On the spectra of striped sign patterns

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June 17, 2002

Abstract

Sign patterns consisting of some positive and some negative columns, with at least one of each kind, are shown to allow any self-conjugate spectrum, and thus to allow any inertia. In the case of the $n \times n$ sign pattern with all columns positive, given any self-conjugate multiset consisting of $n-1$ complex numbers supplemented by a sufficiently large positive number, it is shown how to construct a positive normal matrix whose spectrum is this multiset. Thus, the positive sign pattern allows any inertia with at least one positive eigenvalue.

Keywords: Spectrum, nonnegative matrix, sign pattern, Soules matrix, inertia.
AMS Subject Classifications: 15A18, 15A48

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1 Introduction

Given an $n \times n$ sign pattern matrix S, namely, an array with entries $s_{ij} \in \{+,\ -,\ 0\}$, let its *sign pattern class* be

$$Q(S) \ = \ \{A = [a_{ij}] \in M_n\,(\mathbb{R})\ :\ \operatorname{sign} a_{ij} = s_{ij} \ \text{ for all } i, j\}\,.$$

The *inertia* of an $n \times n$ matrix A is the triple

$$i(A) \ = \ (i_+(A),\ i_-(A),\ i_0(A)),$$

where $i_+(A),\ i_-(A),\ i_0(A)$ are the number of eigenvalues of A with positive, negative and zero real parts, respectively. Clearly, $i_+(A),\ i_-(A),\ i_0(A)$ are nonnegative integers that sum to n. The *inertia* of a sign pattern S is the set of inertias attainable by matrices $A \in Q(S)$, and the *spectrum* of S is the set of spectra attainable by matrices $A \in Q(S)$.

In this paper, we are interested in the inertias and the spectra allowed by an $n \times n$ sign pattern S ($n \geq 2$) having p columns all of whose entries are positive and $n - p$ columns all of whose entries are negative ($0 \leq p \leq n$). We refer to such a pattern as *p-striped*. Notice that the location of the positive columns is immaterial as a permutation similarity can permute the positive columns as desired.

For an n-striped $n \times n$ pattern S, the inertia of S represents the set of inertias attainable by an (entrywise) positive matrix A. By the Perron-Frobenius theorem, the spectral radius, $\rho(A)$, of any (entrywise) nonnegative irreducible matrix A is a simple eigenvalue. A corresponding eigenvector is referred to as a *Perron vector* of A. So for all $A \in Q(S)$, where $Q(S)$ is n-striped, $i_+(A) \ \geq \ 1$. We prove that any given self-conjugate multiset of $n - 1$ complex numbers can be supplemented by a sufficiently large positive number to form the spectrum of an $n \times n$ positive normal matrix, and show how to construct such a matrix. It follows that a positive matrix A exists having any given inertia $i(A)$ with $i_+(A) \geq 1$. In particular, if the $n - 1$ given numbers are real, then the matrix constructed is symmetric. Based on the above, we then show that every p-striped pattern S with $1 \leq p \leq n - 1$ allows any self-conjugate spectrum; that is, the magnitudes of the entries of $A \in Q(S)$ can be chosen to obtain a matrix A with any prescribed eigenvalues provided the nonreal eigenvalues occur in conjugate pairs. It follows that such patterns are *spectrally arbitrary*, that is, any self-conjugate spectrum is possible by an appropriate choice of the signed entries. Clearly such patterns are *inertially arbitrary*, that is, they allow any inertia. Other spectrally and inertially arbitrary patterns are considered in [1, 5], however, we know of no other $n \times n$ sign pattern that has been proved to be spectrally arbitrary for all n. Our study of p-striped patterns is partially motivated by the inertia and spectral problems considered in [1] (see Section 3 for more details) and, more generally, by the inverse eigenvalue problem for matrices over the real field.

2 Spectral and Inertia Results

In [2], a real orthogonal matrix R is called a *Soules matrix* if the first column of R is positive and if for every nonnegative diagonal matrix Λ with its entries arranged in nonincreasing order, $R\Lambda R^T$ is a nonnegative symmetric matrix. A method of constructing all Soules matrices is given in [2], along with links to MMA-matrices and to strictly ultrametric matrices. Here, we use a particular type of Soules matrix, originally introduced by Soules [9]. Its construction begins with an arbitrary positive vector w_1 such that $\|w_1\|_2 = 1$ and proceeds as follows.

Partition $w_1^T = [u^T, v^T]$ so that $u \in \mathbb{R}^{n-1}$ and $v \in \mathbb{R}$. Then form $w_2^T = [\tilde{u}^T, \tilde{v}^T]$, where

$$\tilde{u} = \frac{\|v\|_2}{\|u\|_2} u \quad \text{and} \quad \tilde{v} = -\frac{\|u\|_2}{\|v\|_2} v.$$

Notice that $\|w_2\|_2 = 1$ and $w_2^T w_1 = 0$. Next form $w_3^T = [\hat{u}^T, 0]$, where $\hat{u}$ is obtained from $\tilde{u}/\|\tilde{u}\|_2$ in the same way w_2 was obtained from w_1. The vector w_4 would be constructed similarly by splitting off the last entry of $\hat{u}$, modifying the resulting vector as above, and complementing the outcome by two zero entries. This construction, after $n-1$ steps, yields an orthonormal set $\{w_1, \ldots, w_n\}$ such that $R = [w_1\ w_2\ \ldots\ w_n]$ is a Soules matrix.

We now use R with $w_1 = e_n/\sqrt{n}$, where e_n denotes the all ones vector of dimension n, to construct a positive (n-striped) matrix for which $n-1$ of the eigenvalues are arbitrary complex numbers (subject to the necessary condition that nonreal numbers occur in conjugate pairs).

Theorem 2.1 *Let $\sigma = \{\lambda_2, \ldots, \lambda_n\} \subset \mathbb{C}$ be a self-conjugate multiset. For any $\rho > n \max_{2\le k\le n} |\lambda_k|$, the multiset $\sigma \cup \{\rho\}$ is the spectrum of an $n\times n$ positive normal matrix A with $\rho(A) = \rho$ and Perron vector e_n.*

Proof. Given $n \ge 2$, let $1 \le r \le n$ with r odd (even) if n is odd (even). Let σ consist of $r-1$ real numbers $\lambda_2, \ldots, \lambda_r$ and $(n-r)/2$ pairs of complex conjugate numbers $\lambda_k = a_k + ib_k$ and $\lambda_{k+1} = a_k - ib_k$ with $b_k > 0$, for $k = r+1, r+3, \ldots, n-1$.

(Note that σ contains all real numbers if $r = n$, and contains no real numbers if $r = 1$ and n is odd.) Let the normal matrix D be defined by

$$D = \operatorname{diag}(1, \lambda_2, \ldots, \lambda_r) \oplus \begin{bmatrix} a_{r+1} & -b_{r+1} \\ b_{r+1} & a_{r+1} \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} a_{n-1} & -b_{n-1} \\ b_{n-1} & a_{n-1} \end{bmatrix}.$$

Consider the $n \times n$ orthogonal Soules matrix R with first column $w_1 = e_n/\sqrt{n}$, namely,

$$R = \begin{bmatrix} \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{(n-1)(n-2)}} & \cdots & \cdots & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{(n-1)(n-2)}} & \cdots & \cdots & -\frac{1}{\sqrt{2}} \\ \vdots & \vdots & \vdots & & & 0 \\ \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{(n-1)(n-2)}} & & & \vdots \\ \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n(n-1)}} & \frac{2-n}{\sqrt{(n-1)(n-2)}} & & & \vdots \\ \frac{1}{\sqrt{n}} & \frac{1-n}{\sqrt{n(n-1)}} & 0 & \cdots & \cdots & 0 \end{bmatrix}.$$

Partition the above matrices as

$$D = [1] \oplus \hat{D}, \quad R = [\ e_n/\sqrt{n}\ \mid\ \hat{R}\ ],$$

where $\hat{R}$ is $n \times (n-1)$. Then

$$RDR^T = \frac{1}{n} e_n e_n^T + \hat{R}\hat{D}\hat{R}^T,$$

and since for all $i, j = 1, \ldots, n$,

$$\max_{2 \le k \le n} |\lambda_k| = \|\hat{D}\|_2 = \|\hat{R}\hat{D}\hat{R}^T\|_2 \ge |(\hat{R}\hat{D}\hat{R}^T)_{ij}|,$$

it follows that $(RDR^T)_{ij} > 0$ provided that $\max_{2 \le k \le n} |\lambda_k| < 1/n$. Clearly $B = RDR^T$ is normal with eigenvalues $1, \lambda_2, \ldots, \lambda_n$. Thus any such multiset $\sigma \subset \mathbb{C}$ having $n-1$ elements λ_k with $|\lambda_k| < 1/n$ adjoined by 1 is the spectrum of an $n \times n$ positive normal matrix B.

Given now an arbitrary self-conjugate multiset $\sigma = \{\lambda_2, \ldots, \lambda_n\} \subset \mathbb{C}$, choose $\rho > n \max_{2 \le k \le n} |\lambda_k|$ so that $|\lambda_k/\rho| < 1/n$ for $2 \le k \le n$. Then, there exists a positive normal matrix B as constructed above with spectrum $\{1, \lambda_2/\rho, \ldots, \lambda_n/\rho\}$. Thus $A = \rho B$ has spectrum $\{\rho, \lambda_2, \ldots, \lambda_n\}$. The proof is completed by noting that A is positive, normal, and $Ae_n = \rho RDR^T e_n = \rho e_n$, since $R^T e_n = [\sqrt{n}, 0, \ldots, 0]^T$. □

Referring to the proof of the above theorem, notice that any $n \times n$ orthogonal matrix R with first column $e_n/\sqrt{n}$ would suffice to deduce RDR^T is positive provided that $\max_{2 \le k \le n} |\lambda_k| < 1/n$. We have chosen to use a specific Soules matrix to give an explicit construction. For an analysis of the role of Soules matrices in the inverse eigenvalue problem for nonnegative symmetric matrices see [7, 8].

For the special case in which σ contains all real numbers (i.e., $r = n$), the above construction with D diagonal yields a symmetric matrix.

Corollary 2.2 *Let* $\sigma = \{\lambda_2, \ldots, \lambda_n\} \subset \mathbb{R}$ *be a multiset. For any* $\rho > n \max_{2 \leq k \leq n} |\lambda_k|$, *the multiset* $\sigma \cup \{\rho\}$ *is the spectrum of an* $n \times n$ *positive symmetric matrix* A *with* $\rho(A) = \rho$ *and Perron vector* e_n.

By the Perron-Frobenius theorem, a positive matrix must have at least one positive eigenvalue. Hence, as stated next, Corollary 2.2 implies that positive symmetric matrices have almost arbitrary inertia.

Corollary 2.3 *Given any nonnegative triple* $(n_1,\ n_2,\ n_3)$ *with* $n_1 + n_2 + n_3 = n$ *and* $n_1 \geq 1$, *there exists a positive* $n \times n$ *symmetric matrix* A *with* $i(A) = (n_1,\ n_2,\ n_3)$.

Letting $J_{m \times n} = e_m e_n^T$ denote the $m \times n$ all ones matrix, we now proceed to our results on striped patterns. The construction used in (2.1) is similar in spirit to that used by Fiedler [4].

Lemma 2.4 *If* p *is odd, then every* p*-striped* $n \times n$ *sign pattern with* $1 \leq p \leq n - 1$ *allows any self-conjugate spectrum.*

Proof. We begin with the smallest case, namely, $n = 2$ and $p = 1$. We establish the result in this case by showing that there is a matrix in the p-striped pattern class whose characteristic polynomial is any given monic, real polynomial $p(x) = x^2 + \alpha_1 x + \alpha_0$. Let

$$\hat{A} = \begin{bmatrix} b & -d \\ d & -c \end{bmatrix}.$$

Then the characteristic polynomial of $\hat{A}$ is

$$x^2 + (c - b)x + d^2 - bc.$$

Set $b = c - \alpha_1$ and $d = \sqrt{\alpha_0 + bc} = \sqrt{c^2 - \alpha_1 c + \alpha_0}$. Then for all sufficiently large $c > 0$, both b and d are positive. In fact, for any $\eta > 0$, we can ensure that $b > \eta$ by choosing c sufficiently large. For the choices of b and d above, the characteristic polynomial of $\hat{A}$ is indeed $p(x)$.

We now proceed to the case where $n > 2$ and p is odd. Let S be an $n \times n$ p-striped pattern with columns $1, \ldots, p$ being positive and columns $p + 1, \ldots, n$ being negative. Let $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be a self-conjugate multiset of complex numbers. Since p is odd, $p - 1$ is even and hence, without loss of generality, we can assume that

$$\{\lambda_1, \ldots, \lambda_{p-1}\}, \quad \{\lambda_p, \ldots, \lambda_{n-2}\}, \quad \text{and} \quad \{\lambda_{n-1}, \lambda_n\}$$

are all self-conjugate multisets.

By Theorem 2.1 we can choose a positive normal $p\times p$ matrix B so that the eigenvalues of B are $b, \lambda_1, \ldots, \lambda_{p-1}$, where $b = \rho(B) > p\max_{1\le k\le p-1}|\lambda_k|$. Similarly, letting $q = n-p$, we can choose a positive normal $q\times q$ matrix C so that the eigenvalues of C are $c, -\lambda_p, \ldots, -\lambda_{n-2}$, where $c = \rho(C) > q\max_{p\le k\le n-2}|\lambda_k|$ can be as large as we would like. In fact, we can choose b and c so that d in $\hat{A}$ is positive and the eigenvalues of $\hat{A}$ are λ_{n-1} and λ_n. Note that if $p = 1$ or $n-1$, then the first or second subset of the multiset $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is empty, and $B = b$ or $C = c$, respectively.

Consider next the $n\times n$ matrix

$$A \;=\; \begin{bmatrix} B & -\frac{d}{q}\,J_{p\times q} \\ \frac{d}{p}\,J_{q\times p} & -C \end{bmatrix} \;\in\; Q(S), \tag{2.1}$$

By Theorem 2.1 we know that $B\in M_p(\mathbb{R})$ and $C\in M_q(\mathbb{R})$ are positive matrices such that

$$B\,e_p \;=\; \rho(B)\,e_p = be_p \quad \text{and} \quad C\,e_q \;=\; \rho(C)\,e_q = ce_q.$$

If $By = \lambda y$, where $\lambda\ne\rho(B)$ and $y\ne 0$, then as B is normal, $e_p^T y = 0$ and thus $J_{q\times p}\,y = 0$. From (2.1) it follows that

$$A\begin{bmatrix} y \\ 0 \end{bmatrix} \;=\; \begin{bmatrix} B\,y \\ 0 \end{bmatrix} \;=\; \lambda\begin{bmatrix} y \\ 0 \end{bmatrix}.$$

Thus any eigenvalue $\lambda\ne\rho(B)$ of B is also an eigenvalue of A, i.e., $\lambda_1, \ldots, \lambda_{p-1}$ are eigenvalues of A. Similarly,

$$A\begin{bmatrix} 0 \\ z \end{bmatrix} \;=\; \begin{bmatrix} 0 \\ -C\,z \end{bmatrix} \;=\; -\mu\begin{bmatrix} 0 \\ z \end{bmatrix},$$

where $\mu\ne\rho(C)$ is an eigenvalue of C with $Cz = \mu z$. This shows that $-\mu$ is an eigenvalue of A. Thus $\lambda_1, \ldots, \lambda_{n-2}$ are eigenvalues of A.

Now let $[x, y]^T$ be a right eigenvector of $\hat{A}$ corresponding to an eigenvalue $\nu\in\{\lambda_{n-1}, \lambda_n\}$. Then

$$A\begin{bmatrix} xe_p \\ ye_q \end{bmatrix} = \begin{bmatrix} xBe_p - \frac{dy}{q}J_{p\times q}e_q \\ \frac{dx}{p}J_{q\times p}e_p - yCe_q \end{bmatrix} = \begin{bmatrix} (bx-dy)e_p \\ (dx-cy)e_q \end{bmatrix} = \nu\begin{bmatrix} xe_p \\ ye_q \end{bmatrix}.$$

Thus the eigenvalues of $\hat{A}$ are also eigenvalues of A, completing the proof. □

Lemma 2.5 *If n and p are both even, then every p-striped $n\times n$ sign pattern with $1\le p\le n-1$ allows any self-conjugate spectrum.*

Proof. Once again, we begin with the smallest case, namely, $n = 4$ and $p = 2$. We establish our result in this case by showing that there is a matrix in the p-striped pattern class whose characteristic polynomial is any given monic, real polynomial $p(x) = x^4 + \alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0$. Let

$$\hat{A} = \begin{bmatrix} b & 1 & -1 & -1 \\ f & 1 & -1 & -g \\ f & d & -1 & -g \\ 1 & 1 & -1 & -c \end{bmatrix}.$$

The characteristic polynomial of $\hat{A}$ is given by

$$x^4+(c-b)x^3+(d-bc)x^2+(d-1)(f-g+c-b)x+(d-1)[f(c-1)+g(b-1)-(bc-1)].$$

We claim that positive values for b, c, d, f, g can be chosen so that the characteristic polynomial of $\hat{A}$ is $p(x)$. Set

$$b = c - \alpha_3 \quad \text{and} \quad d = \alpha_2 + bc = c^2 - c\alpha_3 + \alpha_2.$$

Notice that by choosing c sufficiently large, b can be made arbitrarily large (a fact needed later on in the proof) and $d > 2$. Next set

$$f \;=\; \frac{\alpha_1}{d-1} - (c-b) + g \;=\; \frac{\alpha_1}{d-1} - \alpha_3 + g \;\geq\; -|\alpha_1| - \alpha_3 + g.$$

Then f is positive for all sufficiently large g. For the constant term of the characteristic polynomial of $\hat{A}$ to equal α_0, we must have

$$f(c-1) + g(b-1) = \frac{\alpha_0}{d-1} + (bc-1).$$

Using the above set values for b, d, f and solving the latter equation for g in terms of c and $\alpha_0, \alpha_1, \alpha_2, \alpha_3$, we obtain:

$$g = \frac{c^4 - \alpha_3\, c^3 + (\alpha_2 - \alpha_3 - 2)\, c^2 + (\alpha_3^2 + \alpha_3 - \alpha_1)\, c + \alpha_0 + \alpha_1 - \alpha_2 + \alpha_3 - \alpha_3\alpha_2 + 1}{2c^3 - (3\alpha_3 + 2)\, c^2 + (\alpha_3^2 + 2\alpha_2 + 2\alpha_3 - 2)\, c + \alpha_3 - 2\alpha_2 - \alpha_3\alpha_2 + 2},$$

which implies that g can be made arbitrarily large for sufficiently large c. Thus, by choosing c sufficiently large, we have that $b > 0$, $d > 2$, and that $g > 0$ is sufficiently large to ensure $f > 0$. In addition, by construction, the characteristic polynomial of A is $p(x)$.

We now proceed to the case where $n > 4$ and p are both even. Set $q = n - p$. Then q is also even. Let $\{\lambda_1, \ldots, \lambda_n\}$ be any self-conjugate multiset of complex numbers. Since $p - 2$, $q - 2$ and 4 are even, we can assume without loss of generality that the multisets

$$\{\lambda_1, \ldots, \lambda_{p-2}\}, \quad \{\lambda_{p-1}, \ldots, \lambda_{n-4}\} \quad \text{and} \quad \{\lambda_{n-3}, \lambda_{n-2}, \lambda_{n-1}, \lambda_n\}$$

are also self-conjugate.

By Theorem 2.1 we can choose a positive normal $(p-1)\times(p-1)$ matrix B whose eigenvalues are $b, \lambda_1, \ldots, \lambda_{p-2}$, where $b = \rho(B) > (p-1)\max_{1\le k\le p-2}|\lambda_k|$ is as large a positive number as we would like. By Theorem 2.1 we can choose a positive normal $(q-1)\times(q-1)$ matrix C so that the eigenvalues of C are $c, -\lambda_{p-1}, \ldots, -\lambda_{n-4}$, where $c = \rho(C) > (q-1)\max_{p-1\le k\le n-4}|\lambda_k|$ is as large as we would like.

Thus for all sufficiently large c, the values of b, d, f and g in $\hat{A}$ are positive and the eigenvalues of $\hat{A}$ are $\lambda_{n-3}, \lambda_{n-2}, \lambda_{n-1}, \lambda_n$.

Consider next the matrix

$$A = \begin{bmatrix} B & e_{p-1} & -e_{p-1} & -\frac{1}{q-1}J_{(p-1)\times(q-1)} \\ \frac{f}{p-1}e_{p-1}^T & 1 & -1 & -\frac{g}{q-1}e_{q-1}^T \\ \frac{f}{p-1}e_{p-1}^T & d & -1 & -\frac{g}{q-1}e_{q-1}^T \\ \frac{1}{p-1}J_{(q-1)\times(p-1)} & e_{q-1} & -e_{q-1} & -C \end{bmatrix}. \tag{2.2}$$

By Theorem 2.1 we know that $B \in M_{p-1}(\mathbb{R})$ and $C \in M_{q-1}(\mathbb{R})$ are positive matrices such that

$$B\,e_{p-1} = \rho(B)\,e_{p-1} = be_{p-1} \quad \text{and} \quad C\,e_{q-1} = \rho(C)\,e_{q-1} = ce_{q-1}.$$

If $By = \lambda y$, where $\lambda \ne \rho(B)$ and $y \ne 0$, then as B is normal, $e_{p-1}^T y = 0$ and thus $J_{(q-1)\times(p-1)}\,y = 0$. From (2.2) it follows that

$$A\begin{bmatrix} y \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} B\,y \\ 0 \\ 0 \\ 0 \end{bmatrix} = \lambda\begin{bmatrix} y \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus any eigenvalue $\lambda \ne \rho(B)$ of B is also an eigenvalue of A. Similarly,

$$A\begin{bmatrix} 0 \\ 0 \\ 0 \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -C\,z \end{bmatrix} = -\mu\begin{bmatrix} 0 \\ 0 \\ 0 \\ z \end{bmatrix},$$

where $\mu \ne \rho(C)$ is an eigenvalue of C with $Cz = \mu z$. This shows that $-\mu$ is an eigenvalue of A. Thus $\lambda_1, \ldots, \lambda_{n-4}$ are eigenvalues of A.

Now let $[w, x, y, z]^T$ be a right eigenvector of $\hat{A}$ corresponding to an eigenvalue $\nu \in \{\lambda_{n-3}, \lambda_{n-2}, \lambda_{n-1}, \lambda_n\}$. Then

$$A\begin{bmatrix} we_{p-1} \\ x \\ y \\ ze_{q-1} \end{bmatrix} = \begin{bmatrix} wBe_{p-1} + (x-y-z)e_{p-1} \\ fw + x - y - gz \\ fw + dx - y - gz \\ (w + x - y)e_{q-1} - zCe_{q-1} \end{bmatrix}$$

$$= \begin{bmatrix} (bw+x-y-z)e_{p-1} \\ fw+x-y-gz \\ fw+dx-y-gz \\ (w+x-y-cz)e_{q-1} \end{bmatrix} = \nu \begin{bmatrix} we_{p-1} \\ x \\ y \\ ze_{q-1} \end{bmatrix}.$$

Thus the eigenvalues of $\hat{A}$ are also eigenvalues of A, completing the proof. □

Theorem 2.6 *Every p-striped $n \times n$ sign pattern with $1 \le p \le n-1$ is spectrally arbitrary, namely, it allows any self-conjugate spectrum.*

Proof. If p is odd, then the result follows from Lemma 2.4. If p and n are both even, then the result follows from Lemma 2.5. If p is even and n is odd, then $n-p$ is odd and the result follows by applying Lemma 2.4 to the negatives of the desired spectrum elements and negating the constructed matrix. □

Corollary 2.7 *Every p-striped $n \times n$ sign pattern with $1 \le p \le n-1$ is inertially arbitrary.*

The results on the spectra and inertias of p-striped patterns in Theorem 2.6 and Corollary 2.7 can clearly be extended to sign patterns that are obtained by transposition, permutation or signature similarity (i.e., similarity by a diagonal matrix with diagonal entries $\pm$ 1) of p-striped patterns. For example,

$$S = \begin{bmatrix} - & + & - \\ + & - & + \\ + & - & + \end{bmatrix}$$

allows any self-conjugate spectrum.

3 Discussion

Part of our motivation for considering a striped pattern comes from the observation that if matrix A is nonsingular and $A \in Q(T_n)$, where T_n is the *antipodal* tridiagonal pattern in [1],

$$T_n = \begin{bmatrix} - & + & 0 & \dots & \dots & 0 \\ - & 0 & + & \ddots & & \vdots \\ 0 & - & 0 & + & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & - & 0 & + \\ 0 & \dots & \dots & 0 & - & + \end{bmatrix},$$

then $A^{-1} \in Q(S_n)$, where S_n is a p-striped pattern with alternating positive and negative columns. It is conjectured in [1] that T_n is inertially and spectrally arbitrary and this is verified for $n \leq 7$. Although the inverse of any nonsingular matrix in $Q(T_n)$ is in $Q(S_n)$, the inverse of nonsingular matrices from $Q(S_n)$ need not be in $Q(T_n)$, thus the result of Theorem 2.6 cannot be used to show that the invertible matrices in $Q(T_n)$ can achieve any self–conjugate spectrum that does not include zero. The result, however, gives added strength to the conjecture.

Note that there is no result analogous to Corollary 2.3 for irreducible nonnegative sign patterns. On the contrary, some such patterns have fixed inertia. For example,

$$S = \begin{bmatrix} 0 & + & 0 \\ 0 & 0 & + \\ + & 0 & 0 \end{bmatrix}$$

is an irreducible nonnegative sign pattern and it is straightforward to verify that the inertia of S is $\{(1, 2, 0)\}$. In fact, if an $n \times n$ pattern S is n-cyclic, then S has fixed inertia. In [6] symmetric sign patterns that require unique inertia are characterized.

Finally, we note that Theorem 2.6 establishes that p-striped patterns allow nilpotence. The problem of identifying such patterns has been considered in the literature, for example, in [3, 10].

Acknowledgment The authors thank Ludwig Elsner for insightful discussions, especially regarding the bound for ρ in Theorem 2.1, and acknowledge support through research grants from the Natural Sciences and Engineering Research Council of Canada.

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