RAY PATTERNS OF MATRICES AND NONSINGULARITY

(Linear Algebra and Its Applications, 267:359-373, 1997)

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December 6, 2000

Abstract

A complex matrix A is ray–nonsingular if $\det(X \circ A) \neq 0$ for every matrix X with positive entries. A sufficient condition for ray–nonsingularity is that the origin is not in the relative interior of the convex hull of the signed transversal products of A. The concept of an isolated set of transversals is defined and used to obtain a necessary condition for A to be ray–nonsingular. Some fundamental similarities as well as differences between ray–nonsingularity and sign–nonsingularity are illustrated.

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1 Introduction

We are interested in determining the nonsingularity of a complex matrix based solely on the arguments of its nonzero entries. This idea generalizes the notion of sign–nonsingularity of real matrices (discussed e.g., in [1], [7], [2], and [3]), which facilitates the determination of nonsingularity based on the two possible arguments, 0 or π (corresponding to the signs + or −), and without regard to the magnitudes of the nonzero entries. To describe this generalization of sign–nonsingularity to complex matrices, it is convenient to use the Hadamard (entrywise) product $\circ$ of matrices.

Definition 1.1 We call $A \in M_n(\mathbf{C})$ *ray–nonsingular* if $X \circ A$ is a nonsingular matrix for every $X \in M_n(\mathbf{R})$ with positive entries.

If A in the above definition is a real matrix, then A is sign–nonsingular. Ray–nonsingularity amounts to fixing the arguments (mod 2π) of the nonzero entries of a complex matrix, letting the moduli of the nonzero entries vary in $(0, \infty)$, and requiring that all matrices obtained are nonsingular.

We first give a sufficient condition for ray–nonsingularity in terms of the set of the signed transversal products of the matrix (see Theorem 3.1), and an example to show that this condition is not in general necessary. In order to obtain a necessary condition for ray–nonsingularity, we study the range of $\det(X \circ A)$ as a function of a matrix X with positive entries. To achieve this, we introduce the combinatorial concept of an isolated set of transversals (see Section 4). We also observe that ray–nonsingularity can be characterized in ways that naturally extend some of the known characterizations of sign–nonsingularity (see Theorem 3.5). However, there are fundamental structural differences regarding the submatrices of ray–nonsingular matrices and their sparsity patterns from those of sign–nonsingular matrices (see e.g., Theorem 3.6).

2 Definitions and Notation

For any nonzero $z \in \mathbf{C}$, we denote by $\arg(z)$ the argument of z (mod 2π), measured in the interval $(-\pi, \pi]$. Throughout we let $A = (a_{jk}) \in M_n(\mathbf{C})$, and i denote the imaginary unit.

The matrix PAQ, where P and Q are permutation matrices in $M_n(\mathbf{R})$, is *permutation equivalent to A*. A nonzero diagonal matrix $D \in M_n(\mathbf{C})$ whose nonzero entries have modulus 1 is called a *complex signature*. Let $\hat{A}$ be a submatrix of A. The *complementary submatrix* of $\hat{A}$ is defined as the submatrix of A whose rows and columns are indexed by the complements in $\{1, 2, \ldots, n\}$ of the row and column indices, respectively, of $\hat{A}$.

The *ray pattern* of A, denoted by $\mathcal{A} = (\alpha_{jk}) \in M_n(\mathbf{C})$, is defined by

$$\alpha_{jk} = \begin{cases} e^{i\arg(a_{jk})} & \text{if } a_{jk} \neq 0 \\ 0 & \text{otherwise,} \end{cases}$$

and we write $A \in \mathcal{A}$. Clearly, the matrix A is ray–nonsingular if and only if every matrix with the same ray pattern as A is nonsingular. Then the ray pattern $\mathcal{A}$ *requires* the property of nonsingularity (i.e., it requires rank n). Clearly, transposition and multiplication by a permutation matrix or by a nonsingular complex signature leave ray–nonsingularity invariant.

The *digraph* of A, denoted by $G(A) = (V, E)$, consists of the vertex set $V = \{1, 2, \ldots, n\}$ and the set of directed arcs $E = \{(j, k) \mid a_{jk} \neq 0\}$. A *path of length* $m \geq 1$ from j to k in $G(A)$ is a sequence of distinct vertices $j = r_1, r_2, ..., r_{m+1} = k$, such that $(r_s, r_{s+1}) \in E$ for $s = 1, ..., m$. For $m \geq 2$, a sequence of vertices $r_1, r_2, ..., r_m, r_{m+1} = r_1$ such that $(r_s, r_{s+1}) \in E$ for $s = 1, ..., m$ is called a *circuit of length* m. If the vertices $r_1, r_2, ..., r_m$ of the above circuit are distinct, then it is called a *cycle of length* m.

Matrix A is said to be *fully indecomposable* if it is not permutation equivalent to a matrix of the form

$$\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix},$$

where A_{11} is a $k \times k$ square matrix with $1 \leq k \leq n-1$. The matrix A is called *irreducible* if there is a path from any vertex j to any other vertex k in $G(A)$. Equivalently, A is irreducible if and only if it is not permutation similar to a matrix of the form above. Every reducible matrix is permutation similar to its *Frobenius normal form*, namely a block triangular matrix whose diagonal blocks, referred to as the *irreducible components of* A, are irreducible.

We continue with some more combinatorial concepts. Consider $\{j_1, j_2, \ldots, j_n\}$ and $\{k_1, k_2, \ldots, k_n\}$, two permutations of $\{1, 2, \ldots, n\}$. A set of nonzero entries of A, $\{a_{j_1k_1}, a_{j_2k_2}, \ldots, a_{j_nk_n}\}$, is called a *transversal* of A. The set of all transversals of A is denoted by $\tau(A)$. A transversal can be uniquely partitioned into subsets corresponding to the permutation cycles of the permutation σ, where $\sigma(j_s) = k_s$ for $s = 1, 2, \ldots, n$. We refer to this partition as the *cyclic decomposition* of the transversal. For a nonempty set τ of transversals, we denote by $M(\tau)$ the matrix obtained from A when the entries belonging to the transversals in τ are replaced by 1 and the rest are set equal to 0. When $\tau = \{t\}$, we write $M(t)$ instead of $M(\{t\})$. The product of the entries of a transversal of A, weighted by $(-1)^{sgn(\sigma)}$, where σ is the permutation satisfying $\sigma(j_s) = k_s$ for $s = 1, 2, \ldots, n$, is referred to as a *signed transversal product* of A. We let $T(A)$ denote the collection (multiset) of all signed transversal products of A with repetitions allowed. The sum of the elements of $T(A)$ is indeed the standard expansion of det$A$. If A has no transversal, then it is singular; in fact every matrix with the same zero/nonzero pattern as A is singular, in which case A is *combinatorially singular*. If A has exactly one transversal, then it is ray–nonsingular.

To describe our results we use the standard terminology of convex sets (see [9]) that we summarize next. A set $S \subset \mathbf{C}$ is *convex* if $\alpha z_1 + (1-\alpha) z_2 \in S$ for all $z_1, z_2 \in S$ and $\alpha \in (0, 1)$. The *affine hull* of S is

$$\{\sum \alpha_t z_t \mid z_t \in S, \alpha_t \in \mathbf{R}, \sum \alpha_t = 1\}.$$

In contrast to the interior of S (int S), we denote by ri S the *relative interior* of S, namely, the interior of S when it is regarded as a subset of its affine hull. Note that the two concepts coincide except when S is a line segment. The *relative boundary* of S is the set difference of its topological closure and its relative interior. An *extreme point* z of S is a point that can be expressed as $\alpha z_1 + (1-\alpha) z_2$ with $z_1, z_2 \in S$ and $\alpha \in (0, 1)$ only if $z_1 = z_2 = z$. Let $z_j \in \mathbf{C}$, $j = 1, 2, \ldots, k$. The *convex hull* of $\{z_1, z_2, \ldots, z_k\}$ is the convex set

$$\text{conv } \{z_1, z_2, \ldots, z_k\} = \{\sum_{t=1}^{k} \alpha_t z_t \mid \alpha_t \geq 0, \ \sum_{t=1}^{k} \alpha_t = 1\}.$$

The *cone generated by* $\{z_1, z_2, \ldots, z_k\}$ is the convex set

$$\text{cone } \{z_1, z_2, \ldots, z_k\} = \{\sum_{t=1}^{k} \alpha_t z_t \mid \alpha_t \geq 0\}.$$

3 Ray–Nonsingular Matrices

We begin with a basic result regarding ray–nonsingularity.

Theorem 3.1 *Assume that $A \in M_n(\mathbf{C})$ is not combinatorially singular. If $0 \notin$ ri conv $T(A)$, then A is ray–nonsingular (i.e., $\mathcal{A}$ requires nonsingularity).*

Proof. We prove the contrapositive. If $|T(A)| = 1$, then the result is trivially true. Otherwise let $T(A) = \{T_1, T_2, \ldots, T_p\}$ and suppose that $B \in \mathcal{A}$ is singular. Then there exist positive numbers $c_1, c_2, \ldots, c_p$ such that $\det B = \sum_{j=1}^{p} c_j T_j = 0$. Letting $c = \sum_{j=1}^{p} c_j$ and $d_j = c_j/c \in (0,1)$ gives $\sum_{j=1}^{p} d_j T_j = 0$, where $\sum_{j=1}^{p} d_j = 1$. Hence $0 \in \text{ri conv } T(A)$. ∎

The following example illustrates this theorem. We abbreviate e^{i0}, $e^{i\frac{\pi}{2}}$, $e^{i\pi}$, and $e^{-i\frac{\pi}{2}}$ by 1, i, -1, and $-i$, respectively.

Example 3.2 Let

$$A = \begin{pmatrix} -1 & -i & -1 \\ -1 & -1 & -i \\ -1 & i & -1 \end{pmatrix}.$$

The signed transversal products in this case are 1, -1, and i. Notice that the origin lies on the relative boundary of the convex hull of these numbers and hence, by Theorem 3.1, A is ray–nonsingular. Moreover, the determinant of every matrix whose ray pattern is $\mathcal{A}$ always has positive imaginary part.

In contrast, consider the following.

Example 3.3 Let

$$\mathcal{A} = \begin{pmatrix} 1 & 1 & e^{i\frac{3\pi}{4}} \\ -1 & 1 & 1 \\ e^{-i\frac{3\pi}{4}} & -1 & 1 \end{pmatrix}.$$

The signed transversal products of $\mathcal{A}$ are equal to 1, -1, $e^{i\frac{3\pi}{4}}$, and $e^{-i\frac{3\pi}{4}}$, thus $0 \in$ ri conv $T(\mathcal{A})$. Indeed $\mathcal{A}$ does not require nonsingularity, since if $A \in \mathcal{A}$ has its (1,1) entry equal to $-\frac{1}{2}(e^{-i\frac{3\pi}{4}} + e^{i\frac{3\pi}{4}}) = 1/\sqrt{2}$ and the remaining entries as in $\mathcal{A}$, then A is singular.

The converse of Theorem 3.1 is not in general true, as can be seen by the following.

Example 3.4 Let

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ e^{i\frac{3\pi}{4}} & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & e^{-i\frac{3\pi}{4}} & 1 \end{pmatrix}.$$

A is ray–nonsingular since it is the direct sum of ray–nonsingular matrices. However, it is easy to see that $0 \in \text{ri conv } T(A)$.

Note that the matrix A in Example 3.4 is not fully indecomposable. We do not know whether or not the converse of Theorem 3.1 holds when A is fully indecomposable.

In the spirit of Remark 1.1 in [7] and of Theorem 5.1 in [5], we prove the following.

Theorem 3.5 *The matrix $A \in M_n(\mathbf{C})$ is ray–nonsingular if and only if for every complex signature $D \in M_n(\mathbf{C})$, the relative interior of the convex hull of the nonzero entries of at least one column (resp. row) of DA (resp. AD) does not contain the origin.*

Proof. We prove the contrapositives of the implications in the statement of the theorem. First suppose that a matrix with ray pattern $\mathcal{A}$, denoted for convenience by A, is singular. That is, there exists nonzero $x \in \mathbf{C}^n$ such that $\sum_{j=1}^n |x_j| = 1$, and $x^T A = 0$. Define the complex signature $D = (d_{ij})$ by

$$d_{jj} = \begin{cases} x_j/|x_j| & \text{if } x_j \neq 0 \\ 0 & \text{otherwise} \end{cases} \quad (j = 1, 2 \ldots, n).$$

Let $y = (|x_1|, |x_2|, \ldots, |x_n|)^T$. Then $0 = x^T A = y^T DA$. Suppose that only one entry of y, say $|x_k|$, is nonzero. Then d_{kk} is the only nonzero entry of D. From $x^T A = 0$, it follows that the k–th row of A is zero and thus $DA = 0$. On the other hand, if more than one entry of y is nonzero, then the relative interior of the convex hull of the nonzero entries of every column of DA contains the origin. Conversely, if for some complex signature D the relative interior of the convex hull of the nonzero entries of every column of DA contains the origin, then the entries in each column of a matrix with ray pattern $\mathcal{A}$ can be chosen so that their sum is zero, meaning that A is not ray–nonsingular. (To prove the respective column scaling part of the theorem apply the above steps to A^T.) ∎

For a sign–nonsingular matrix it is well known that every square submatrix is sign–nonsingular, combinatorially singular, or else its complementary submatrix is combinatorially singular (see [6]). The situation with ray–nonsingularity includes another alternative that is described in the next theorem.

Theorem 3.6 *Let $A \in M_n(\mathbf{C})$ with $0 \notin ri\ conv\ T(A)$. Let $\hat{A}$ be a square submatrix of A and $\tilde{A}$ be its complementary submatrix. Then, either*

(1) $\hat{A}$ is ray–nonsingular, or

(2) $\hat{A}$ is combinatorially singular, or

(3) conv $T(\hat{A})$ *is a line segment whose relative interior includes the origin, or*

(4) $\tilde{A}$ *is combinatorially singular.*

Proof. Let A, $\hat{A}$, and $\tilde{A}$ be as prescribed and suppose $\hat{A}$ is not ray–nonsingular. Then, either conv $T(\hat{A})$ is empty (case (2)) or, by Theorem 3.1, we have that $0 \in$ ri conv $T(\hat{A})$. Two cases can then occur: either conv $T(\hat{A})$ is a line segment through the origin (case (3)) or conv $T(\hat{A})$ has three or more extreme points. In the latter case, $\tilde{A}$ must be combinatorially singular (case (4)); otherwise, since $T(A)$ contains the products between elements of $T(\hat{A})$ and properly signed elements of $T(\tilde{A})$, $T(A)$ contains a rotation about the origin of $T(\hat{A})$, and hence $0 \in$ ri conv $T(A)$, which is a contradiction. ■

Remark 3.7 Contrary to the case of sign–nonsingularity, alternative (3) of the above theorem may hold even when $\tilde{A}$ is not combinatorially singular. In fact, when (3) holds and since A is ray–nonsingular by Theorem 3.1, the origin lies on the relative boundary of conv $T(A)$. An occurrence of this situation is found in Example 3.2, when $\hat{A}$ is taken to be the trailing 2–by–2 principal submatrix of the given ray–nonsingular matrix.

Though we have so far concentrated on square ray patterns, it is possible to extend some of our results to rectangular patterns. In fact, the proof of Theorem 3.5 yields the following result.

Theorem 3.8 *The* n*–by–*m *ray pattern* $\mathcal{A}$ *requires rank* n *if and only if for every complex signature* $D \in M_n(\mathbf{C})$ *the relative interior of the convex hull of the nonzero entries of at least one column of* $D\mathcal{A}$ *does not contain the origin.*

Moreover, we have the following theorem, whose proof technique is identical to the one of Theorem 5.3 in [5], except for the equivalence of clauses (2) and (3), which now follows from our Theorem 3.8.

Theorem 3.9 *Let* $\mathcal{A}$ *be an* n*–by–*m *ray pattern and let* $r \geq 0$ *be an integer. Then the following are equivalent:*

(1) $\mathcal{A}$ *requires rank* r*.*

(2) For some integer $0 \le k \le r$*,* $\mathcal{A}$ *is permutationally equivalent to a ray pattern*

$$\begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & 0 \end{pmatrix}, \quad \textit{where} \quad \mathcal{A}_{11} \quad \textit{is } k\textit{–by–}(r-k),$$

$\mathcal{A}_{12}$ *requires rank* k*, and* $\mathcal{A}_{21}$ *requires rank* $r-k$*.*

(3) For some integer $0 \le k \le r$*,* $\mathcal{A}$ *is a ray pattern as in* (2) *above satisfying the following: every row (resp. column) scaling of* $\mathcal{A}_{12}$ *(resp.* $\mathcal{A}_{21}$*) by a complex signature has a column (resp. row) whose relative interior of the convex hull of the nonzero entries does not contain the origin.*

We continue with a few comments on the sparsity of ray–nonsingular patterns. As it is easily checked, every sign–nonsingular 3–by–3 pattern must have at least one zero entry. In [4] and in subsequent papers (e.g., [2]) the maximum number of nonzero entries allowable in a sign–nonsingular pattern, as well as the properties of the maximal, in this sense, sign patterns are studied. In the case of ray–nonsingularity the situation is, in general, different. It is evident from Example 3.2 that 3–by–3 ray–nonsingular patterns can have all entries nonzero. The following is an interesting full 4–by–4 example.

Example 3.10 The signed transversal products of

$$A = \begin{pmatrix} i & 1 & 1 & 1 \\ 1 & i & 1 & 1 \\ 1 & 1 & i & 1 \\ 1 & 1 & 1 & i \end{pmatrix}$$

are equal to $1, -1$, and i. Hence, by Theorem 3.1, $\mathcal{A}$ is ray–nonsingular.

It is natural to ask whether the above example can be generalized to $n \ge 5$. However, it can be shown that there is no n–by–n, $n \ge 5$, ray–nonsingular matrix all of whose off–diagonal entries are equal to 1. Indeed, suppose that A is of the form

$$A = \begin{pmatrix} e^{i\theta_1} & 1 & \dots & 1 \\ 1 & e^{i\theta_2} & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & e^{i\theta_n} \end{pmatrix},$$

and that $D = \mathrm{diag}(e^{i\alpha_1}, e^{i\alpha_2}, \dots, e^{i\alpha_n})$. If we choose $\alpha_j = j\ (\frac{2\pi}{n})$ for $j = 1, 2, \dots, n$, then the origin belongs to the relative interior of the convex hull of the entries of every column of DA. (For example, let $n = 5$. Then for the first column,

$$0 \in \mathrm{ri\ conv}\ \{e^{i(\theta_1 + 2\pi/5)},\ e^{i4\pi/5},\ e^{i6\pi/5},\ e^{i8\pi/5},\ 1\}$$

for all $\theta_1 \in (-\pi, \pi]$.) Hence, by Theorem 3.5, $\mathcal{A}$ is not ray–nonsingular.

The maximum number of nonzero entries allowable in an n–by–n ray–nonsingular matrix is currently unknown for $n \geq 5$.

4 Range of the Determinant of a Ray Pattern

Now we study the range of the determinant as a function with domain the set of all matrices of a given ray pattern. The *range of the determinant* of a ray pattern $\mathcal{A}$ is formally defined and denoted by $\mathcal{R}(\mathcal{A}) = \{\det A \mid A \in \mathcal{A}\}$. Next we introduce a combinatorial notion to help us study $\mathcal{R}(\mathcal{A})$.

Definition 4.1 Let $A \in M_n(\mathbf{C})$. We say that $\tau = \{t_1, t_2, \dots t_r\} \subseteq \tau(A)$ is an *isolated set of transversals* if

(a) every transversal in $\tau(A) \setminus \tau$ contains an entry of A that is not in any of the transversals in τ, and

(b) every $t_j \in \tau$ contains an entry that is not in any of the transversals in $\tau \setminus \{t_j\}$.

Note that any proper subset of an isolated set τ is isolated, and that $M(\tau)$ contains exactly r transversals.

Theorem 4.2 *Let $A \in M_n(\mathbf{C})$ and τ be a set of r transversals of A. If τ is isolated, then the following hold:*

(1) For every matrix $\hat{M}$ that is permutation equivalent to $M(\tau)$ and has nonzero diagonal entries, $G(\hat{M})$ contains exactly $r - 1$ cycles of length ≥ 2 and every pair of these cycles has a common vertex.

(2) $M(\tau)$ is permutation equivalent to a matrix with nonzero diagonal entries such that its irreducible components, except possibly one, are 1–by–1.

Proof.

(1): Suppose P and Q are permutation matrices such that $\hat{M} = PM(\tau)Q$ has nonzero diagonal entries. Clearly, the transversals in τ correspond to an isolated set τ' of r transversals of $\hat{M}$. The diagonal entries of $\hat{M}$ constitute a transversal, $t_d \in \tau'$. There are $r-1$ transversals of $\hat{M}$ left to account for. Suppose that the cyclic decomposition of some $t \in \tau'$ consists of more than one cycle of length ≥ 2. Let $\hat{t} \in \tau'$ be formed by replacing one of these cycles with the corresponding diagonal entries. Then every element of $\hat{t}$ is either in t_d or t, contradicting that τ' is isolated. Thus every transversal in τ' except t_d consists of a cycle of length ≥ 2 and the complementary diagonal entries. It follows that any pair of cycles of length ≥ 2 in $G(\hat{M})$ must have a vertex in common, otherwise we can form a transversal of $\hat{M}$ consisting of the entries corresponding to these two cycles and of the complementary diagonal entries.

(2): For any $t \in \tau$, notice that $P = M(t)^{-1}$ is a permutation matrix and that $PM(\tau)$ has nonzero diagonal entries. Hence by (1), any pair of cycles of length ≥ 2 in $G(PM(\tau))$ has a vertex in common. It follows that $PM(\tau)$ has at most one irreducible component larger than 1–by–1. ■

The converse of Theorem 4.2 is not in general true, as can be seen by considering a full 3×3 matrix A, and $\tau = \tau(A)$. Thus $r = 6$, $G(A)$ contains 5 cycles of length 2 or 3, and (1) is satisfied. Also, as each matrix that is permutation equivalent to A is irreducible, (2) is satisfied. However, τ is not isolated as part (b) of Definition 4.1 fails.

In the results that follow we are primarily concerned with isolated sets of two, three or four transversals, hence the following characterizations are useful.

Theorem 4.3 *Let $A \in M_n(\mathbf{C})$ and $\tau = \{t_1, t_2, \ldots, t_r\}$ be a set of transversals of A. If $r \leq 3$ then the following are equivalent:*

(1) The set τ is isolated.

(2) For every matrix $\hat{M}$ that is permutation equivalent to $M(\tau)$ and has nonzero diagonal entries, $G(\hat{M})$ contains exactly $r-1$ cycles of length ≥ 2 and (if $r = 3$) these cycles have a common vertex.

(3) $G(M(t_k)^{-1}M(\tau))$, for some $1 \leq k \leq r$, contains exactly $r-1$ cycles of length ≥ 2 and (if $r = 3$) these cycles have a common vertex.

(4) The matrix $M(\tau)$ contains exactly r transversals.

Proof. If $r = 1$ the equivalences are obvious.

(1) implies (2): Follows from (1) of Theorem 4.2.

(2) implies (3): Follows from the first sentence of the proof of (2) of Theorem 4.2.

(3) implies (4): The union of any pair of cycles of length ≥ 2 in $G(M(t_k)^{-1}M(\tau))$ contains a vertex that is the terminal vertex of two arcs. Hence each transversal can contain at most one cycle of length ≥ 2. Thus the transversals of $M(t_k)^{-1}M(\tau)$ consist of the transversal with n diagonal entries, and of transversals formed from any cycles of length ≥ 2 complemented by diagonal entries. Thus there are exactly r of them in $M(\tau)$.

(4) implies (1): Since $M(\tau)$ contains exactly r transversals, every transversal not in τ contains an entry not in any of the transversals in τ, hence condition (a) of Definition 4.1 is satisfied. If $r = 2$, (b) of Definition 4.1 follows easily. Consider the case where $r = 3$. Suppose that condition (b) of Definition 4.1 does not hold. We can assume without loss of generality that each element of t_1 is contained in either t_2 or t_3. Notice that every row and every column of $M(\tau)$ contains either 1 or 2 nonzero entries. Consider the set t created by choosing the nonzero entry from each of the rows with only one nonzero entry, and the nonzero entry not in t_1 from each of the rows with two nonzero entries. We claim that t is a transversal of $M(\tau)$. For if not, suppose that for some k, column k of $M(t)$ has two nonzero entries. One nonzero entry belongs only to t_2 and the other only to t_3. Hence the rows corresponding to the nonzero entries in column k of $M(\tau)$ must contain two nonzero entries; by the construction of t, the nonzero entries in each of these rows that are not in column k must belong to t_1. But then there is no entry from column k in t_1, contradicting that t_1 is a transversal. Thus, by the pigeon hole principle, t also has exactly one element from every column and hence it is a transversal of $M(\tau)$. By the construction of t, t cannot be equal to t_1. If the elements of t coincide with those of t_2 then it must be that $t_1 = t_3$, contradicting that there are exactly three transversals in $M(\tau)$. Hence $t \neq t_2$. Similarly $t \neq t_3$. Thus t must be distinct from t_1, t_2, and t_3. This contradicts that there are exactly three transversals in $M(\tau)$. Hence condition (b) of Definition 4.1 must hold and (1) follows. ■

To show that the restriction of $r \leq 3$ is required in order for Theorem 4.3 to hold, we let A be as in Example 3.4, and $\tau = \tau(A)$. Then $M(\tau)$ consists of exactly 4 transversals, however τ is not isolated since part (b) of Definition 4.1 is not satisfied.

Lemma 4.4 *Let* $z_1, z_2, \ldots, z_r \in \mathbf{C}$ *with* $r \geq 2$. *Then there exist* $j_1, j_2, \ldots, j_m \in \{1, 2, \ldots, r\}$ *with* $m \leq 4$ *such that*

$$K = \mathit{cone}\ \{z_1, z_2, \ldots, z_r\} = \mathit{cone}\ \{z_{j_1}, z_{j_2}, \ldots, z_{j_m}\}.$$

In particular, if $K \neq \mathbf{C}$, *then* $m \leq 3$.

Proof. If $K \neq \mathbf{C}$, then one of the following holds:

K is a half–line and thus $K = \text{cone}\,\{z_j\}$ for some $j \in \{1, 2, \ldots, r\}$,
K is a line through 0 and thus $K = \text{cone}\,\{z_j, z_k\}$ for some $j, k \in \{1, 2, \ldots, r\}$,
K is properly contained in a half–space and thus $K = \text{cone}\,\{z_j, z_k\}$ for some $j, k \in \{1, 2, \ldots, r\}$, or
K is a closed half–space and thus $K = \text{cone}\,\{z_j, z_k, z_\ell\}$ for some $j, k, \ell \in \{1, 2, \ldots, r\}$.

If $K = \mathbf{C}$, then the proof proceeds by induction. Clearly, the result is true for $r \leq 4$. Suppose it is true for all positive integers $\leq r-1$. Let $K' = \text{cone}\,\{z_1, z_2, \ldots, z_{r-1}\}$. If $K' \neq \mathbf{C}$, then by the analysis above $K' = \text{cone}\,\{z_{j_1}, z_{j_2}, \ldots, z_{j_m}\}$ with $j_1, j_2, \ldots, j_m \in \{1, 2, \ldots, r-1\}$ and $m \leq 3$. Hence $K = \text{cone}\,\{z_{j_1}, z_{j_2}, \ldots, z_{j_m}, z_r\}$. If $K' = \mathbf{C}$, then $K = K' = \text{cone}\,\{z_1, z_2, \ldots, z_{r-1}\}$ and thus by the inductive assumption the result is true. ∎

Lemma 4.5 *Let $\mathcal{A}$ be a ray pattern such that there is an $A \in \mathcal{A}$ with $detA = \alpha$. Then for any $\beta > 0$, there is a matrix $B \in \mathcal{A}$ such that $detB = \beta\alpha$.*

Proof. Form the matrix B by multiplying the first row of A by β. ∎

Theorem 4.6 *Let $\mathcal{A} \in M_n(\mathbf{C})$ be a ray pattern and let $\tau = \{t_1, t_2, \ldots, t_r\}$ be an isolated set of transversals of $\mathcal{A}$, with $T = \{T_1, T_2, \ldots, T_r\}$ the corresponding signed transversal products. Then int cone $T \subseteq \mathcal{R}(\mathcal{A})$.*

Proof. If all the signed transversal products in T lie on a line through 0, then int cone T is empty and the result follows trivially. Otherwise, by Lemma 4.4, without loss of generality we can assume that $2 \leq r \leq 4$. We can also assume that $T_1, T_2, \ldots, T_r$ are distinct. Suppose $T(\mathcal{A}) = \{T_1, T_2, \ldots, T_r, T_{r+1}, \ldots, T_k\}$ and consider the terms in the standard expansion of $\det(X \circ \mathcal{A})$, where X is a matrix with (variable) positive entries. Since τ is isolated we can make the following selections. For each $j > r$, select p_j, q_j so that $x_{p_j q_j}$ is a factor in the term with coefficient T_j, but not in the terms with transversal products in T as coefficients. Set $x_{p_j q_j} = \delta$. For each $j \leq r$, select p_j, q_j such that $x_{p_j q_j}$ is a factor of the term with coefficient T_j but not a factor of any other term with a coefficient in T. Set $x_{p_1, q_1} = \alpha_1$ and for $1 < j < r$, set $x_{p_j, q_j} = \alpha_j \Pi_{l=1}^{j-1}(1 - \alpha_l)$. Also set $x_{p_r, q_r} = \Pi_{l=1}^{r-1}(1 - \alpha_l)$. Set all other entries of X equal to 1. Let $f(\delta, \alpha_1, \ldots, \alpha_{r-1})$ equal $\det(X \circ A)$ with the above substitutions. If $r = 2$, then

$$f(\delta, \alpha_1) = \delta\; g(\delta, \alpha_1) + \alpha_1 T_1 + (1 - \alpha_1) T_2,$$

where $g(\delta, \alpha_1)$ is a complex polynomial, which is bounded for $\delta, \alpha_1 \in [0,1]$. For small fixed δ, as α_1 varies from 0 to 1, $f(\delta, \alpha_1)$ traces out a continuous connected curve that approximates the line from T_2 to T_1. The result now follows from Lemma 4.5.

If $r = 3$, then

$$f(\delta, \alpha_1, \alpha_2) = \delta \; g(\delta, \alpha_1, \alpha_2) + \alpha_1 T_1 + (1-\alpha_1)\alpha_2 T_2 + (1-\alpha_1)(1-\alpha_2)T_3.$$

For any fixed small positive value of δ define

$$F_\delta(s,t) = \begin{cases} f(\delta, 0, 4ts) & 0 \le t \le \frac{1}{4} \\ f(\delta, (4t-1)s, s) & \frac{1}{4} < t \le \frac{1}{2} \\ f(\delta, s, (3-4t)s) & \frac{1}{2} < t \le \frac{3}{4} \\ f(\delta, (4-4t)s, 0) & \frac{3}{4} < t \le 1. \end{cases}$$

Notice that for small values of δ and as t varies from 0 to 1, $F_\delta(1,t)$ is a continuous connected curve that approximates the line segments from T_3 to T_2, T_2 to T_1, and T_1 to T_3. Hence by Lemma 4.5, all nonzero points in int cone T are in $\mathcal{R}(\mathcal{A})$. If $0 \in$ int cone T, then notice that for sufficiently small δ, $F_\delta(s,t)$ is a continuous retraction of the closed curve $F_\delta(1,t)$, which has zero in its interior, to the point $F_\delta(0,t) \approx T_3$. Thus we can conclude that 0 is an interior point of the range of g (see for example [8, Chapter 8]). Hence 0 is in the range of f and thus $0 \in \mathcal{R}(\mathcal{A})$.

If $r = 4$ then

$$f(\delta, \alpha_1, \alpha_2, \alpha_3) =$$

$$\delta \; g(\delta, \alpha_1, \alpha_2) + \alpha_1 T_1 + (1-\alpha_1)\alpha_2 T_2 + (1-\alpha_1)(1-\alpha_2)\alpha_3 T_3 + (1-\alpha_1)(1-\alpha_2)(1-\alpha_3)T_4.$$

We can assume, without loss of generality, that T_1, T_2, T_3, T_4 are ordered in a clockwise fashion. Analogous to the above, for any fixed small positive value of δ define

$$F_\delta(s,t) = \begin{cases} f(\delta, 0, 0, 5ts) & 0 \le t \le \frac{1}{5} \\ f(\delta, 0, (5t-1)s, s) & \frac{1}{5} < t \le \frac{2}{5} \\ f(\delta, (5t-2)s, s, s) & \frac{2}{5} < t \le \frac{3}{5} \\ f(\delta, s, (4-5t)s, (4-5t)s) & \frac{3}{5} < t \le \frac{4}{5} \\ f(\delta, (5-5t)s, 0, 0) & \frac{4}{5} < t \le 1. \end{cases}$$

Notice that for small values of δ and as t varies from 0 to 1, $F_\delta(1,t)$ is a continuous connected curve that approximates the line segments from T_4 to T_3, T_3 to T_2, T_2 to T_1, and T_1 to T_4. Hence by Lemma 4.5, all nonzero points in int cone T are in $\mathcal{R}(\mathcal{A})$. If $0 \in$ int cone T, then notice that for sufficiently small δ, $F_\delta(s,t)$ is a continuous retraction of the closed curve $F_\delta(1,t)$, which has zero in its interior, to the point $F_\delta(0,t) \approx T_4$. Thus, as before, we can conclude that $0 \in \mathcal{R}(\mathcal{A})$. ∎

Example 4.7 Consider the matrix A of Example 3.4. Then, if X has positive entries,

$$det(X \circ A) = (1)x_{11}x_{22}x_{33}x_{44} + (e^{-i\frac{3\pi}{4}})x_{11}x_{22}x_{34}x_{43}$$
$$+(e^{i\frac{3\pi}{4}})x_{12}x_{21}x_{33}x_{44} + (1)x_{12}x_{21}x_{34}x_{43}.$$

Since A is block diagonal it is easy to see that

$$\mathcal{R}(\mathcal{A}) = \{\alpha e^{i\theta} \mid \alpha > 0,\ -\frac{3\pi}{4} < \theta < \frac{3\pi}{4}\}.$$

The signed transversal products of A are $T_1 = 1, T_2 = e^{-i\frac{3\pi}{4}}, T_3 = e^{i\frac{3\pi}{4}}$, and $T_4 = 1$, corresponding to transversals t_1, t_2, t_3, t_4, respectively. Notice that $\{t_1, t_2\}$ and $\{t_3, t_4\}$ are isolated. By Theorem 4.6, int cone $\{T_1, T_2\} \subseteq \mathcal{R}(\mathcal{A})$ and int cone $\{T_3, T_4\} \subseteq \mathcal{R}(\mathcal{A})$. However, $\{t_2, t_3\}$ is not isolated and in fact

$$\text{int cone } \{T_2, T_3\} \cap \mathcal{R}(\mathcal{A}) = \emptyset.$$

The following corollary provides a connection between ray–nonsingularity and isolated sets of transversals.

Corollary 4.8 *Let $\mathcal{A}$ be a ray pattern with an isolated set τ of ≥ 2 transversals, and let T be the corresponding set of signed transversal products of $\mathcal{A}$. If $0 \in$ int cone T, then $\mathcal{A}$ is not ray–nonsingular.*

As an application of Corollary 4.8 let $A \in \mathcal{A}$, where $\mathcal{A}$ is as in Example 3.3. Then

$$\{\{a_{12}, a_{23}, a_{31}\}, \{a_{13}, a_{21}, a_{32}\}, \{a_{11}, a_{23}, a_{32}\}\}$$

is an isolated set of transversals. The corresponding signed transversal products are equal to $e^{-i\frac{3\pi}{4}}, e^{i\frac{3\pi}{4}}$, and 1, respectively. Hence by Corollary 4.8 $\mathcal{A}$ is not ray–nonsingular.

Acknowledgment

The concept of extending sign–nonsingularity to complex matrices was discussed by some of the authors with C. R. Johnson. We also thank B. L. Shader for providing an example showing that the converse of Theorem 3.1 is false, and a referee for constructive comments.

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