characterization will be fully stated and explained in Section 3, but only outlined now. Let $\beta_1, \ldots, \beta_m$ be the singular classes of $-A$. We look at a representation of v_+ as a linear combination $v_+ = \sum c_i y^{(i)}$ of a preferred basis $y^{(1)}, \ldots, y^{(m)}$ of nonnegative vectors for $\mathcal{N} = \ker((-A)^{p_1})$, where $p_1 = \mathrm{index}_0(-A) = \mathrm{index}_0(A)$. Define $T_{v_+} := \{k | c_k \neq 0\}$. Schneider shows that $v_+ \in X_A(R^n_+)$ if and only if all coefficients $c_k, \ k \in T_{v_+}$, corresponding to singular classes which are final in the subgraph of the singular graph induced by the vertices $\{\beta_k | k \in T_{v_+}\}$, are positive. Schneider terms the latter set $Top(v_+)$. Let $G^*(\mathrm{Top}(v_+))$ denote the set of all classes of A which have access to a class $\beta_k, \ k \in \mathrm{Top}(v_+)$. We shall show that now (in the weakly stable case) the set $\mathcal{I}(v)$ of (1.5) is given by

$$\{i \in \langle n \rangle \mid i \in \beta_k, \ k \in \mathrm{Top}(v_+)\} \bigcup \{i \in \langle n \rangle \mid i \notin \alpha_k \text{ for some } \alpha_k \in G^*(\mathrm{Top}(v_+))\}.$$

It thus follows from the aforementioned that $v \in X_A(R^n_+)$ is a symbiosis point for (1.1) if and only if

$$j \in \{i \in \langle n \rangle \mid i \notin \alpha_k \text{ for some } \alpha_k \in G^*(\mathrm{Top}(v_+))\} \ \Rightarrow \ (x(t))_j = 0, \ \forall t \geq 0$$

and there exists a time $t_0 \geq 0$ such that

$$j \in \{i \in \langle n \rangle \mid i \in \beta_k, \ k \in \mathrm{Top}(v_+)\} \ \Rightarrow \ 0 \leq (x_-(t))_j \ \downarrow 0, \quad \forall t \geq t_0.$$

The last result of this paper concerns the characterization of vectors $v \in X_A(R^n_+)$ such that $A^i v \in X_A(R^n_+)$ for all $0 \leq i \leq p_1 + 1$. In other words we require the derivatives up to the order of nilpotency plus one of the restriction of A to $\mathcal{N}$ of the trajectory emanating from v to eventually become and remain nonnegative. We shall show that a necessary and sufficient condition for this to happen is that $v = v_+$.

2 CHARACTERIZATION

We begin by showing that a necessary condition for a point $v \in R^n$ to be in $X_A(R^n_+)$ is that $v_+ \in X_A(R^n_+)$. For this purpose it will be convenient to let

$$\mathcal{N}_{\lambda_i} := N((A - \lambda_i))^{p_i}), \ \lambda_i \in \sigma(A), \tag{2.1}$$

where p_i =index$_{\lambda_i}(A)$ is the multiplicity of λ_i in the minimal polynomial, $i = 1, \dots, n$. Then

$$\mathcal{N} \ = \oplus\{\mathcal{N}_{\lambda_i} \mid \Re(\lambda_i) \geq 0\}. \tag{2.2}$$

LEMMA 2.1 *Let* $A \overset{e}{\geq} 0$. *Then,*

$$v \in X_A(R^n_+) \implies v_+ \in X_A(R^n_+). \tag{2.3}$$

Proof

Let $v \in X_A(R^n_+)$. If $v_+ = 0$ there is nothing to prove. Suppose that $v_+ \neq 0$ and decompose

$$v_+ = \sum_{\Re(\lambda_i) \geq 0} v_+^{(i)}, \qquad v_+^{(i)} \in N_{\lambda_i}. \tag{2.4}$$

Then, since $(A - \lambda_i I)$ restricted to $\mathcal{N}_{\lambda_i}$ is nilpotent of order p_i, we have that

$$(A - \lambda_i I)^m v_+^{(i)} = 0, \ \ \forall m \geq p_i, \tag{2.5}$$

and therefore for any $1 \leq j \leq n$, the j–th entry of $x_+(t) = e^{tA} v_+$ is of the form

$$(x_+(t))_j = \sum_{\Re(\lambda_i) \geq 0} e^{t\lambda_i}([v_+^{(i)} \ + \ t(A - \lambda_i I)v_+^{(i)} \ + \ \frac{t^2(A - \lambda_i I)^2}{2!} v_+^{(i)} \ + \ \dots$$

$$\dots \ + \ \frac{t^{p_i - 1}(A - \lambda_i I)^{p_i - 1}}{(p_i - 1)!} v_+^{(i)}])_j. \tag{2.6}$$

Now let $p_1(j)$ denote the largest power $k \geq 0$, *if such exists*, such that

$$((A - \lambda_1 I)^k v_+^{(1)})_j \neq 0. \tag{2.7}$$

We claim that,

$$((A - \lambda_1 I)^{p_1(j)} v_+^{(1)})_j > 0. \tag{2.8}$$

To prove this observe that since e^{tA} restricted to $\mathcal{R}$ is a stability matrix, we have that as $t \to \infty$,

$$(x_-(t))_j \longrightarrow 0. \tag{2.9}$$

In addition, observe that

$$\frac{e^{t\lambda_1}t^{p_1(j)}}{(p_1(j))!}((A-\lambda_1 I)^{p_1(j)}v_+^{(1)})_j$$

asymptotically dominates the contribution of the summand corresponding to λ_1 in (2.6). This summand, in turn, asymptotically dominates the entire j-th entry of $x_+(t)$. As a consequence, if (2.8) is not true, then for t sufficiently large, by (2.9), the assumption that $v \in X_A(R_+^n)$ is contradicted. We must now consider the case when

$$((A-\lambda_1 I)^k v_+^{(1)})_j = 0, \quad \forall k \geq 0,$$

namely, when (for all $t \geq 0$) there is no contribution to $(x_+(t))_j$ from the eigenspace corresponding to λ_1. In this case we use a deflation argument as follows: Let μ be an eigenvalue of A with the second largest nonnegative real part. If no other eigenvalue of A has modulus equal to $|\mu|$ so that μ is real, then apply to the summand of (2.6) which corresponds to μ the same analysis as we have just applied to the summand corresponding to λ_1. On the other hand, if A has more than one eigenvalue whose real part is equal to $\Re(\mu)$, then consider the sum of the summands of (2.6) corresponding to such eigenvalues, which is real, and apply to this sum the same analysis as was applied to the summand corresponding to λ_1. It is clear that we can proceed with this repeated deflation argument to show that there exists a sufficiently large time $t_j \geq 0$ such that

$$(x_+(t))_j \geq 0, \quad \forall t \geq t_j,$$

or equivalently, that

$$v_+ \in X_A(R_+^n). \tag{2.10}$$

□

We are now ready to characterize all symbiosis points for the differential system (1.1).

THEOREM 2.1 *Let* $A \overset{e}{\geq} 0$. *Then a vector* $v \in X_A(R_+^n)$ *is a symbiosis point for* (1.1) *if and only if there exists a sufficiently large time* t_0 *such that*

$$j \in \mathcal{I}(v) \implies 0 \le (x_-(t))_j \downarrow 0, \quad \forall t \ge t_0, \tag{2.11}$$

where $\mathcal{I}(v)$ *is given in* (1.5)*. Furthermore, if* $v \in X_A(R^n_+)$ *is a symbiosis point, then for any* $j \in \mathcal{I}(v)$*,* $(v_+)_j = 0$ *if and only if* $(x(t))_j = 0$ *for all* $t \ge 0$*.*

Proof

We begin by noting that since $v \in X_A(R^n_+)$, then, by Lemma 2.1, $v_+ \in X_A(R^n_+)$. Let us consider first indices $j \notin \mathcal{I}(v)$. For such an index j it follows on close inspection of the proof of Lemma 2.1 that $(x_+(t))_j = (e^{tA}v_+)_j \to \infty$ as $t \to \infty$. Moreover, A has an eigenvalue with a nonnegative real part, say λ_k, such that for

$$s(t) := \frac{e^{t\lambda_k}}{(p_k(j))!}((A - \lambda_k I)^{p_k(j)} v_+)_j, \tag{2.12}$$

the sum

$$\frac{1}{2}[s(t) + \overline{s(t)}] \tag{2.13}$$

is positive and asymptotically dominates $(x_+(t))_j$. Let us now expand $(x_-(t))_j$ similarly to the expansion for $(x_+(t))_j$ given in (2.6). One gets

$$(x_-(t))_j = \sum_{\Re(\lambda_i)<0} e^{t\lambda_i}([v_-^{(i)} + t(A - \lambda_i I)v_-^{(i)} + \frac{t^2(A-\lambda_i I)^2}{2!} v_-^{(i)} + \ldots$$

$$\ldots + \frac{t^{p_i-1}(A-\lambda_i I)^{p_i-1}}{(p_i-1)!} v_-^{(i)}])_j. \tag{2.14}$$

Upon differentiating $(x_+(t))_j$ and $(x_-(t))_j$ it is now clear from (2.6), (2.12), (2.13), and (2.14), that there exists a sufficiently large time t_j, such that $(\dot{x}(t))_j \ge 0, \ \forall t \ge t_j$.

The arguments presented in the above paragraph show that in order to prove the equivalence of our theorem we need only confine our attention to the indices j which belong to $\mathcal{I}(v)$. Suppose then that $j \in \mathcal{I}(v)$. If (2.11) holds, then

$$(\dot{x}(t))_j = -(\dot{x}_-(t))_j \ge 0, \ \forall t \ge t_j.$$

This, together with the observations in the opening paragraph of the proof, have the implication that there exists a t_0 such that $\dot{x}(t) \ge 0, \ \forall t \ge t_0$. Conversely, if $\dot{x}(t) \ge 0, \ \forall t \ge t_0$, then for any $j \in \mathcal{I}(v)$ we have that

$$0 \le (\dot{x}(t))_j = (\dot{x}_+(t))_j - (\dot{x}_-(t))_j = 0 - (\dot{x}_-(t))_j.$$

This shows the $(x_-(t))_j$ is a decreasing function for any $t \geq t_0$. However we already know that $(x_-(t))_j \to 0$ as $t \to \infty$. Together these two properties of $(x_-(t))_j$ imply that there must come a time such that from this time on $(x_-(t))_j$ must also be nonnegative giving us (2.11).

Suppose now that $v \in X_A(R^n_+)$ is a symbiosis point for (1.1), $j \in \mathcal{I}(v)$, and $(v_+)_j = 0$. Let $t_0 \geq 0$ be a time such that $x(t)$ is nonnegative, nondecreasing, and has attained maximal number of positive components. Then first, as $(x(t))_j = (x_+(t))_j - (x_-(t))_j = (v_+)_j - (x_-(t))_j = -(x_-(t))_j$, we must have that $(x_-(t))_j \leq 0$, $\forall t \geq t_0$. But then, by (2.11), $(x_-(t))_j = 0$, $\forall t \geq t_0$, showing that $(x(t))_j = 0$, $\forall t \geq t_0$. Now $x(t) = e^{(t-t_0)A}x(t_0)$, $\forall t \geq t_0$ and it is well known that for all $t > 0$ the directed graph (please see paragraph 2 in Section 3 for definition) $\Gamma(e^{tA})$ of e^{tA} is equal to its reflexive transitive closure and is a constant not dependent on t. Let us denote this closure by $\overline{\Gamma}$. Thus there is no edge $(j,k) \in \overline{\Gamma}$ linking vertex j to any vertex k in the support of $x(t_0)$. As $\Gamma(e^{-tA}) \subseteq \overline{\Gamma}$, $\forall t \geq 0$ and $x(t) = e^{(t-t_0)A}x(t_0)$, $\forall 0 \leq t \leq t_0$, we must have that $(x(t))_j = 0$ also for all $0 \leq t \leq t_0$. Conversely, suppose that $j \in \mathcal{I}(v)$ is an index such that $(x(t))_j = 0$, $\forall t \geq 0$. Then, because $0 = (x_+(t))_j - (x_-(t))_j = (v_+)_j - (x_-(t))_j$, we first have that $(x_-(t))_j = (v_+)_j$, $\forall t \geq 0$. However, we know that $(x_-(t))_j \to 0$ as $t \to \infty$. Whence $(v_+)_j = 0$ and the proof is complete. □

We conclude this section with an observation concerning the sequence of finite differences approximations emanating at an initial point $v \in X_A(R^n_+)$ which is a symbiosis points. We require some preliminaries. Let $\hat{x}_0 = v \in R^n$ be an initial point of the trajectory $x(t) = e^{tA}\hat{x}_0$ and, for $h > 0$, consider the sequence of finite differences approximations generated by the quotient

$$\frac{\hat{x}_k \; - \; \hat{x}_{k-1}}{h} \;=\; A\hat{x}_{k-1}, \quad k = 1, 2, \ldots.$$

Then

$$\hat{x}_k \;=\; (I \;+\; hA)\hat{x}_{k-1} \;= \cdots =\; (I + hA)^k\hat{x}_0, \quad k = 1, 2, \ldots. \tag{2.15}$$

Next let

$$\begin{aligned} h(A) \;&=\; \sup\{h > 0 \mid I + hA \geq 0\} \\ &=\; \sup\{h > 0 \mid 1 + ha_{i,i} \geq 0,\; i = 1, \ldots, n\}. \end{aligned} \tag{2.16}$$

Note that $h(A)$ is only dependent on the size of the diagonal entries of A. In Neumann, Stern, and Tsatsomeros [**?**, Theorems 3.1 and 3.3] it was shown that for any $0 < h < h(A)$ such that $\det(I+hA) \neq 0$, a point $\hat{x}_0 \in X_A(R^n_+)$ *if and only if* there exists an exponent $k_0 = k_0(\hat{x}_0)$ such that $\hat{x}_k \geq 0$, $\forall k \geq k_0$. In our next result we shall show that initial points in $X_A(R^n_+)$ which are symbiosis points give rise to sequences of finite differences which become nondecreasing in the partial ordering on R^n induced by R^n_+.

THEOREM 2.2 *Let* $A \overset{e}{\geq} 0$ *and suppose that* $\hat{x}_0 \in X_A(R^n_+)$*. Then for any* h *satisfying* $0 < h < h(A)$ *such that det*$(I + hA) \neq 0$*, where* $h(A)$ *given in* (2.16)*,* $A\hat{x}_0 \in X_A(R^n_+)$ *if and only if there exists an exponent* k_0*, which depends on* $A\hat{x}_0$*, such that*

$$\hat{x}_{k+1} \geq \ \hat{x}_k, \quad \forall k \geq k_0. \tag{2.17}$$

Proof

From (2.15) we readily have that

$$\hat{x}_{k+1} = \hat{x}_k + hA\hat{x}_k = \hat{x}_k + h(I + hA)^k A\hat{x}_0, \quad \forall k \geq 1,$$

yielding that

$$\hat{x}_{k+1} - \hat{x}_k = h(I + hA)^k A\hat{x}_0, \quad \forall k \geq 1. \tag{2.18}$$

Now by the results in [**?**] mentioned above if $A\hat{x}_0 \in X_A(R^n_+)$, then there exists an index k_0 which depends on $A\hat{x}_0$ such that

$$(I + hA)^k A\hat{x}_0 \ \geq 0, \quad \forall k \geq k_0$$

and so, by (2.18),

$$\hat{x}_{k+1} \geq \hat{x}_k, \ \forall k \geq k_0.$$

Conversely, if there exists an exponent k_0 such that $\hat{x}_{k+1} \geq \hat{x}_k$ for all $k \geq k_0$, then, by (2.18), the sequence of finite differences generated from the vector $A\hat{x}_0$ eventually becomes and remains nonnegative. Thus, once again by the results in [**?**] mentioned above, $A\hat{x}_0 \in X_A(R^n_+)$. This completes the proof.
□

3 THE WEAKLY STABLE CASE

In this section we consider the case when $\lambda_1 = 0$, the so called *weakly stable case.* We shall first interpret from a matrix combinatorial point of view the results of Theorem 2.1 for this case. Then we shall consider characterizing trajectories of weakly stable systems whose higher order derivatives are required to lie in $X_A(R_+^n)$.

Assume then that A is an $n \times n$ essentially nonnegative and weakly stable matrix. Then $-A$ has nonpositive off–diagonal entries and all its eigenvalues have a nonnegative real part. Hence $-A$ is a singular M–matrix (cf. Berman and Plemmons [**?**]), in which case $-A$ admits the representation $-A = \rho(B)I - B$, where B is an $n \times n$ nonnegative matrix whose spectral radius is $\rho(B)$. Recall that the *directed graph of* A, $\Gamma(A)$, is a set of n vertices $\langle n \rangle = \{1, \ldots, n\}$ and a set of edges with a directed edge linking vertex i to vertex j if and only if $a_{i,j} \neq 0$. We identify $\Gamma(A)$ with its edge set. Denote by $\overline{\Gamma(A)}$ the *reflexive transitive closure of* $\Gamma(A)$. For $S, T \subseteq \langle n \rangle$ we say that S *has access to* T, and write $S \succeq T$, if there exists a path in $\Gamma(A)$ from a member of S to a member of T. A *class of* A is the vertex set corresponding to a strongly connected component of $\Gamma(A)$. We denote the classes of A by $\alpha_1, \ldots, \alpha_q$. If $q = 1$ then, as is well known, A is irreducible. For $T \subseteq \langle n \rangle$ we denote by $A[T]$ the principal submatrix of A whose rows and columns are indexed by T. Similarly, if $v \in R^n$, we denote by $v[T]$ the subvector of v whose entries are indexed by T. If A is reducible, then A is permutationally similar to a block upper triangular Frobenius normal form whose diagonal blocks are *successively* the irreducible matrices $A[\alpha_{i_k}]$, $k = 1, \ldots, q$. We call a class α of A *(non) singular* if $A[\alpha]$ is (non) singular. According to a result of Cooper [**?**] the number of singular classes of A is, precisely, $m :=\dim(\mathcal{N})$. A class of A is called *final* if it has access to no other class. The *reduced graph of* A, $R(A)$, is the set $\{\alpha_{i_1}, \ldots, \alpha_{i_q}\}$ – which we now identify with $\langle q \rangle$ – *together with the partial order of access that is induced from* $\Gamma(A)$. A chain of classes (vertices) in $R(A)$ is a sequence $(k_1, \ldots, k_s)$ such that $\alpha_{k_j} \succeq \alpha_{k_{j+1}}$, $j = 1, \ldots, s-1$. The *length* of a chain connecting vertex i to vertex j in $R(A)$, denoted by $d(i,j)$, is the maximal number of singular vertices lying on a chain connecting vertex i to vertex j. Let $\mathcal{S} = \{i \in \langle q \rangle \mid \alpha_i \text{ is a singular vertex}\}$. The *singular graph of* A, $\mathcal{S}(A)$, has the vertex set $\mathcal{S}$ together with the partial order of access induced by $R(A)$, viz., $\alpha_i \succeq \alpha_j$ in $\mathcal{S}(A)$ if and only if the same holds true in $R(A)$.

For convenience let us rename the singular classes of A by $\beta_1, \ldots, \beta_m$. In the papers Rothblum [?] and Richman and Schneider [?] the authors prove the existence of a nonnegative basis for $\mathcal{N}$. Their results are summarized in a survey paper of Schneider [?, Theorem 7.1] from which we now paraphrase:

" $\mathcal{N}$ *possesses a basis of nonnegative vectors* $y^{(1)},\ldots,y^{(m)}$ *having the following properties:*

$$y^{(i)}[\alpha_j] \begin{cases} \gg 0 & \text{if } \alpha_j \succeq \beta_i \\ = 0 & \text{otherwise} \end{cases} \tag{3.1}$$

for $j = 1,\ldots,q$ *and* $i = 1,\ldots,m$. *Moreover, a nonnegative basis* $y^{(1)},\ldots,y^{(m)}$ *satisfying* (3.1) *can be chosen so that*

$$c_{i,k} \begin{cases} > 0 & \text{if } \beta_k \succeq \beta_i, \beta_k \neq \beta_i \\ 0 & \text{otherwise,} \end{cases} \tag{3.2}$$

$i, k = 1,\ldots,q$, *where the* $c_{i,k}$*'s are determined by*

$$Ay^{(i)} \;=\; \sum_{k\in\langle m\rangle} c_{i,k}y^{(k)}, \quad i = 1,\ldots,m. \tag{3.3}$$

More generally,

$$(A^r y^{(i)})[\alpha_j] \;=\; \begin{cases} \gg 0 & \text{if and only if } d(\alpha_j,\beta_i) \geq r+1, \\ 0 & \text{otherwise."} \end{cases} \tag{3.4}$$

A nonnegative basis $y^{(1)},\ldots,y^{(m)}$ satisfying (3.1), (3.2), and (3.3) is called a *preferred basis for* $\mathcal{N}$.

Hans Schneider [?, Lemma 6.4.32] has used preferred bases to characterize when a vector $v = v_+$ is in $X_A(R^n_+)$ as follows. Assume that $y^{(1)},\ldots,y^{(m)}$ is a preferred basis for $\mathcal{N}$ and represent v_+ as the linear combination

$$v_+ \;=\; \sum_{i\in\langle m\rangle} c_i y^{(i)}. \tag{3.5}$$

Let

$$T_{v_+} \;= \{i \mid c_i \neq 0\}.$$

Furthermore let

$$\mathrm{Top}(v_+) \;=\; \{k \in T_{v_+} \mid \beta_k \succeq \beta_i \text{ and } i \in T_{v_+} \Rightarrow i = k\}.$$

We remark that the vertices $\beta_k,\ k \in$Top(v_+), correspond to the final classes of the induced subgraph of $\mathcal{S}(A)$ on the vertices $\beta_k,\ k \in T_{v_+}$. Schneider proves the following characterization:

THEOREM S *Let* $A \overset{e}{\geq} 0$ *be weakly stable. Let* $y^{(1)}, \ldots, y^{(m)}$ *be a preferred basis for* $\mathcal{N}$ *and let* (3.5) *be the representation of* v_+ *in this basis. Then* $v_+ \in X_A(R^n_+)$ *if and only if*

$$k \in \operatorname{Top}(v_+) \;\Rightarrow\; c_k > 0.$$

Let us denote by $G^*(\operatorname{Top}(v_+))$ the set of all classes of A which have access to a class $\beta_k,\ k \in \operatorname{Top}(v_+)$. In essence Schneider's proof consists of observing, using (3.1)–(3.4), that if $k \in \operatorname{Top}(v_+)$, then $(e^{tA}v_+)[\beta_k] = v_+[\beta_k],\ \forall t \geq 0$, while if $k \in G^*(\operatorname{Top}(v_+)) \setminus \operatorname{Top}(v_+)$, then all the components of $v_+[\beta_k]$ tend to $+\infty$ as $t \to \infty$.

Then Theorem S together with our Theorem 2.1 yield the following characterization for symbiosis points in the weakly stable case:

COROLLARY 3.1 *Let* $A \overset{e}{\geq} 0$ *and let* $v \in X_A(R^n_+)$. *Then* v *is a symbiosis point for* (1.1) *if and only if*

$$j \in \{i \in \langle n \rangle \mid i \notin \alpha_k \text{ for some } \alpha_k \in G^*(\mathit{Top}(v_+))\} \;\Rightarrow\; (x(t))_j = 0,\ \forall t \geq 0$$

and there exists a time $t_0 \geq 0$ *such that*

$$j \in \{i \in \langle n \rangle \mid i \in \beta_k,\ k \in \mathit{Top}(v_+)\} \;\Rightarrow\; 0 \leq (x_-(t))_j \downarrow 0, \quad \forall t \geq t_0.$$

In the last result of our paper we characterize points $v \in X_A(R^n_+)$ such the derivatives of all orders $1 \leq i \leq p_1$ of the trajectories emanating from them belong to $X_A(R^n_+)$.

THEOREM 3.1 *Let* $A \overset{e}{\geq} 0$ *with* $\lambda_1 = 0$ *and with* $\mathit{index}_0(A) = p_1$. *Then, for any* $v \in X_A(R^n_+)$ *the following are equivalent.*

(i) $A^{p_1+1}v \in X_A(R^n_+)$.

(ii) $v_- = 0$.

(iii) $A^m v \in X_A(R^n_+), \quad \forall m \geq 0$.

Proof

(i) ⇒ *(ii)* Since $v_+ \in \mathcal{N}$, $A^{p_1+1}v_+ = 0$ and so from $A^{p_1+1}v \in X_A(R^n_+)$ we have that

$$-A^{p_1+1}v_- \in X_A(R^n_+). \tag{3.6}$$

This means that there exists a finite time $t_0 \geq 0$ such that, for all $t \geq t_0$, each component of $A^{p_1}x_-(t) = e^{tA}A^{p_1}v_-$ becomes nonincreasing and at the same time, as A is a stability matrix on $\mathcal{R}$, it converges to 0. As a consequence, for all $1 \leq j \leq n$,

$$(A^{p_1}x_-(t))_j \geq 0, \quad \forall t \geq t_0$$

or equivalently,

$$A^{p_1}v_- \in X_A(R^n_+). \tag{3.7}$$

Similarly, by (3.7), all components of $A^{p_1-1}x_-(t)$ become nondecreasing and converge to 0, which has the implication that

$$-A^{p_1-1}v_- \in X_A(R^n_+). \tag{3.8}$$

Continuing along the same line, for *sufficiently large time*, we have that the sequence of vectors

$$x_-(t),\ Ax_-(t), \ldots, A^{p_1}x_-(t),\ A^{p_1+1}x_-(t) \tag{3.9}$$

forms, in the language of Hershkowitz, Rothblum, and Schneider [**?**], a nontrivial *alternating sequence* of length $p_1 + 1$ for A. But $-A$ is a singular M–Matrix, and so by Theorem 3.4 of [**?**], the length of such a nontrivial alternating sequence cannot exceed the index p_1, namely, we necessarily have that for all sufficiently large time, $x_-(t) = 0$ and so, by nonsingularity of e^{tA},

$$v_- = 0. \tag{3.10}$$

(ii) ⇒ *(iii)* If $v_- = 0$, then $v = v_+ \in X_A(R^n_+)$. We shall now show that $Av_+ \in X_A(R^n_+)$. From (3.10) and the definition of p_1 (viz. Lemma 2.1) and since $\lambda_1 = 0$, we obtain that

$$Ax_+(t) = \frac{1}{t}[tAv_+ + t^2A^2v_+ + \ldots + \frac{t^{p_1-1}A^{p_1-1}}{(p_1-2)!}v_+]. \tag{3.11}$$

Consider now the j–th entry of Av_+ and let $p_1(j)$ be as defined in equation (2.7).

If $p_1(j) = 0$ then $(Ax_+(t))_j = 0$. If $1 \leq p_1(j) \leq p_1 - 1$, then on examining the asymptotic effect of (2.8) in (3.11), we obtain that there exists a sufficiently large $t_0(j) \geq 0$ such that

$$(Ax_+(t))_j > 0, \quad \forall t \geq t_0(j) \tag{3.12}$$

or equivalently,

$$Av_+ \in X_A(R^n_+). \tag{3.13}$$

The proof that $A^m x_+ \in X_A(R^n_+)$ for any $m \geq 2$ follows similarly.
$(iii) \Rightarrow (i)$ On letting $m = p_1+1$, this implication is trivial. □

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