COMBINATORIAL EIGENVALUES OF MATRICES

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Abstract

Let S be a subset of diagonal entries of an $n \times n$ complex matrix A. When the members of S have a common value which is equal to an eigenvalue of A, then S is a critical diagonal set of A. The existence of such a set is equivalent to the matrix $\tilde{A}$, obtained from A by setting its diagonal entries equal to zero, having an $s \times t$ zero submatrix with $s + t \geq n + 1$. If S is a minimal critical diagonal set, then equality holds. A zero submatrix of $\tilde{A}$ is used to identify the elements in a critical diagonal set. Then a combinatorial approach is taken in order to study the eigenspace of an eigenvalue associated with a critical diagonal set and to make observations regarding the Frobenius normal form of A and reducibility of A.

1 Introduction

Let $A = [a_{ij}]$ be an $n \times n$ matrix with entries from the complex field $\mathcal{C}$; it is convenient to let $d_i \equiv a_{ii}$ for $i = 1, 2, \ldots, n$. We are interested in discovering when it happens that at least one diagonal entry of A is equal to an eigenvalue of A. An obvious example of this occurence is a triangular matrix. A less obvious example is the 3×3 tridiagonal matrix

$$A = \begin{bmatrix} d_1 & a_{12} & 0 \\ a_{21} & d_2 & a_{23} \\ 0 & a_{32} & d_3 \end{bmatrix}. \tag{1.1}$$

If $d_1 = d_3$, their common value is an eigenvalue of A regardless of how values are assigned to the remaining nonzero entries of the matrix. Thus, for both the triangular and for the 3×3 tridiagonal matrix we can conclude the existence of a set of diagonal entries equal to an eigenvalue of the matrix.

It is from a combinatorial point of view that we study this problem; specifically, for what given zero/nonzero off–diagonal patterns of a matrix A does there exist a set of diagonal entries of A whose common value is an eigenvalue of A?

We let $N \equiv \{1, 2, \ldots, n\}$, $\delta \subseteq N$ and $\delta^c = \delta \setminus N$, and denote the spectrum of A by $\sigma(A)$.

Definition 1.1 A set $\{d_i : i \in \delta\}$ is called a *critical diagonal set* of an $n \times n$ matrix A with a specified zero/nonzero off–diagonal pattern if for any $\hat{\lambda} \in \mathcal{C}$, when $d_i = \hat{\lambda}$ for all $i \in \delta$, then $\hat{\lambda} \in \sigma(A)$, regardless of how the nonzero off–diagonal entries and the remaining diagonal entries of A are chosen. The (arbitrary) value $\hat{\lambda}$ is called a *combinatorial eigenvalue* of A.

There are certainly matrices which do not have a critical diagonal set based upon their zero/nonzero off–diagonal patterns. For example, the 2×2 matrix with both off–diagonal entries different from zero, the 4×4 tridiagonal matrix and the $n \times n$ (irreducible) weakly cyclic matrices of index n (i.e., matrices whose zero/nonzero off–diagonal pattern is a cycle of length n, see [V]). Also, a matrix may have more than one critical diagonal set; for example, any subset of the diagonal entries of a triangular matrix is a critical diagonal set. In general, any superset of a critical diagonal set is a critical diagonal set, but a proper subset of a critical diagonal set is not necessarily a critical diagonal set. For example, $\{d_1, d_3\}$ and $\{d_1, d_2, d_3\}$ are critical diagonal sets of the matrix A in (1.1), but $\{d_1\}$, $\{d_2\}$, $\{d_3\}$, $\{d_1, d_2\}$ and $\{d_2, d_3\}$ are not.

We are also interested in the algebraic and geometric multiplicity of a combinatorial eigenvalue. An important point needs to be made and we illustrate it with two examples. We let $A_\lambda \equiv A - \lambda I$. The tridiagonal matrix in (1.1) above has the characteristic polynomial

$$det A_\lambda = -(d_1-\lambda)a_{23}a_{32} - (d_3-\lambda)a_{12}a_{21} + (d_1-\lambda)(d_2-\lambda)(d_3-\lambda).$$

Obviously $d_1 = d_3 = \hat{\lambda}$ implies that $\hat{\lambda}$ is a simple eigenvalue for almost all values a_{ij}. But if, in addition, we have the quantitative condition $a_{23}a_{32} = -a_{12}a_{21}$, then $\hat{\lambda}$ becomes instead a double eigenvalue (algebraic multiplicity 2). Because of the required extra condition, we see that, for almost all choices of the off–diagonal entries of the matrix A the eigenvalue $\hat{\lambda}$ is simple. Our second example is the 3×3 lower triangular matrix

$$L = \begin{bmatrix} d_1 & 0 & 0 \\ a_{21} & d_2 & 0 \\ a_{31} & a_{32} & d_3 \end{bmatrix},$$

with $det L_\lambda = (d_1-\lambda)(d_2-\lambda)(d_3-\lambda)$. Suppose we choose $d_1 = d_3 = \hat{\lambda}$, but $d_2 \neq \hat{\lambda}$. Then $\hat{\lambda}$ is a double eigenvalue of L. If $u = (u_1, u_2, u_3)^T$ is an eigenvector belonging to $\hat{\lambda}$, then

$$a_{21}u_1 + (d_2 - \hat{\lambda})u_2 = 0 \quad \text{and} \quad a_{31}u_1 + a_{32}u_2 = 0.$$

Except for the very special choice $a_{21}a_{32} = a_{31}(d_2 - \hat{\lambda})$, these equations imply that $u_1 = u_2 = 0$ so that the only eigenvector belonging to $\hat{\lambda}$ is a multiple of $(0,0,1)^T$. For almost all choices of the off–diagonal entries, the eigenvalue $\hat{\lambda}$ has geometric multiplicity one. From these two examples it is evident that there lurks the possibility that, because of numerical coincidence, the stated algebraic (or geometric) multiplicity of an eigenvalue may be different from its *combinatorial* multiplicity which we now define.

Definition 1.2 Let A be a matrix with a critical diagonal set $\{d_i : i \in \delta\}$. The combinatorial eigenvalue $\hat{\lambda} = d_i,\ i \in \delta$, has *combinatorial algebraic multiplicity* r if the algebraic multiplicity of the eigenvalue $\hat{\lambda}$ is at least r, regardless of the choice of the nonzero off–diagonal entries and of $d_j,\ j \in \delta^c$, and equals r for some such choice of entries of A. The *combinatorial geometric multiplicity* of $\hat{\lambda}$ is similarly defined. We denote these by $cam_\delta(\hat{\lambda})$ and $cgm_\delta(\hat{\lambda})$ respectively.

These definitions incorporate the fact that off–diagonal nonzero entries can be arbitrarily chosen and therefore, as we state our results in terms of $cam_\delta(\hat{\lambda})$ and $cgm_\delta(\hat{\lambda})$, we do not have to add a caveat to this effect.

Suppose that $d_i = \hat{\lambda}$ for all d_i in a critical diagonal set of a matrix A. Then, by definition, $\hat{\lambda}$ is a combinatorial eigenvalue of A or equivalently 0 is a combinatorial eigenvalue of $A_{\hat{\lambda}}$. The characterization of sign–pattern matrices requiring k eigenvalues equal to 0 is considered in [E, Section 4.3]. The relationship between our results and those of [E] is discussed in Section 3.

2 Some Fundamental Combinatorial Ideas

For an $n \times n$ matrix A, let $D = diag(d_1, d_2, \ldots, d_n)$ be the diagonal matrix with diagonal entries equal to those of A and let $\tilde{A} \equiv A - D$; thus, with $\tilde{A} = [\tilde{a}_{ij}]$, $\tilde{a}_{ij} = a_{ij}$ for $i \neq j$ and $\tilde{a}_{ii} = 0$. Recall that $N = \{1, 2, \ldots, n\}$. For $\alpha, \beta \subseteq N$ we let $A[\alpha|\beta]$ denote the submatrix in rows $\alpha \subseteq N$ and columns $\beta \subseteq N$. When $\alpha = \beta$, we denote the principal submatrix $A[\alpha|\alpha]$ of order $|\alpha|$ by $A[\alpha]$, where $|\alpha|$ denotes the cardinality of α. We also let $A(\alpha)$ denote the principal submatrix in the rows and columns α^c of A. If $A[\alpha|\beta] = 0$, we call this a *zero submatrix of size* $|\alpha| + |\beta|$. We define the *principal rank* of a matrix A as the order of the largest nonsingular principal submatrix of A.

To treat our problem combinatorially, the following concepts are introduced. Let $T = \{a_{i_1,j_1}, a_{i_2,j_2}, \ldots, a_{i_k,j_k}\}$ be a set of nonzero entries of A. The set T is called a *transversal* of A if the sets $\{i_1, i_2, \ldots, i_k\} \subseteq N$ and $\{j_1, j_2, \ldots, j_k\} \subseteq N$ both consist of distinct integers. The length of the transversal T is k. A transversal is called *maximal* if its length is greater than or equal to the length of any other transversal of A. A transversal is called a *partial factor* of A if the row and column sets are equal.

We say that the matrix A with a given zero/nonzero pattern has *generic rank* k if the rank of A is less than or equal to k regardless of how the nonzero entries of A are chosen and is equal to k for some choice of these entries. Similarly we say A has *generic principal rank* k_0 if the principal rank of A is less than or equal to k_0 regardless of how the nonzero entries of A are chosen and is equal to k_0 for some choice of these entries. Note that for a given $n \times n$ matrix :

- principal rank $\leq$ generic principal rank $\leq$ generic rank;
- generic rank $= n$ if and only if generic principal rank $= n$.

Also, from the definitions we have that :

- A has generic rank k if and only if the length of any maximal transversal of A is k. The generic rank of A is therefore equal to the *term rank* of A, namely, the minimum number of lines (rows or columns) that cover all the nonzero entries of A (see e.g. [BR] Theorem 9.2.1).
- A has generic principal rank k_0 if and only if the length of any maximal partial factor of A is k_0.

Definition 2.1 An $n \times n$ matrix with generic principal rank less than n is called *combinatorially singular.*

Note that an $n \times n$ combinatorially singular matrix A cannot have a transversal of length n and that any matrix with the same zero/nonzero pattern as A must have zero determinant. The following equivalent condition for combinatorial singularity is classical; see [F] where it first appeared, or [MP] for an interesting discussion.

Theorem 2.2 **[Frobenius–König]** *An $n \times n$ matrix A is combinatorially singular if and only if there exists a zero submatrix $A[\alpha|\beta]$ of size* $|\alpha| + |\beta| \geq n + 1$.

In addition to the above combinatorial ideas, we use the following identity, recalling that $A_\lambda = A - \lambda I$.

$$\begin{aligned} det A_\lambda &= det\tilde{A} + \sum_{i=1}^{n}(d_i - \lambda) det\tilde{A}(i) + \sum_{1 \leq i_1 < i_2 \leq n} (d_{i_1} - \lambda)(d_{i_2} - \lambda) det\tilde{A}(i_1, i_2) \\ &\quad + \ldots + \sum_{1 \leq i_1 < \ldots < i_{n-2} \leq n} (d_{i_1} - \lambda) \ldots (d_{i_{n-2}} - \lambda) det\tilde{A}(i_1, \ldots, i_{n-2}) \\ &\qquad + \prod_{i=1}^{n}(d_i - \lambda). \end{aligned} \tag{2.1}$$

This formula is easy to verify (see, e.g., [PR]) and is particularly useful because of the way it relates an eigenvalue of A to its diagonal entries. Note that there is no term involving the product of exactly $(n-1)$ diagonal entries of A_λ. Specifically, (2.1) is useful in studying the combinatorial algebraic multiplicity. By definition, if $cam_\delta(\hat{\lambda}) = r$, then (barring accidental numeric cancellations) each nonzero term of the characteristic equation $det A_\lambda = 0$ has at least r factors $(\hat{\lambda} - \lambda)$ and there is one such term with exactly r factors. By (2.1), this can occur if and only if $\tilde{A}$ has generic principal rank less than or equal to $n - r$. In addition, given a combinatorial eigenvalue $\hat{\lambda} = d_i,\ i \in \delta$, it is a consequence of (2.1) that $cam_\delta(\hat{\lambda}) \leq |\delta|$.

3 Critical Diagonal Sets

The following two theorems connect the existence of a critical diagonal set with combinatorial singularity.

Theorem 3.1 *Let $\delta \subseteq N$ and $\hat{\lambda} \in \mathcal{C}$, and set $d_i = \hat{\lambda}$ for all $i \in \delta$. Then $S = \{d_i : i \in \delta\}$ is a critical diagonal set of A if and only if $A_{\hat{\lambda}}$ is combinatorially singular.*

Proof. This follows immediately from the definitions of a critical diagonal set and combinatorial singularity. $\square$

Theorem 3.2 *For an $n \times n$ matrix A, the matrix $\tilde{A}$ is combinatorially singular if and only if A has a critical diagonal set.*

Proof. The proof follows from (2.1) and the fact that $\tilde{A}$ is combinatorially singular if and only if every matrix with the same zero/nonzero pattern as $\tilde{A}$ has zero determinant. □

In the following result, the existence of a zero submatrix in $\tilde{A}$ is used to be more specific about its generic principal rank and the cardinality of a critical diagonal set of A.

Theorem 3.3 *Let A be an $n \times n$ matrix and suppose that $\tilde{A}[\alpha|\beta] = 0$ where $|\alpha| + |\beta| = n + p,\ p \geq 1$. Let $\gamma \equiv \alpha \cap \beta$. Then,*

(i) *$\tilde{A}$ has generic principal rank $\leq n - p$.*

(ii) *$\{d_i : i \in \gamma\}$ is a critical diagonal set of A with $|\gamma| \geq p$.*

(iii) *If $d_i = \hat{\lambda}$ for all $i \in \gamma$, then $cgm_\gamma(\hat{\lambda}) \geq p$.*

Proof.

(i) Observe that all of the nonzero entries of A are contained in at most $n - |\alpha|$ rows and $n - |\beta|$ columns of $\tilde{A}$. Thus the order of the largest principal submatrix of $\tilde{A}$ containing a partial factor is less than or equal to $n - |\alpha| + n - |\beta|$. Since $|\alpha| + |\beta| = n + p$, this order must be $\leq n - p$.

(ii) It follows from Theorem 3.2 that A has a critical diagonal set. If $d_i = \hat{\lambda}$ for all $i \in \gamma$, then $\{d_i : i \in \gamma\}$ is a critical diagonal set of A, because $A_{\hat{\lambda}}$ contains the same zero submatrix as $\tilde{A}$ and so, by Theorem 2.2, is combinatorially singular. Since $|\alpha \cup \beta| \leq n$ and $|\alpha| + |\beta| = n + p$, it follows that $|\gamma| \geq p$.

(iii) If $\hat{\lambda}$ is a combinatorial eigenvalue of A, the diagonal entries of $A_{\hat{\lambda}}[\gamma]$ are 0. Therefore $A_{\hat{\lambda}}[\alpha|\beta] = 0$. Thus the nonzero entries in $A_{\hat{\lambda}}$ can be covered with $n - p$ lines, and the term rank of $A_{\hat{\lambda}}$ is $\leq n - p$. Therefore, the rank of $A_{\hat{\lambda}}$ is $\leq n - p$, so the nullity of $A_{\hat{\lambda}}$ is $\geq p$, that is, $cgm_\gamma(\hat{\lambda}) \geq p$. □

As mentioned in the introduction, the above results can also be stated in terms of 0 being a combinatorial eigenvalue of $A_{\hat{\lambda}}$, or equivalently $A_{\hat{\lambda}}$ being combinatorially singular. This is the approach taken in [E, Section 4.3], where results equivalent to our Theorems 3.1, 3.2 and 3.3 (i) are proved. In our results the emphasis is placed on zero submatrices in $\tilde{A}$, leading to the identification of the elements of a critical diagonal set. In [E] the proofs are based on a continuity argument and the elements of a critical diagonal set are not specified.

Consider the matrix

$$A = \begin{bmatrix} d_1 & 0 & 0 & a_{14} \\ 0 & d_2 & 0 & a_{24} \\ a_{31} & 0 & d_3 & 0 \\ a_{41} & 0 & 0 & d_4 \end{bmatrix}. \tag{3.1}$$

Since $\tilde{A}[\{1,2\}|\{1,2,3\}]=0$ and $\tilde{A}[\{1,2,3,4\}|\{2,3\}]=0$, by Theorem 3.3(ii), we have that $\{d_1,d_2\}$ and $\{d_2,d_3\}$ are critical diagonal sets of A. Recall that any superset of a critical diagonal set is also a critical diagonal set. For instance, $\{d_1,d_2,d_3\}$ is a critical diagonal set of A. Notice now that $\tilde{A}[\{1,2,3\}]\neq 0$. As a consequence, no converse of Theorem 3.3(ii) exists; i.e., given a critical diagonal set S of an $n\times n$ matrix A, there may not be a zero submatrix $\tilde{A}[\alpha|\beta]$ of size $|\alpha|+|\beta|=n+p,\ p\geq 1$, so that $S=\{d_i : i\in\alpha\cap\beta\}$. Observe also that, since the characteristic polynomial of A in (3.1) is

$$detA_\lambda=\prod_{i=1}^{4}(d_i-\lambda)-a_{41}a_{14}(d_2-\lambda)(d_3-\lambda),$$

a critical diagonal set of A *must* contain either d_2 or d_3. These observations suggest the following definition.

Definition 3.4 Let $S=\{d_i : i\in\delta\}$ be a critical diagonal set of a matrix A. We call S a *minimal critical diagonal set* of A if no proper subset of S is a critical diagonal set of A.

For example, $\{d_1,d_3\}$ is a minimal critical diagonal set of A in (1.1) but $\{d_1,d_2,d_3\}$ is not. Also $\{d_2\}$ and $\{d_3\}$ are the only minimal critical diagonal sets of A in (3.1). The next theorem provides us with a converse of Theorem 3.3(ii) when the critical diagonal set S is minimal.

Theorem 3.5 *If $S=\{d_i : i\in\delta\}$ is a minimal critical diagonal set of an $n\times n$ matrix A, then there exists a zero submatrix $\tilde{A}[\alpha|\beta]$ of size $|\alpha|+|\beta|=n+1$ such that $\delta=\alpha\cap\beta$.*

Proof. Suppose that $S=\{d_i : i\in\delta\}$ is a minimal critical diagonal set of A and let $d_i=\hat{\lambda}$ for all $i\in\delta$. Then, by Theorem 3.1, the matrix $A_{\hat{\lambda}}$ is combinatorially singular and so, by Theorem 2.2, it must have a zero submatrix $A_{\hat{\lambda}}[\alpha|\beta]$ of size $|\alpha|+|\beta|=n+1$. We then have that $|\alpha\cap\beta|\geq 1$ and hence $A_{\hat{\lambda}}$ must have a zero diagonal entry. Furthermore, since we may in particular choose $d_j\neq\hat{\lambda}$ for all $j\in\delta^c$, we have that $\alpha\cap\beta\subseteq\delta$. But, by minimality of S, it follows that $\alpha\cap\beta=\delta$, or otherwise $S\supset\{d_i : i\in\alpha\cap\beta\}$, which by Theorem 3.3(ii) is indeed a critical diagonal set of A. Also observe that $\tilde{A}[\alpha|\beta]=0$, completing the proof of the theorem. □

Note that if $S=\{d_i : i\in\alpha\cap\beta\}$, where $\tilde{A}[\alpha|\beta]=0$ and $|\alpha|+|\beta|=n+1$, then S is not necessarily a minimal critical diagonal set, as the following example shows. Let

$$A=\begin{bmatrix} d_1 & 0 & 0\\ 0 & d_2 & a_{23}\\ a_{31} & a_{32} & d_3\end{bmatrix}.$$

The critical diagonal set $\{d_1, d_2\}$ is obtained from the zero submatrix $\tilde{A}[\{1,2\}]$ of size 4, but it is not minimal since $\{d_1\}$ is also a critical diagonal set obtained from $\tilde{A}[\{1\}|\{1,2,3\}]$. A zero submatrix $\tilde{A}[\alpha|\beta]$ of size $|\alpha| + |\beta| = n+1$ yields a minimal critical diagonal set if no diagonal entry of $\tilde{A}[\alpha|\beta]$ is contained in another zero submatrix of A of size $\geq n+1$.

We can obtain, as a corollary to Theorem 3.5, the following bound on the cardinality of a minimal critical diagonal set.

Corollary 3.6 *Let S be a minimal critical diagonal set of an $n \times n$ matrix A. Then,*

$$|S| \leq \frac{n+1}{2}.$$

Proof. Let $S = \{d_i : i \in \delta\}$ be a minimal critical diagonal set of A. By Theorem 3.5, there exists a zero submatrix $\tilde{A}[\alpha|\beta]$ of size $|\alpha|+|\beta| = n+1$ such that $\delta = \alpha \cap \beta$. Observe then that

$$|S| = |\delta| \leq \min(|\alpha|, |\beta|) \leq \frac{n+1}{2}. \quad \square$$

Example 3.7 Consider the matrix

$$A = \begin{bmatrix} d_1 & a_{12} & 0 & 0 \\ a_{21} & d_2 & a_{23} & a_{24} \\ a_{31} & a_{32} & d_3 & 0 \\ 0 & a_{42} & 0 & d_4 \end{bmatrix} \quad \text{with} \quad \tilde{A} = \begin{bmatrix} 0 & a_{12} & 0 & 0 \\ a_{21} & 0 & a_{23} & a_{24} \\ a_{31} & a_{32} & 0 & 0 \\ 0 & a_{42} & 0 & 0 \end{bmatrix}.$$

Here $\tilde{A}$ has generic principal rank 3 and zero submatrices $\tilde{A}[\{1,4\}|\{1,3,4\}]$ and $\tilde{A}[\{1,3,4\}|\{3,4\}]$. For both of these submatrices $p = 1$. Observe that, by Theorem 3.5, d_2 cannot be contained in any minimal critical diagonal set of A. The sets $\gamma_1 = \{1,4\}$, $\gamma_2 = \{3,4\}$ and $\gamma_3 = \gamma_1 \cup \gamma_2 = \{1,3,4\}$ all yield critical diagonal sets of A. Only the first two are minimal. It is easy to verify that $cam_{\gamma_i}(\hat{\lambda}) = cgm_{\gamma_i}(\hat{\lambda}) = 1$, for $i = 1,2,3$.

In the following result we show that the combinatorial eigenvalue associated with a minimal critical diagonal set is always simple.

Theorem 3.8 *Let $\{d_i : i \in \delta\}$ be a minimal critical diagonal set of a matrix A. If $d_i = \hat{\lambda}$ for all $i \in \delta$, then $cam_\delta(\hat{\lambda}) = 1$.*

Proof. Let $S = \{d_i : i \in \delta\}$ be a minimal critical diagonal set of A. If $|\delta| = 1$, then the result is trivially true. If $|\delta| > 1$ and $cam_\delta(\hat{\lambda}) = m \geq 2$, then each term on the righthand side of (2.1) has a factor of the form $\prod_{j=1}^{m}(d_{i_j} - \lambda)$, where

$d_{i_j} \in S$. In particular, for any fixed $l \in \delta$ each term on the righthand side of (2.1) contains a factor of the form $\prod_{k=1}^{m-1}(d_{i_k} - \lambda)$, where $d_{i_k} \in S \setminus \{d_l\}$. Consequently, $\{d_i : i \in \delta \setminus \{l\}\}$ is a critical diagonal set of A, contradicting the minimality of S. Thus $cam_\delta(\hat{\lambda}) = 1$. $\square$

In the next theorem, we investigate the lower bound for the combinatorial algebraic multiplicity, when the critical diagonal set is obtained from two distinct zero submatrices of $\tilde{A}$.

Theorem 3.9 *Let A be an $n \times n$ matrix and suppose that $\tilde{A}[\alpha_j|\beta_j] = 0$, $|\alpha_j| + |\beta_j| = n + p_j$, with $p_j \geq 1$, $\gamma_j = \alpha_j \cap \beta_j$ for $j = 1, 2$. If $d_i = \hat{\lambda}$ for all $i \in \gamma = \gamma_1 \cup \gamma_2$, then*

$$cam_\gamma(\hat{\lambda}) \geq \begin{cases} p_1 + p_2 - |\gamma_1 \cap \gamma_2| & \textit{if } |\gamma_1 \cap \gamma_2| < \min(p_1, p_2) \\ \max(p_1, p_2) & \textit{otherwise.} \end{cases}$$

Proof. Clearly $\{d_i : i \in \gamma_1 \cup \gamma_2\}$ is a critical diagonal set of A. In addition, we can show that every term of the characteristic polynomial of A which does not vanish has at least p_1 factors $(\hat{\lambda} - \lambda)$ from the set γ_1 and at least p_2 factors $(\hat{\lambda} - \lambda)$ from the set γ_2. Observe that since $|\gamma_i| \geq p_i$, $i = 1, 2$, we have that if

$$\mathcal{M} \equiv \max\{\max(p_1, p_2),\ p_1 + p_2 - |\gamma_1 \cap \gamma_2|\}, \tag{3.2}$$

then $cam_\gamma(\hat{\lambda}) \geq \mathcal{M}$. Observe now that if $|\gamma_1 \cap \gamma_2| \leq \min(p_1, p_2)$, then $\mathcal{M} = p_1 + p_2 - |\gamma_1 \cap \gamma_2|$, and if $|\gamma_1 \cap \gamma_2| > \min(p_1, p_2)$, then $\mathcal{M} = \max(p_1, p_2)$. $\square$
We illustrate this result with the following example.

Example 3.10 Let

$$A = \begin{bmatrix} d_1 & a_{12} & a_{13} & a_{14} & 0 & 0 & a_{17} \\ a_{21} & d_2 & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} \\ 0 & a_{32} & d_3 & a_{34} & 0 & 0 & a_{37} \\ 0 & a_{42} & 0 & d_4 & 0 & 0 & 0 \\ 0 & a_{52} & a_{53} & a_{54} & d_5 & 0 & a_{57} \\ 0 & a_{62} & a_{63} & a_{64} & 0 & d_6 & a_{67} \\ 0 & a_{72} & 0 & 0 & 0 & 0 & d_7 \end{bmatrix}.$$

$\tilde{A}$ has zero submatrices $\tilde{A}[\{4,7\}|\{1,3,4,5,6,7\}]$ and $\tilde{A}[\{1,3,4,5,6,7\}|\{1,5,6\}]$ with $p_1 = 1$ and $p_2 = 2$ respectively. Here $\gamma_1 = \{4,7\}$ and $\gamma_2 = \{1,5,6\}$. Observe that γ_1 yields a minimal critical diagonal set, but γ_2 does not (since $\{d_1, d_5\}$, $\{d_1, d_6\}$ and $\{d_5, d_6\}$ are also minimal critical diagonal sets of A). It is easy to see that $\tilde{A}$ has generic principal rank 4. If $d_4 = d_7 = \hat{\lambda}$, then $\hat{\lambda}$ is a combinatorial eigenvalue of A with $cam_{\gamma_1}(\hat{\lambda}) = cgm_{\gamma_1}(\hat{\lambda}) = 1$. If $d_1 = d_5 = d_6 = \hat{\lambda}$, then $cam_{\gamma_2}(\hat{\lambda}) = cgm_{\gamma_2}(\hat{\lambda}) = 2$. If $d_i = \hat{\lambda}$ for all $i \in \gamma = \gamma_1 \cup \gamma_2$, then by Theorem 3.9 $cam_{\gamma}(\hat{\lambda}) \geq 3$. In fact, in this case, $cam_{\gamma}(\hat{\lambda}) = 3$ and $cgm_{\gamma}(\hat{\lambda}) = 2$, illustrating that $cam_{\gamma}(\hat{\lambda})$ in Theorem 3.9 cannot be replaced by $cgm_{\gamma}(\hat{\lambda})$, although it is always true that $cgm_{\gamma}(\hat{\lambda}) \geq \max(p_1, p_2)$.

In the next theorem we connect the notion of a minimal critical diagonal set with the Frobenius normal form of a matrix. Recall that for any $n \times n$ matrix A, there exists a permutation matrix $P = [p_{ij}]$ such that PAP^T is a block upper triangular matrix, where each diagonal submatrix A_{kk} is either 1×1 or is irreducible (see [V 2.3]). This is called the Frobenius normal form of A. Since $\sigma(A) = \bigcup_k \sigma(A_{kk})$ and a permutation similarity simply permutes the diagonal entries of A, a critical diagonal set $\{d_i : i \in \delta\}$ of A corresponds to a critical diagonal set $\{d_{\pi(i)} : i \in \delta\}$ of PAP^T, where π is a permutation of N and $p_{i,\pi(i)} = 1,\ 1 \leq i \leq n$.

Theorem 3.11 *Let $\{d_i : i \in \delta\}$ be a minimal critical diagonal set of A. Then, all entries of the corresponding minimal critical diagonal set $\{d_{\pi(i)} : i \in \delta\}$ of the Frobenius normal form of A are in the same diagonal submatrix.*

Proof. Suppose $S = \{d_i : i \in \delta\}$ is a minimal critical diagonal set of A and let $d_i = \hat{\lambda}$ for all $i \in \delta$. Then $\hat{\lambda} \in \sigma(A)$. Consider now a diagonal submatrix A_{kk} of the Frobenius normal form of A, such that $d_{\pi(j)}$ is in A_{kk}, for some $j \in \delta$. We claim that $\hat{\lambda} \in \sigma(A_{kk})$ and all $d_{\pi(i)},\ i \in \delta$ belong to A_{kk}. To prove these claims, observe that the spectrum of any one diagonal submatrix of the Frobenius normal form of A does not depend on the choice of entries of any other submatrix. As a consequence, if $\hat{\lambda} \notin \sigma(A_{kk})$, then $\hat{\lambda} \in \sigma(A)$ independently of the choice of $d_{\pi(j)}$, so that $S \setminus \{d_{\pi(j)}\}$ is a critical diagonal set of A, contradicting minimality of S. Similarly, since now $\hat{\lambda} \in \sigma(A_{kk})$, if for some $l \in \delta\ \ d_{\pi(l)}$ is not in A_{kk}, then the diagonal entries of A_{kk} in δ form a critical diagonal set of A, a contradiction to the minimality of S, completing the proof of the theorem. $\square$

By Theorem 3.11, S is a minimal critical diagonal set of A if and only if S is a minimal critical diagonal set of some diagonal submatrix of the Frobenius normal form of A. As illustrated by Example 3.10, one such submatrix can contain more than one minimal critical diagonal set of A.

The following result characterizes the case when an $n \times n$ matrix has n minimal critical diagonal sets. An equivalent statement in terms of 0 eigenvalues is contained in [E], so we omit the proof.

Theorem 3.12 *Every diagonal entry is a (minimal) critical diagonal set of a matrix A if and only if A is permutationally similar to a triangular matrix.*

Note that if A is an $n \times n$ matrix with $\tilde{A}[\alpha|\beta] = 0$ of size $|\alpha| + |\beta| = n + p$ and $|\alpha \cap \beta| = p \geq 1$, then A is reducible. Also observe that if A is an $n \times n$ irreducible matrix, then there cannot be a zero submatrix $\tilde{A}[\alpha|\beta]$ with $|\alpha| + |\beta| = n + p$ and $p > n - 2$. However, the matrix A having off-diagonal entries a_{ij} nonzero if and only if $i = 1$ or $j = 1$ has a zero submatrix in $\tilde{A}$ with $p = n - 2$.

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