Bounds for Levinger's Function of Nonnegative Almost Skew-symmetric Matrices[1]

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Abstract

The analysis of the Perron eigenspace of a nonnegative matrix A whose symmetric part has rank one is continued. Improved bounds for the Perron root of Levinger's transformation $(1-\alpha)A + \alpha A^t$ $(\alpha \in [0, 1])$ and its derivative are obtained. The relative geometry of the corresponding left and right Perron vectors is examined. The results are applied to tournament matrices to obtain a comparison result for their spectral radii.

Keywords: Perron root, Perron vector, Levinger's function, tournament.

AMS Subject Classifications: 15A18, 15A42, 15A60, 05C20.

1 Introduction

Our main goal is to study the spectrum and, in particular, the spectral radius of an entrywise nonnegative matrix whose symmetric part has rank one. We refer to such a matrix A as "nonnegative almost skew-symmetric". Our motivation lies in the fundamental nature of this problem within the realm of nonnegative matrix theory, as well as the relation and applications of such matrices to the theory of tournaments; see [6, 8, 9, 11]. A key role in these papers was played by Levinger's transformation, *i.e.,* $(1-\alpha)A + \alpha A^t$ $(\alpha \in [0, 1])$.

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This transformation has also proven useful in answering fundamental questions about the geometry of the numerical range of a (general or nonnegative) matrix [10].

In Sections 3 and 4, we continue the analysis of the Perron eigenspace of a nonnegative almost skew-symmetric matrix started in [11] and obtain new bounds for the Perron root of Levinger's transformation and its derivative. These concepts were first systematically studied by Fiedler [4]. Section 5 comprises an illustrative example. The association to tournament matrices, relevant definitions and some observations relating to their spectral radii and the Brualdi-Li conjecture are presented in Section 6.

2 Preliminaries

Let $x, y \in \mathbb{R}^n$ be two real vectors. We call x a *unit* vector if its Euclidean norm is $\|x\|_2 = 1$. The angle between x and y is defined by

$$(\widehat{x,y}) \;=\; \cos^{-1}\left(\frac{x^t y}{\|x\|_2\|y\|_2}\right) \;\in\; [0,\, \pi].$$

Consider an $n \times n$ real matrix A and denote its *spectrum* by $\sigma(A)$ and its *spectral radius* by $\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$.

For every real square matrix A, we write $A \,=\, S(A) + K(A)$, where

$$S(A) \;=\; \frac{A + A^t}{2} \quad \text{and} \quad K(A) \;=\; \frac{A - A^t}{2}$$

are the (real) *symmetric part* and the (real) *skew-symmetric part* of A, respectively.

Given any square matrix A, we also consider *Levinger's transformation*,

$$\mathcal{L}(A, \alpha) \;=\; (1 - \alpha)A + \alpha A^t\,; \quad \alpha \in [0,\, 1],$$

as well as *Levinger's function*,

$$\phi(A, \alpha) \;=\; \rho(\mathcal{L}(A, \alpha)) \;=\; \rho((1 - \alpha)A + \alpha A^t)\,; \quad \alpha \in [0,\, 1]. \tag{1}$$

Note that as Levinger's function is symmetric about $\alpha \,=\, 1/2,$ without loss of generality, we will hereon restrict our attention to $\alpha \in [0,\, 1/2]$. One can also readily see that for every $\alpha \in [0,\, 1/2]$,

$$\mathcal{L}(A, \alpha) \;=\; S(A) + (1 - 2\alpha)K(A).$$

Given an (entrywise) nonnegative square matrix A, by the Perron-Frobenius theory, we know that $\rho(A)$ is an eigenvalue of A, to which we shall refer as the *Perron root* of A. Corresponding to $\rho(A)$ are nonnegative unit right and left eigenvectors referred to as *Perron vectors* and denoted, respectively, by $x_r(A)$ and $x_l(A)$.

Continuing to assume that A is an $n \times n$ nonnegative matrix, we of course have that $S(A)$ is also nonnegative. Let us further assume that $\operatorname{rank} S(A) = 1$; *i.e.,* there exists a nonzero nonnegative vector $w \in \mathbb{R}^n$ such that $S(A) = w\, w^t$. This means that $\sigma(S(A))$ consists of the eigenvalue 0 with multiplicity $n-1$ and a simple positive eigenvalue $\delta(A) = w^t w$. We refer to a matrix A having all of the features in this paragraph as a *nonnegative almost skew-symmetric matrix*. We will from now on presume that the notation associated with such a matrix A is readily recalled.

Note that when A is an (irreducible) nonnegative almost skew-symmetric matrix, all the matrices $\mathcal{L}(A,\alpha)$ ($\alpha \in [0,\, 1/2]$) are also (irreducible) nonnegative almost skew-symmetric, having the same symmetric part, $S(A)$. Following the terminology in [9, 11], we define the *variance* of A by

$$\operatorname{var}(A) \;=\; \frac{\|K(A)w\|_2^2}{\|w\|_2^2} \;=\; \frac{w^t(K(A)^t K(A))w}{w^t w}$$

and observe that the variance of $\mathcal{L}(A,\alpha)$ is

$$\operatorname{var}(\mathcal{L}(A,\alpha)) \;=\; (1-2\alpha)^2 \operatorname{var}(A).$$

Furthermore,

$$\phi(A,0) \;=\; \rho(A)\;, \quad \phi(A,1/2) \;=\; \rho(S(A)) \;=\; \delta(A)$$

and

$$x_r(\mathcal{L}(A,1/2)) \;=\; x_l(\mathcal{L}(A,1/2)) \;=\; x_r(S(A)) \;=\; x_l(S(A)) \;=\; w/\|w\|_2.$$

Next, we summarize some results in [9, 11] needed in our discussion.

Theorem 1 *Consider an $n \times n$ nonnegative almost skew-symmetric matrix A with irreducible symmetric part. Then Levinger's function $\phi(A,\alpha)$ defined in* (1) *and the corresponding unit Perron vectors $x_r(\alpha) = x_r(\mathcal{L}(A,\alpha))$ and $x_l(\alpha) = x_l(\mathcal{L}(A,\alpha))$ satisfy the following:*

(a) *For every $\alpha \in [0,\, 1/2]$ such that $\delta(A)^2 > 4(1-2\alpha)^2 \operatorname{var}(A)$,*

$$\phi(A,\alpha) \;\geq\; \frac{\delta(A) + \sqrt{\delta(A)^2 - 4(1-2\alpha)^2 \operatorname{var}(A)}}{2}\,.$$

(b) *For every $\alpha \in (0,\, 1/2]$ such that $\delta(A)^2 > 4(1-2\alpha)^2 \operatorname{var}(A)$, the cosine of the angle $(\widehat{w, x_r(\alpha)}) = (\widehat{w, x_l(\alpha)})$ is greater than or equal to the quantity*

$$\sqrt{\frac{1}{2} + \sqrt{\frac{\delta(A)^2 - 4(1-2\alpha)^2 \operatorname{var}(A)}{4\delta(A)^2}}}\,.$$

(c) *If we let* $s_1 = \|K(A)\|_2 = \rho(K(A))$, *then for every* $\alpha \in (0,\, 1/2)$ *such that* $\delta(A)^2 > 4(1-2\alpha)^2 \mathrm{var}(A)$,

$$0 \,\le\, \phi^{'}(A,\alpha) \,\le\, \frac{4\, s_1\,(1-2\alpha)\delta(A)\sqrt{\mathrm{var}(A)}}{\delta(A)^2 - 4(1-2\alpha)^2\mathrm{var}(A)}\,.$$

(d) *If* $\delta(A)^2 > 4\mathrm{var}(A)$, *then for every* $\alpha \in [0,\, 1/2)$,

$$\phi(A,\alpha) \,\le\, \rho(A) + \frac{s_1 \delta(A)}{4\sqrt{\mathrm{var}(A)}}\ \ln\left(\frac{\delta(A)^2 - 4(1-2\alpha)^2\mathrm{var}(A)}{\delta(A)^2 - 4\mathrm{var}(A)}\right).$$

Finally in this section, we recall a useful inequality of Rojo, Soto and Rojo [12].

Theorem 2 *For any eigenvalue* λ *of an* $n \times n$ *real matrix* A,

$$\left|\mathrm{Re}\lambda - \frac{\mathrm{trace}(A)}{n}\right| \,\le\, \sqrt{\frac{n-1}{n}\left(\|S(A)\|_2^2 - \frac{\|AA^t - A^tA\|_F^2}{12\,\|A\|_F^2} - \frac{\mathrm{trace}(A)^2}{n}\right)},$$

where $\|\cdot\|_F$ *denotes the Frobenius norm.*

3 Perron roots and Perron vectors

Let A be an $n\times n$ nonnegative almost skew-symmetric matrix with symmetric part $S(A) = w\,w^t$, skew-symmetric part $K(A)$ and variance $\mathrm{var}(A)$.

The quantity $\|AA^t - A^tA\|_F$ is known as the *(Frobenius) distance to normality* of A. Next we obtain a formula for this distance that is necessary for the remainder and of independent interest.

Proposition 3 *The distance to normality of an* $n\times n$ *nonnegative almost skew-symmetric matrix* A *is*

$$\|AA^t - A^tA\|_F \,=\, 2\delta(A)\sqrt{2\mathrm{var}(A)}.$$

Proof Let $y = w/\|w\|_2 \in \mathbb{R}^n$ be the unit eigenvector of $S(A) = ww^t$ corresponding to $\delta(A)$. Then there is an $n \times n$ real unitary matrix V, whose first column is y, such that

$$V^tS(A)V \,=\, \mathrm{diag}\{\delta(A), 0, \ldots, 0\}.$$

Moreover, since $V^tK(A)V$ is real skew-symmetric, it follows

$$V^tK(A)V \,=\, \left[\begin{array}{cc} 0 & -u^t \\ u & K_1 \end{array}\right],$$

where K_1 is $(n-1)\times(n-1)$ real skew-symmetric and $u \in \mathbb{R}^{n-1}$.

Observe now that

$$\begin{aligned} AA^t - A^tA &= (S(A)+K(A))(S(A)-K(A)) - (S(A)-K(A))(S(A)+K(A)) \\ &= 2\,(K(A)S(A) - S(A)K(A)), \end{aligned}$$

and since the Frobenius norm is invariant under unitary similarities,

$$\begin{aligned} \|AA^t - A^tA\|_F^2 &= 4\left\|V^tK(A)VV^tS(A)V - V^tS(A)VV^tK(A)V\right\|_F^2 \\ &= 4\left\|\begin{bmatrix} 0 & -u^t \\ u & K_1 \end{bmatrix}\begin{bmatrix} \delta(A) & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \delta(A) & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & -u^t \\ u & K_1 \end{bmatrix}\right\|_F^2 \\ &= 4\left\|\begin{bmatrix} 0 & 0 \\ \delta(A)u & 0 \end{bmatrix} + \begin{bmatrix} 0 & \delta(A)u^t \\ 0 & 0 \end{bmatrix}\right\|_F^2 \\ &= 4\delta(A)^2\left\|\begin{bmatrix} 0 & u^t \\ u & 0 \end{bmatrix}\right\|_F^2 \\ &= 8\delta(A)^2\|u\|_2^2. \end{aligned}$$

Denoting by e_1 the first standard basis vector in $\mathbb{R}^n$, we have

$$\begin{aligned} \|u\|_2^2 &= \left\|\begin{bmatrix} 0 \\ u \end{bmatrix}\right\|_2^2 = \left\|V\begin{bmatrix} 0 & -u^t \\ u & K_1 \end{bmatrix}e_1\right\|_2^2 \\ &= \left\|V\begin{bmatrix} 0 & -u^t \\ u & K_1 \end{bmatrix}V^tVe_1\right\|_2^2 \\ &= \|K(A)y\|_2^2 \;=\; \mathrm{var}(A). \end{aligned}$$

As a consequence, $\|AA^t - A^tA\|_F^2 = 8\delta(A)^2\mathrm{var}(A)$. $\square$

Corollary 4 *An $n\times n$ nonnegative almost skew-symmetric matrix A is normal if and only if* $\mathrm{var}(A) = 0$.

The Frobenius norm of A may be written as

$$\|A\|_F^2 \;=\; \|S(A)+K(A)\|_F^2 \;=\; \|S(A)\|_F^2 + \|K(A)\|_F^2,$$

and one can verify that

$$\|A\|_F^2 \;=\; \delta(A)^2 + \|K(A)\|_F^2.$$

Then Theorem 2 yields the following result.

Theorem 5 *The spectral radius of an $n\times n$ nonnegative almost skew-symmetric matrix A satisfies*

$$\rho(A) \;\le\; \delta(A)\left(\frac{1}{n} + \sqrt{\frac{n-1}{n}\left(\frac{n-1}{n} - \frac{2\,\mathrm{var}(A)}{3(\delta(A)^2 + \|K(A)\|_F^2)}\right)}\right). \tag{2}$$

Proof By Proposition 3, we have

$$\|AA^t - A^tA\|_F^2 \;=\; 8\delta(A)^2 \mathrm{var}(A).$$

Since $\|A\|_F^2 = \delta(A)^2 + \|K(A)\|_F^2$ and $\mathrm{trace}(A) = \mathrm{trace}(S(A)) = \delta(A) = \|S(A)\|_2 = \|w\|_2^2$, the proof is complete by Theorem 2. $\square$

Notice that if $\mathrm{var}(A) \to 0$, the upper bound of the spectral radius $\rho(A)$ in (2) approaches $\delta(A)$, which is the maximum possible value for $\rho(A)$. Furthermore, for any $\alpha \in [0,\, 1/2]$, $\|\mathcal{L}(A,\alpha)\|_F^2 = \delta(A)^2 + (1-2\alpha)^2\|K(A)\|_F^2$.

Corollary 6 *Let A be an $n\times n$ nonnegative almost skew-symmetric matrix. Then for every $\alpha \in [0,\, 1/2]$,*

$$\phi(A,\alpha) \;\le\; \delta(A)\left(\frac{1}{n} + \sqrt{\frac{n-1}{n}\left(\frac{n-1}{n} - \frac{2\,(1-2\alpha)^2\mathrm{var}(A)}{3\,(\delta(A)^2 + (1-2\alpha)^2\|K(A)\|_F^2)}\right)}\right).$$

Equality holds when $\alpha = 1/2$.

Next is a comparison result for the spectral radii of nonnegative almost skew-symmetric matrices (to be cited in Section 6).

Proposition 7 *Let A and B be $n\times n$ nonnegative almost skew-symmetric matrices with variances* $\mathrm{var}(A)$ *and* $\mathrm{var}(B)$, *and assume that $\delta(A) = \delta(B)$. If $\delta(B)^2 > 4\mathrm{var}(B)$ and if*

$$\frac{1}{n} + \sqrt{\frac{n-1}{n}\left(\frac{n-1}{n} - \frac{2\,\mathrm{var}(A)}{3\,\|A\|_F^2}\right)} \;<\; \frac{1+\sqrt{1-\frac{4\mathrm{var}(B)}{\delta(B)^2}}}{2}\,, \tag{3}$$

then $\rho(A) < \rho(B)$.

Proof It follows readily from Theorem 1 (a) and Theorem 5. $\square$

By the results in [11], we know that if the symmetric part of A is irreducible, then for every $\alpha \in [0,\, 1/2]$, the right and left (unit) Perron vectors $x_r(\alpha)$ and $x_l(\alpha)$ of $\mathcal{L}(A,\alpha)$ have the same orthogonal projection onto the vector w (see also Theorem 1 (b)) and satisfy

$$\phi(A,\alpha) \;=\; \delta(A)\left(\frac{w^t x_r(\alpha)}{\|w\|_2}\right)^2 \;=\; \delta(A)\left(\frac{w^t x_l(\alpha)}{\|w\|_2}\right)^2. \tag{4}$$

Note that all the bounds in Proposition 8 and in Corollaries 9 and 10 below approach 1 as $\alpha \to 1/2$.

Proposition 8 *Let A be an $n \times n$ nonnegative almost skew-symmetric matrix. If $S(A)$ is irreducible, then for every $\alpha \in [0,\, 1/2]$, the cosine of the angle $(\widehat{w, x_r(\alpha)}) = (\widehat{w, x_l(\alpha)})$ is less than or equal to*

$$\sqrt{\frac{1}{n} + \sqrt{\frac{n-1}{n}\left(\frac{n-1}{n} - \frac{2\,(1-2\alpha)^2 \mathrm{var}(A)}{3\,(\delta(A)^2 + (1-2\alpha)^2 \|K(A)\|_F^2)}\right)}}\,.$$

Proof By Corollary 6, for every $\alpha \in [0,\, 1/2]$,

$$\phi(A,\alpha) \;\le\; \delta(A)\left(\frac{1}{n} + \sqrt{\frac{n-1}{n}\left(\frac{n-1}{n} - \frac{2\,(1-2\alpha)^2 \mathrm{var}(A)}{3\,(\delta(A)^2 + (1-2\alpha)^2 \|K(A)\|_F^2)}\right)}\right).$$

Hence, by (4),

$$\frac{w^t x_r(\alpha)}{\|w\|_2} \;\le\; \sqrt{\frac{1}{n} + \sqrt{\frac{n-1}{n}\left(\frac{n-1}{n} - \frac{2\,(1-2\alpha)^2 \mathrm{var}(A)}{3\,(\delta(A)^2 + (1-2\alpha)^2 \|K(A)\|_F^2)}\right)}}\,.$$

The Perron vector $x_r(\alpha)$ is a unit vector and the proof is complete. $\square$

For a nonnegative matrix A, by results found *e.g.,* in [5, 10],

$$\|\mathcal{L}(A,\alpha)\|_F \;\le\; 2\sqrt{n}\, \max\{|x^* A x| : x \in \mathbb{C}^n,\ \|x\|_2 = 1\} \;=\; 2\sqrt{n}\, \delta(A).$$

Thus, the above proposition and Theorem 1 (b) imply the following corollaries.

Corollary 9 *Let A be an $n \times n$ nonnegative almost skew-symmetric matrix. If $S(A)$ is irreducible, then for every $\alpha \in [0,\, 1/2]$, the cosine of the angle $(\widehat{w, x_r(\alpha)}) = (\widehat{w, x_l(\alpha)})$ is less than or equal to*

$$\sqrt{\frac{1}{n} + \sqrt{\frac{n-1}{n}\left(\frac{n-1}{n} - \frac{(1-2\alpha)^4 \mathrm{var}(A)}{6\, n\, \delta(A)^2}\right)}}\,.$$

Corollary 10 *Let A be an $n \times n$ nonnegative almost skew-symmetric matrix. If $S(A)$ is irreducible, then for every $\alpha \in [0,\, 1/2]$ such that $\delta(A)^2 > 4(1-2\alpha)^2 \mathrm{var}(A)$, the cosine of the angle $(\widehat{w, x_r(\alpha)}) = (\widehat{w, x_l(\alpha)})$ lies in the interval*

$$\left[\sqrt{\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{(1-2\alpha)^2 \mathrm{var}(A)}{\delta(A)^2}}}\,,\quad \sqrt{\frac{1}{n} + \sqrt{\left(\frac{n-1}{n}\right)^2 - \frac{(n-1)(1-2\alpha)^4 \mathrm{var}(A)}{6\, n^2 \delta(A)^2}}}\,\right].$$

4 Bounds for Levinger's function and its derivative

Let A be an $n \times n$ nonnegative almost skew-symmetric matrix. Consider Levinger's function $\phi(A,\alpha) = \rho(\mathcal{L}(A,\alpha)) = \rho((1-\alpha)A + \alpha A^t)$ $(\alpha \in [0,\, 1/2])$ and the open interval

$$\mathcal{X}_A \;=\; \left(\max\left\{0,\, \frac{1}{2} - \sqrt{\frac{\delta(A)^2}{16\,\mathrm{var}(A)}}\right\},\, \frac{1}{2} \right). \tag{5}$$

Notice that $\alpha \in (0,\, 1/2)$ lies in $\mathcal{X}_A$ if and only if $\delta(A)^2 > 4(1-2\alpha)^2 \mathrm{var}(A)$. We shall first obtain a new upper bound for $\phi^{'}(A,\alpha)$ that is tighter than the one in Theorem 1 (c).

Note that, by the proof of [4, Theorem 1.2], for every $\alpha \in [0,\, 1/2)$,

$$\begin{aligned} 0 \;\leq\; \phi^{'}(A,\alpha) \;&=\; \frac{1}{1-2\alpha}\, \frac{x_l(\alpha)^t(\mathcal{L}(A,\alpha)^t - \mathcal{L}(A,\alpha))x_r(\alpha)}{x_l(\alpha)^t x_r(\alpha)} \\ &=\; -\,2\, \frac{x_l(\alpha)^t K(A) x_r(\alpha)}{\cos(\widehat{x_l(\alpha), x_r}(\alpha))}\,. \end{aligned} \tag{6}$$

Theorem 11 *Let A be an $n \times n$ nonnegative almost skew-symmetric matrix with irreducible symmetric part. Then for every $\alpha \in \mathcal{X}_A$,*

$$\begin{aligned} \phi^{'}(A,\alpha) \;&=\; \frac{2\phi(A,\alpha)}{1-2\alpha} \left(\frac{1}{\cos(\widehat{x_r(\alpha), x_l}(\alpha))} - 1 \right) \\ &\leq\; \frac{2\delta(A)\left(\delta(A) - \sqrt{\delta(A)^2 - 4(1-2\alpha)^2 \mathrm{var}(A)}\right)}{(1-2\alpha)\sqrt{\delta(A)^2 - 4(1-2\alpha)^2\mathrm{var}(A)}}\,. \end{aligned} \tag{7}$$

Proof. By (6) and the definitions, the following equalities ensue:

$$\begin{aligned} \phi^{'}(A,\alpha) \;&=\; -2\, \frac{x_l(\alpha)^t K(A) x_r(\alpha)}{\cos(\widehat{x_r(\alpha), x_l}(\alpha))} \\ &=\; -\,\frac{2}{1-2\alpha}\, \frac{x_l(\alpha)^t(\mathcal{L}(A,\alpha) - S(A))x_r(\alpha)}{\cos(\widehat{x_r(\alpha), x_l}(\alpha))} \\ &=\; -\,\frac{2}{1-2\alpha} \left(\phi(A,\alpha) - \frac{x_l(\alpha)^t S(A) x_r(\alpha)}{\cos(\widehat{x_r(\alpha), x_l}(\alpha))} \right). \end{aligned}$$

Letting $z(\alpha)$ denote the orthogonal projection of $x_r(\alpha)$ and $x_l(\alpha)$ onto w, by [11, Proposition 3.1], we have

$$\begin{aligned} x_l(\alpha)^t S(A) x_r(\alpha) \;&=\; z(\alpha)^t S(A) z(\alpha) \\ &=\; \delta(A)(z(\alpha)^t z(\alpha)) \\ &=\; \delta(A)\, \|z(\alpha)^t\|_2^2 \\ &=\; \phi(A,\alpha), \end{aligned}$$

and hence,

$$\phi^{'}(A,\alpha) \;=\; \frac{-2}{1-2\alpha}\left(\phi(A,\alpha) - \frac{\phi(A,\alpha)}{\cos(\widehat{x_r(\alpha), x_l(\alpha)})}\right).$$

By the above equality, the fact that $\phi(A,\alpha) \leq \delta(A)$, and by [11, Theorem 3.6], the proof is complete. $\square$

It is worth noting that the upper bound in (7) for the derivative of Levinger's function is independent of $s_1 = \|K(A)\|_2$. Furthermore, since

$$s_1 \;\geq\; \frac{\|K(A)w\|_2}{\|w\|_2} \;=\; \sqrt{\mathrm{var}(A)},$$

by straightforward computations, one can see that for every $\alpha \in \mathcal{X}_A$,

$$s_1 \;\geq\; \sqrt{\mathrm{var}(A)} \;>\; \frac{\left(\delta(A) - \sqrt{\delta(A)^2 - 4(1-2\alpha)^2\mathrm{var}(A)}\right)\sqrt{\delta(A)^2 - 4(1-2\alpha)^2\mathrm{var}(A)}}{2(1-2\alpha)^2\sqrt{\mathrm{var}(A)}}.$$

As a consequence, the upper bound in the previous theorem is an improvement over the upper bound provided in Theorem 1 (c) for all $\alpha \in \mathcal{X}_A$.

Integrating through (7) with respect to α in an interval $(\alpha_1,\, \alpha_2) \subseteq \mathcal{X}_A$, we obtain the following result.

Theorem 12 *Let A be an $n\times n$ nonnegative almost skew-symmetric matrix with irreducible symmetric part. Then for every $\alpha_1, \alpha_2 \in \mathcal{X}_A$ with $\alpha_1 < \alpha_2$,*

$$\phi(A,\alpha_2) - \phi(A,\alpha_1) \;\leq\; \delta(A)\;\ln\left(\frac{(1-2\alpha_2)^2\left(\delta(A) - \sqrt{\delta(A)^2 - 4(1-2\alpha_1)^2\mathrm{var}(A)}\right)}{(1-2\alpha_1)^2\left(\delta(A) - \sqrt{\delta(A)^2 - 4(1-2\alpha_2)^2\mathrm{var}(A)}\right)}\right).$$

Moreover, if $\mathcal{X}_A = (0,\, 1/2)$, i.e., *if $\delta(A)^2 \geq 4\mathrm{var}(A)$, then for every $\alpha \in (0,\, 1/2]$,*

$$\phi(A,\alpha) \;\leq\; \rho(A) + \delta(A)\;\ln\left(\frac{(1-2\alpha)^2\left(\delta(A) - \sqrt{\delta(A)^2 - 4\mathrm{var}(A)}\right)}{\delta(A) - \sqrt{\delta(A)^2 - 4(1-2\alpha)^2\mathrm{var}(A)}}\right).$$

Theorem 13 *Let A be an $n\times n$ nonnegative almost skew-symmetric matrix with irreducible symmetric part. Then for every $\alpha_1 < \alpha_2$ in $\mathcal{X}_A$,*

$$\frac{\phi(A,\alpha_2)}{\phi(A,\alpha_1)} \;\leq\; \frac{(1-2\alpha_2)^2\left(\delta(A) - \sqrt{\delta(A)^2 - 4(1-2\alpha_1)^2\mathrm{var}(A)}\right)}{(1-2\alpha_1)^2\left(\delta(A) - \sqrt{\delta(A)^2 - 4(1-2\alpha_2)^2\mathrm{var}(A)}\right)}.$$

Proof. By Theorem 11, we have

$$\frac{(1-2\alpha)\phi^{'}(A,\alpha)}{2\,\phi(A,\alpha)} \;=\; \frac{1}{\cos(\widehat{x_r(\alpha), x_l(\alpha)})} - 1,$$

or equivalently,

$$\frac{1}{\cos(\widehat{x_r(\alpha), x_l(\alpha)})} \;=\; \frac{2\,\phi(A,\alpha)+(1-2\alpha)\phi^{'}(A,\alpha)}{2\,\phi(A,\alpha)}\,.$$

Furthermore, by [11, Theorem 3.6],

$$\begin{aligned}\sqrt{\frac{\delta(A)^2-4(1-2\alpha)^2\mathrm{var}(A)}{\delta(A)^2}} \;&\le\; \cos(\widehat{x_r(\alpha), x_l(\alpha)})\\ &=\; \frac{2\,\phi(A,\alpha)}{2\,\phi(A,\alpha)+(1-2\alpha)\phi'(A,\alpha)}\,,\end{aligned}$$

and hence,

$$2+(1-2\alpha)\frac{\phi^{'}(A,\alpha)}{\phi(A,\alpha)} \;\le\; \frac{2\,\delta(A)}{\sqrt{\delta(A)^2-4(1-2\alpha)^2\mathrm{var}(A)}}\,.$$

Thus for every $\alpha \in \mathcal{X}_A$,

$$0 \;\le\; \frac{\phi^{'}(A,\alpha)}{\phi(A,\alpha)} \;\le\; -\frac{2}{1-2\alpha} + \frac{2\,\delta(A)}{(1-2\alpha)\sqrt{\delta(A)^2-4(1-2\alpha)^2\mathrm{var}(A)}}\,.$$

Integrating through the above inequality with respect to α in the interval $(\alpha_1,\,\alpha_2) \subseteq \mathcal{X}_A$ obtains

$$\ln\left(\frac{\phi(A,\alpha_2)}{\phi(A,\alpha_1)}\right) \;\le\; \ln\left(\frac{(1-2\alpha_2)^2\left(\delta(A)-\sqrt{\delta(A)^2-4(1-2\alpha_1)^2\mathrm{var}(A)}\right)}{(1-2\alpha_1)^2\left(\delta(A)-\sqrt{\delta(A)^2-4(1-2\alpha_2)^2\mathrm{var}(A)}\right)}\right)$$

and consequently,

$$\frac{\phi(A,\alpha_2)}{\phi(A,\alpha_1)} \;\le\; \frac{(1-2\alpha_2)^2\left(\delta(A)-\sqrt{\delta(A)^2-4(1-2\alpha_1)^2\mathrm{var}(A)}\right)}{(1-2\alpha_1)^2\left(\delta(A)-\sqrt{\delta(A)^2-4(1-2\alpha_2)^2\mathrm{var}(A)}\right)}\,.$$

The proof is complete. $\square$

If $\delta(A)^2 > 4\mathrm{var}(A)$, then for $\alpha_2 = \alpha \in (0,\,1/2]$ and $\alpha_1 \to 0$,

$$\phi(A,\alpha) \;\le\; \rho(A)\,\frac{(1-2\alpha)^2\left(\delta(A)-\sqrt{\delta(A)^2-4\mathrm{var}(A)}\right)}{\delta(A)-\sqrt{\delta(A)^2-4(1-2\alpha)^2\mathrm{var}(A)}}\,. \tag{8}$$

Furthermore, by (4), the cosine of the angle $(\widehat{w, x_r(\alpha)}) = (\widehat{w, x_l(\alpha)})$ is less than or equal to

$$\sqrt{\frac{\rho(A)(1-2\alpha)^2\left(\delta(A)-\sqrt{\delta(A)^2-4\mathrm{var}(A)}\right)}{\delta(A)\left(\delta(A)-\sqrt{\delta(A)^2-4(1-2\alpha)^2\mathrm{var}(A)}\right)}}\,. \tag{9}$$

Letting now $\alpha_2 \to 1/2$ and $\alpha_1 = \alpha \in \mathcal{X}_A$, it follows

$$\phi(A, \alpha) \geq \frac{2\mathrm{var}(A)}{\delta(A) - \sqrt{\delta(A)^2 - 4(1-2\alpha)^2 \mathrm{var}(A)}}.$$

Note also that by straightforward computations, one can see that the latter lower bound of $\phi(A, \alpha)$ is exactly the same as the bound provided in Theorem 1 (a).

5 An illustration

The 6×6 (irreducible) nonnegative matrix

$$A = \begin{bmatrix} 1 & 0.2 & 0.4 & 1.8 & 1.4 & 1 \\ 1.8 & 1 & 0 & 2 & 0.6 & 0.6 \\ 1.6 & 2 & 1 & 2 & 2 & 0.2 \\ 0.2 & 0 & 0 & 1 & 1.6 & 1.8 \\ 0.6 & 1.4 & 0 & 0.4 & 1 & 1.6 \\ 1 & 1.4 & 1.8 & 0.2 & 0.4 & 1 \end{bmatrix}$$

is almost skew-symmetric and has irreducible symmetric part $S(A) = \mathbf{1}\mathbf{1}^t$, where $\mathbf{1} \in \mathbb{R}^n$ denotes the all ones vector. The Perron root of A is $\rho(A) = 5.7159$, its variance is $\mathrm{var}(A) = 1.8133$, and $\delta(A) = 6$. The skew-symmetric part of A is

$$K(A) = \begin{bmatrix} 0 & -0.8 & -0.6 & 0.8 & 0.4 & 0 \\ 0.8 & 0 & -1 & 1 & -0.4 & -0.4 \\ 0.6 & 1 & 0 & 1 & 1 & -0.8 \\ -0.8 & -1 & -1 & 0 & 0.6 & 0.8 \\ -0.4 & 0.4 & -1 & -0.6 & 0 & 0.6 \\ 0 & 0.4 & 0.8 & -0.8 & -0.6 & 0 \end{bmatrix}$$

and the Frobenius norms of the matrices A, $S(A)$ and $K(A)$ are $\|A\|_F = 7.2277$, $\|S(A)\|_F = 6$ and $\|K(A)\|_F = 4.0299$. The condition $\delta(A)^2 > 4\mathrm{var}(A)$ is satisfied.

In Figure 1, our bounds for Levinger's function $\phi(A, \alpha)$ are illustrated. The Perron roots $\phi(A, \alpha) = \rho(\mathcal{L}(A, \alpha))$ for $\alpha = 0, 0.05, 0.1, \ldots, 0.5$ are plotted by '+'. The curve (a) is the lower bound in Theorem 1 (a), the curve (b) is the upper bound in Corollary 6, the curve (c) is the upper bound in the second part of Theorem 12, and the curve (d) is the upper bound in (8).

Our bounds for the cosine of the angle $(\widehat{w, x_r(\alpha)}) = (\widehat{w, x_l(\alpha)})$ are verified in Figure 2, where the values of the cosine for $\alpha = 0, 0.05, 0.1, \ldots, 0.5$ are also plotted by '+'. The curve (A) is the lower bound in Theorem 1 (b), the curve (B) is the upper bound in Proposition 8, and the curve (D) is the upper bound in (9). Notice that these three bounds have exactly the same behaviour as the bounds provided by the curves (a), (b) and (d) in Figure 1, respectively.

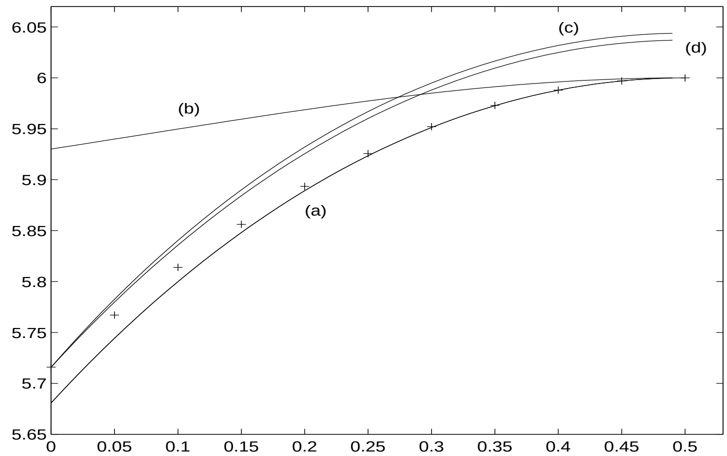

Figure 1: The bounds for $\phi(A, \alpha)$.

6 Tournament matrices

One of the most intriguing problems in combinatorial matrix theory regards a conjecture posed by Brualdi and Li [2]. It states that among all tournament matrices of a given even order, the maximal spectral radius is attained by the Brualdi-Li matrix, defined below. This conjecture has been confirmed for small sizes, and there is supporting evidence for its validity asymptotically (as the order grows large); see [3, 7]. In this section, we suggest an alternative approach and report some related progress. We provide a comparison result for spectral radii and show that the spectral radius of the Brualdi-Li matrix is maximum among $n \times n$ tournament matrices whose "score variance" exceeds a certain function of n.

We begin by reviewing some basic facts about tournament matrices. Recall that an $n \times n$ *tournament matrix* T is a $(0, 1)$-matrix such that $A + A^t = J - I_n$, where $J = \mathbf{1}\mathbf{1}^t$. (Recall that $\mathbf{1}$ denotes an all ones vector.) For odd n, T is called *regular* if all its row sums equal $(n-1)/2$, *i.e.,* $T\mathbf{1} = [(n-1)/2]\mathbf{1}$. For even n, T cannot have all row sums equal; in this case, T is called *almost regular* if half its row sums equal $n/2$ and the others equal $(n-2)/2$.

To make the connection to our previous sections, note that if T is an $n \times n$ tournament matrix, then $A = T + (1/2)I_n$ is nonnegative almost skew-symmetric with $S(A) = (1/2)J$. The *score variance* of T is defined by

$$\mathrm{sv}(T) \;=\; \frac{1}{n} \left\| T\mathbf{1} - \left(\frac{n-1}{2}\right)\mathbf{1} \right\|_2^2 .$$

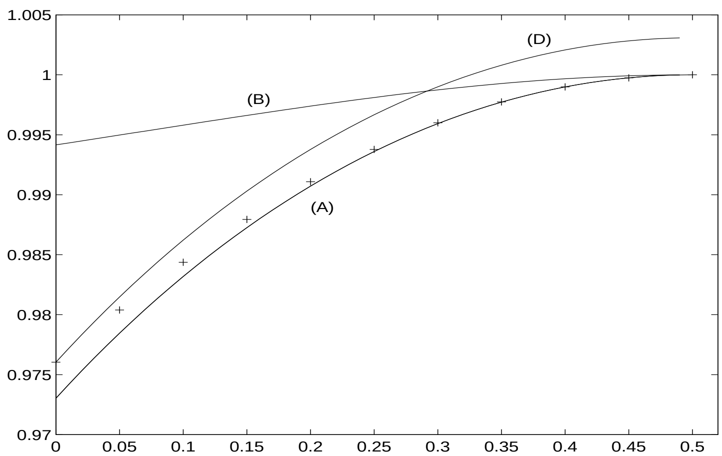

Figure 2: The bounds for the cosine of $(\widehat{w, x_r(\alpha)}) = (\widehat{w, x_l(\alpha)})$.

Consequently,

$$\text{sv}(T) \;=\; \frac{1}{n}\,\|K(T)\mathbf{1}\|_2^2 \;=\; \frac{1}{n}\,\|K(A)\mathbf{1}\|_2^2 \;=\; \text{var}(A).$$

Moreover, $\delta(A) = n/2$, $\rho(T) = \rho(A) - 1/2$ and

$$\|A\|_F^2 \;=\; \frac{n(n-1)}{2} + \frac{n}{4} \;=\; \frac{2n^2 - n}{4}\,.$$

Clearly, the score variance of a regular tournament equals 0 (and hence, by Corollary 4, all regular tournaments are normal matrices) and the score variance of an almost regular tournament is 1/4. In general, $\text{sv}(T) \le (n^2-1)/12$ with equality holding when T is triangular.

It has been shown that for sufficiently large even n, the tournament matrix attaining the maximum Perron root among all $n \times n$ tournament matrices must be almost regular; see Kirkland [7, Theorem 3]. In addition, as shown in [3], among the Perron roots of all almost regular $n \times n$ tournament matrices of the form

$$\mathcal{B}(T) = \begin{bmatrix} T & T^t \\ T^t + I_{n/2} & T \end{bmatrix},$$

where T is itself a tournament matrix of order $n/2$, the maximum is attained when T is the strictly upper triangular matrix U all of whose entries above the main diagonal are equal to 1. We then refer to $\mathcal{B} = \mathcal{B}(U)$ as the Brualdi-Li matrix. The Brualdi-Li conjecture states that $\rho(\mathcal{B})$ maximizes the Perron root among all $n \times n$ tournament matrices of even order n.

Consider now two $n \times n$ tournament matrices T_1 and T_2, as well as the corresponding almost skew-symmetric matrices $A_1 = T_1 + (1/2)I_n$ and $A_2 = T_2 + (1/2)I_n$. By the above discussion, inequality (3) in Proposition 7 can be re-written as

$$2\sqrt{(n-1)\left(n-1-\frac{8\text{var}(A_1)}{6n-3}\right)} \;<\; n-2+\sqrt{n^2-16\text{var}(A_2)}\,,$$

or equivalently,

$$\text{var}(A_1) \;>\; \frac{6n-3}{16(n-1)}\left(n^2-2n+8\text{var}(A_2)-(n-2)\sqrt{n^2-16\text{var}(A_2)}\right).$$

Thus, the following comparison result for the spectral radii of tournaments is valid as a consequence of Proposition 7.

Theorem 14 *Let T_1 and T_2 be two $n\times n$ tournament matrices with score variances* $\text{sv}(T_1)$ *and* $\text{sv}(T_2)$ *such that* $n^2 > 16\,\text{sv}(T_2)$. *If*

$$\text{sv}(T_1) \;>\; \frac{6n-3}{16(n-1)}\left(n^2-2n+8\,\text{sv}(T_2)-(n-2)\sqrt{n^2-16\,\text{sv}(T_2)}\right), \tag{10}$$

then $\rho(T_1) < \rho(T_2)$.

Next, notice that the function

$$h(v) \;=\; \frac{6n-3}{16(n-1)}\left(n^2-2n+8v-(n-2)\sqrt{n^2-16v}\right) \;;\quad 0 \;\le\; v \;\le\; \frac{n^2}{16}$$

that appears on the right hand side of (10) is increasing. In fact,

$$h'(v) \;=\; \frac{6n-3}{2(n-1)}\left(1+\frac{n-2}{\sqrt{n^2-16v}}\right) \;>\; 4\;;\quad 0 \;<\; v \;<\; \frac{n^2}{16}\,,$$

and $h(v)$ satisfies

$$h(0) \;=\; 0 \quad \text{and} \quad h\!\left(\frac{n^2}{16}\right) \;=\; \frac{3n(2n-1)(3n-4)}{32(n-1)}\,.$$

For even n, the smallest possible score variance of an $n\times n$ tournament matrix is $1/4$ and almost regular tournament matrices attain this value. Thus, Proposition 3 implies that almost regular tournaments attain the minimum distance to normality, $(n\sqrt{2})/2$, among all $n\times n$ tournament matrices. Furthermore, the smallest score variance among the rest of $n\times n$ tournament matrices (that is, excluding the almost regular ones) is $1/4+2/n$. As a consequence, keeping in mind that the sequence

$$\frac{6n-3}{16(n-1)}\left(n^2-2n+2-(n-2)\sqrt{n^2-4}\right) \;;\quad n=2,3,\ldots$$

is increasing and converges to $3/2$, Theorem 14 yields the following result related to the Brualdi-Li conjecture; see also [7, Theorem 3].

Corollary 15 *Let* $n \geq 4$ *be even, and let* T_0 *be an* $n \times n$ *almost regular tournament matrix. For every* $n \times n$ *tournament matrix* T *with*

$$\mathrm{sv}(T) \;>\; \frac{6n-3}{16(n-1)}\left(n^2-2n+2-(n-2)\sqrt{n^2-4}\right),$$

we have $\rho(T) < \rho(T_0)$*. Moreover, if* $\mathrm{sv}(T) > 3/2$*, then* $\rho(T) < \rho(T_0)$*.*

Next, we mention that our results can be used to prove another result related to the Brualdi-Li conjecture, which is already known; see [6, Corollary 1.4 and preceding commentary].

Corollary 16 [6, Kirkland] *Let* T_0 *be an* $n \times n$ *(*n *even) almost regular tournament matrix. Then,*

$$\rho(T_0) \;\geq\; \frac{n-2+\sqrt{n^2-4}}{4}$$

and

$$\lim_{n\to\infty}\left(\rho(T_0)-\frac{n-1}{2}\right) \;=\; 0;$$

that is, as the order n *increases, the spectral radius of an almost regular tournament of order* n *approaches the maximum possible value among the spectral radii of all tournament matrices of order* n*.*

Proof. It is known that the real part of every eigenvalue of an $n \times n$ tournament matrix (and thus its spectral radius) is bounded above by $(n-1)/2$ [1]. Hence, by the discussion so far and in conjunction with Theorem 1 (a) applied to $T_0 + (1/2)I_n$, we have

$$\frac{n-2+\sqrt{n^2-4}}{4} \;\leq\; \rho(T_0) \;\leq\; \frac{n-1}{2}.$$

However, it is easy to verify that

$$\lim_{n\to\infty}\left(\frac{n-2+\sqrt{n^2-4}}{4}-\frac{n-1}{2}\right) \;=\; 0,$$

completing the proof. $\square$

In conclusion, we note that our treatment of tournament matrices does not depend strongly on their special $(0,1)$-structure; rather, it is based on analytic techniques that suggest a new possible approach toward the Brualdi-Li conjecture. In particular, it is of interest (and would possibly lead to a positive resolution of the conjecture) if one were able to reach the conclusion of Theorem 14 under a relaxed upper bound for the score variance of T_1 in (10).

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