CONVEX SETS OF NONSINGULAR AND P–MATRICES

(Linear and Multilinear Algebra, 38(3) : 233-239, 1995)

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December 6, 2000

Abstract

We show that the set $r(A, B)$ (resp. $c(A, B)$) of square matrices whose rows (resp. columns) are independent convex combinations of

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[*] Work supported in part by Office of Naval Research contract N00014–90–J–1739 and NSF grant DMS 90–00899.

[†] Work partially supported by NSERC grant 6–53121. Part of the work was done while this author was a Visiting Assistant Professor at the College of William and Mary.

the rows (resp. columns) of real matrices A and B consists entirely of nonsingular matrices if and only if BA^{-1} (resp. $B^{-1}A$) is a P–matrix. This improves a theorem on P–matrices proven in [2] and [3], in the context of interval nonsingularity. We also show that every real P–matrix admits a representation BA^{-1} with the above property. These results are only partially true for complex P–matrices. Based on them we obtain a characterization of complex P–matrices in terms of block partitions.

1 Introduction

The subset of the n–by–n complex matrices ($M_n(\mathcal{C})$), all of whose principal minors are positive is commonly referred to as the *P–matrices* and is denoted here by $\mathcal{P}_n(\mathcal{C})$. Clearly, $\mathcal{P}_n(\mathcal{C}) \subset GL_n(\mathcal{C})$ (the complex n–by–n nonsingular matrices), and it is known to include several important classes of matrices, such as the positive definite matrices, the M–matrices and the totally positive matrices, arising in a variety of applications. In [2] and [3] it was observed that real P–matrices also arise in the theory of systems of linear interval equations and are intimately related to interval nonsingularity. More specifically, in the language used in [3], given $A = (a_{ij}),\ B = (b_{ij}) \in M_n(\mathcal{R})$, if

$$\{C = (c_{ij}) \mid min\{a_{ij}, b_{ij}\} \leq c_{ij} \leq max\{a_{ij}, b_{ij}\}\} \subset GL_n(\mathcal{R}), \tag{1.1}$$

then $BA^{-1},\ AB^{-1},\ B^{-1}A,\ A^{-1}B \in \mathcal{P}_n(\mathcal{R})$. In [2] it was also shown that (1.1) holds if a certain 2^n matrices, constructed from matrices such as C in (1.1), are P–matrices (see Theorem 3.2(b)).

In Section 3 of the present paper, in the interest of interpolating between the results in [2] and [3] and a necessary and sufficient condition for the convex hull of two matrices to be a subset of $GL_n(\mathcal{C})$ (see Observation 3.1), we show that $BA^{-1},\ AB^{-1},\ B^{-1}A,\ A^{-1}B \in \mathcal{P}_n(\mathcal{R})$ under conditions weaker than (1.1). Moreover, these conditions are not only sufficient but also necessary (see Theorems 3.3, 3.4, and Corollary 3.5). We also show that every P–matrix admits a representation BA^{-1} (or $B^{-1}A$), such that every matrix whose rows (or columns) are convex combinations of the corresponding rows (or columns) of A and B, is nonsingular (see Theorem 3.8). In Section 4 we discuss how these results carry over to complex P–matrices and obtain another characterization of P–matrices (see Theorem 4.4).

We begin with notation, definitions, and some basic background.

2 Notation and Definitions

Let $diag(t_1, \ldots, t_n)$ denote the diagonal matrix whose diagonal entries are $t_1, \ldots, t_n$. Let J denote the all ones matrix, and let $\circ$ denote the Hadamard (entrywise) product of matrices. For any $A, B \in M_n(\mathcal{C})$ we define the following sets:

$$h(A,B) = \{C \mid C = tA + (1-t)B,\ t \in [0,1]\},$$

$$r(A,B) = \{C \mid C = TA + (I-T)B,\ T = diag(t_1,\dots,t_n),\ t_i \in [0,1]\},$$

$$c(A,B) = \{C \mid C = AT + B(I-T),\ T = diag(t_1,\dots,t_n),\ t_i \in [0,1]\},$$

$$i(A,B) = \{C \mid C = T \circ A + (J-T) \circ B,\ T = (t_{ij}),\ t_{ij} \in [0,1]\}.$$

We note that $r(A,B)$ (resp. $c(A,B)$) denotes the set of matrices whose rows (resp. columns) are independent convex combinations of the corresponding rows (resp. columns) of A and B. When $A, B \in M_n(\mathcal{R})$, $i(A,B)$ coincides with the set of matrices in the left hand side of (1.1) and is usually referred to as the *interval of* A *and* B. It is clear that $h(A,B)$, $r(A,B)$, $c(A,B)$ and $i(A,B)$ are convex sets and that

$$h(A,B) \subseteq r(A,B) \subseteq i(A,B) \ \text{ and } \ h(A,B) \subseteq c(A,B) \subseteq i(A,B).$$

Notice that if $A, B \in M_n(\mathcal{R})$, then there always exist matrices $C_L,\ C_R \in i(A,B)$ such that $C_L \leq C \leq C_R$ (entrywise), for all $C \in i(A,B)$. We then write $i(A,B) = [C_L, C_R]$, and associate with this interval *extremal matrices* $C_m,\ m = 1, \dots 2^n$, defined as follows:

$$C_m = T_m C_L + (I - T_m) C_R, \tag{2.2}$$

where the $T_m,\ m = 1, \dots 2^n$, are all possible $0,1$ n–by–n diagonal matrices, ordered lexicographically.

A brief review of the basic properties of P–matrices is now in order. Given $X \in \mathcal{P}_n(\mathcal{C})$, it is well known that $X^{-1},\ X^T$, and any permutation similarity of X is also in $\mathcal{P}_n(\mathcal{C})$. Also $X + D,\ DX,\ XD \in \mathcal{P}_n(\mathcal{C})$ for every diagonal matrix D with positive diagonal entries. In addition, $X \in \mathcal{P}_n(\mathcal{C})$ does not reverse the sign of any real vector, namely, if $Xu = w$, $u, w \in \mathcal{R}^n$ and $u \neq 0$, then $u \circ w$ has at least one positive entry. Conversely, any real matrix that does not reverse the sign of any real vector is a P–matrix. Finally, if $X \in \mathcal{P}_n(\mathcal{C})$ then every real eigenvalue of every principal submatrix of X is positive. This condition is also sufficient for real P–matrices. For more background material on P–matrices and the proofs of these facts the reader is referred to [1].

3 Real P–matrices

We are interested in comparing the conditions for $h(A,B),\ r(A,B),\ c(A,B)$, or $i(A,B)$ to consist entirely of nonsingular matrices, and in better understanding the role that the P–matrices play in such an occurence. We begin with an observation regarding $h(A,B)$.

Observation 3.1 *Let $A,B \in GL_n(\mathcal{C})$. Then, $h(A,B) \subset GL_n(\mathcal{C})$ if and only if BA^{-1} has no negative eigenvalues.*

Proof:
First notice that $tA + (1-t)B \in GL_n(\mathcal{C})$, for all $t \in [0,1]$, if and only if $tI + (1-t)BA^{-1} \in GL_n(\mathcal{C})$, for all $t \in [0,1]$. If $tI + (1-t)BA^{-1}$ is singular for some $t \in (0,1)$, then BA^{-1} has the negative eigenvalue $-\frac{t}{1-t}$. Conversely, if $\lambda < 0$ is an eigenvalue of BA^{-1} with eigenvector x, then

$$(tI + (1-t)BA^{-1})x = (t + (1-t)\lambda)x = 0,$$

for $t = -\frac{\lambda}{1-\lambda} \in (0,1)$. □

In the next theorem we paraphrase the results from [2, Theorems 1.2 and 5.1] and [3], regarding $i(A,B)$ and P–matrices.

Theorem 3.2 *Let $A,B \in M_n(\mathcal{R})$.*

- *(a) If $i(A,B) \subset GL_n(\mathcal{R})$, then $BA^{-1},\ AB^{-1},\ B^{-1}A,\ A^{-1}B \in \mathcal{P}_n(\mathcal{R})$.*
- *(b) Let C_m, $m = 1,\ldots,2^n$, be the extremal matrices associated with $i(A,B)$, given in (2.2). If $C_{m_1}^{-1}C_{m_2} \in \mathcal{P}_n(\mathcal{R})$ for all m_1, m_2 for which $T_{m_1} + T_{m_2} = I$, then $i(A,B) \subset GL_n(\mathcal{R})$.*

A closer look at the proof of part (a) of Theorem 3.2, given in [2] and [3], reveals that indeed weaker conditions are necessary for $BA^{-1},\ AB^{-1},\ B^{-1}A$, and $A^{-1}B$ to be P–matrices. In the next two theorems and their corollary we conclude that these necessary conditions involve $r(A,B)$ and $c(A,B)$ (rather than $i(A,B)$) and, moreover, that these conditions are also sufficient.

Theorem 3.3 *Let $A, B \in M_n(\mathcal{R})$. The following are equivalent:*

(a) $r(A, B) \subset GL_n(\mathcal{R})$.

(b) $BA^{-1} \in \mathcal{P}_n(\mathcal{R})$.

Proof:
$(a) \implies (b)$. Let $r(A, B) \subset GL_n(\mathcal{R})$ and suppose, by way of contradiction, that $BA^{-1} \notin \mathcal{P}_n(\mathcal{R})$. Then there exists $x = (x_1, \ldots, x_n)^T \neq 0$ such that $BA^{-1}x = y = (y_1, \ldots, y_n)^T$ and $x_i y_i \leq 0$ for all $i = 1, \ldots, n$. We can then choose $t_i \in [0, 1]$ such that $t_i x_i + (1 - t_i)y_i = 0$, for all $i = 1, \ldots, n$. Thus, on letting $T = diag(t_1, \ldots, t_n)$,

$$Tx + (I - T)BA^{-1}x = 0,$$

namely $TA + (I - T)B$ is not invertible, contradicting our assumption.
$(b) \implies (a)$. Let $BA^{-1} \in \mathcal{P}_n(\mathcal{R})$ and let $T = diag(t_1, \ldots, t_n)$ with $t_i \in [0, 1]$. Then $C = TA + (I - T)B \in r(A, B)$ is invertible if and only if

$$CA^{-1} = T + (I - T)BA^{-1}$$

is invertible. Let $N = \{1, 2, \ldots, n\}$ and denote by $A[\alpha]$ the principal submatrix of A whose rows and columns are indexed by $\alpha \subseteq N$. Since T and $I - T$ are diagonal matrices we have that

$$det(CA^{-1}) = \sum_{\alpha \subseteq N} \prod_{i \notin \alpha} t_i \; det\left(\left((I - T)BA^{-1}\right)[\alpha]\right).$$

Since $t_i \in [0, 1]$ and $BA^{-1} \in \mathcal{P}_n(\mathcal{R})$, all summands in this determinantal expansion are nonnegative. Unless $T = 0$, in which case $C = B$, one of these summands is positive. Hence $det(CA^{-1}) > 0$, implying that C is nonsingular. $\square$

In a similar way we obtain the following theorem.

Theorem 3.4 *Let $A, B \in M_n(\mathcal{R})$. The following are equivalent:*

(a) $c(A, B) \subset GL_n(\mathcal{R})$.

(b) $B^{-1}A \in \mathcal{P}_n(\mathcal{R})$.

The next corollary provides us with the desired improvement of Theorem 3.2(a). It follows from Theorems 3.3 and 3.4 and the fact that $\mathcal{P}_n(\mathcal{R})$ is closed under inversion.

Corollary 3.5 *Let* $A, B \in M_n(\mathcal{R})$*. The following are equivalent:*

(a) $r(A,B) \subset GL_n(\mathcal{R})$ *and* $c(A,B) \subset GL_n(\mathcal{R})$*.*

(b) $BA^{-1},\ AB^{-1},\ B^{-1}A,\ A^{-1}B \in \mathcal{P}_n(\mathcal{R})$*.*

Remark 3.6 Referring to part (b) of Theorem 3.2 and comparing it with Theorem 3.3, we must clarify that even though $C_{m_1}, C_{m_2} \in r(A,B)$, the assumption $r(A,B) \subset GL_n(\mathcal{R})$ does not imply $i(A,B) \subset GL_n(\mathcal{R})$, because the matrices $C_{m_1}^{-1}C_{m_2}$, $m_1, m_2 = 1, 2, \ldots, 2^n$, are not necessarily P–matrices (only $C_{m_1}C_{m_2}^{-1}$ are P–matrices). In general, $r(A,B) \subset GL_n(\mathcal{R})$ and $c(A,B) \subset GL_n(\mathcal{R})$ (separately or simultaneously) do not imply that $i(A,B) \subset GL_n(\mathcal{R})$. This is illustrated in Example 3.10.

Remark 3.7 Let $A, B \in M_n(\mathcal{R})$. If A and B commute, then $BA^{-1} = A^{-1}B$. In this case, it follows from Theorems 3.3 and 3.4, and the fact that $\mathcal{P}_n(\mathcal{R})$ is closed under inversion, that $r(A,B) \subset GL_n(\mathcal{R})$ if and only if $c(A,B) \subset GL_n(\mathcal{R})$.

In the following theorem we investigate to what extent are the conditions in Theorems 3.3 and 3.4 characteristic of all real P–matrices.

Theorem 3.8 *The following are equivalent:*

(a) $P \in \mathcal{P}_n(\mathcal{R})$*.*

(b) There exist $A, B \in M_n(\mathcal{R})$ *such that* $P = BA^{-1}$*, and* $r(A,B) \subset GL_n(\mathcal{R})$*.*

(c) There exist $A, B \in M_n(\mathcal{R})$ *such that* $P = B^{-1}A$*, and* $c(A,B) \subset GL_n(\mathcal{R})$*.*

Proof:
$(a) \Longrightarrow (b)$. Let $P \in \mathcal{P}_n(\mathcal{R})$ and B be any nonsingular matrix. On letting $A^{-1} = B^{-1}P$ we have that $P = BA^{-1} \in \mathcal{P}_n(\mathcal{R})$. By Theorem 3.3, it follows that $r(A, B) \subset GL_n(\mathcal{R})$.
$(b) \Longrightarrow (a)$. Follows from Theorem 3.3.
$(a) \Longleftrightarrow (c)$. Follows similarly, using Theorem 3.4. □

Implicit in the proof of the preceding theorem is the following result.

Theorem 3.9 *Let $D \in GL_n(\mathcal{R})$ be any diagonal matrix. Then $P \in \mathcal{P}_n(\mathcal{R})$ if and only if $r(D, DP) \subset GL_n(\mathcal{R})$. Moreover, in such a case, if D has positive diagonal entries, then $r(D, DP) \subset \mathcal{P}_n(\mathcal{R})$. Similar assertions hold for $c(D, PD)$.*

We illustrate some of our observations in the following example.

Example 3.10 Consider the matrices

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

Since

$$BA^{-1} = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \in \mathcal{P}_3(\mathcal{R}) \text{ and } A^{-1}B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 2 \end{bmatrix} \notin \mathcal{P}_3(\mathcal{R}),$$

we have by Theorems 3.3 and 3.4, respectively, that $r(A, B) \subset GL_3(\mathcal{R})$ and $c(A, B) \not\subset GL_3(\mathcal{R})$. Also by Theorem 3.9, $r(I, A) \subset \mathcal{P}_3(\mathcal{R})$ and $c(I, A) \subset \mathcal{P}_3(\mathcal{R})$ since $A \in \mathcal{P}_3(\mathcal{R})$. However, the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \in i(I, A)$$

is singular and hence $i(I, A) \not\subset GL_3\mathcal{R}$.

We continue with propositions connecting our results with M–matrices and sign–nonsingularity. For the definition of an M–matrix and the relevant theory see [1, Chapter 2.5].

Proposition 3.11 *Let $A \in M_n(\mathcal{R})$ be an M–matrix. Then,*

(a) every matrix in $r(I, A)$ and $c(I, A)$ is an M–matrix,

(b) every matrix in $r(I, A^{-1})$ and $c(I, A^{-1})$ is an inverse M–matrix.

Proof:
(a) Since $A \in \mathcal{P}_n(\mathcal{R})$ and since all matrices in $r(I, A)$ and $c(I, A)$ have non–positive off–diagonal entries, the conclusion follows from Theorem 3.9 and a well known characterization of M–matrices.
(b) This implication follows by recalling that the principal submatrices of an inverse M–matrix are inverse M–matrices, and that if A is an inverse M–matrix, then so are the matrices $DA, AD, A + D$, where $D \in GL_n(\mathcal{R})$ is a nonnegative diagonal matrix. $\square$

A matrix $A \in M_n(\mathcal{R})$ is called *sign nonsingular* if every matrix with the same sign–pattern $(0, +, -)$ as A is also nonsingular. The following proposition follows from Corollary 3.5 and the fact that if A is sign nonsingular and B has the same sign–pattern as A, then all matrices in $c(A, B)$ and $r(A, B)$ have the same sign–pattern as A.

Proposition 3.12 *Let $A \in M_n(\mathcal{R})$ be sign nonsingular and $B \in M_n(\mathcal{R})$ any matrix with the same sign–pattern as A. Then BA^{-1}, AB^{-1}, $B^{-1}A$, $A^{-1}B \in \mathcal{P}_n(\mathcal{R})$.*

4 On Complex P–matrices

The results of the previous section depend on the realness of the entries of a P–matrix, specifically, on the fact that a real P–matrix does not reverse the sign of a real vector and, conversely, that any real matrix that does not reverse the sign a real vector is a P–matrix. As a consequence, for complex P–matrices these results are only partially true. For example, the proof of $(b) \Longrightarrow (a)$ of Theorem 3.3 and of $(a) \Longrightarrow (b)$ of Theorem 3.8, respectively, provide us with the following two theorems.

Theorem 4.1 *Let $A, B \in M_n(\mathcal{C})$. If $BA^{-1} \in \mathcal{P}_n(\mathcal{C})$ then $r(A, B) \subset GL_n(\mathcal{C})$. Similarly, if $B^{-1}A \in \mathcal{P}_n(\mathcal{C})$ then $c(A, B) \subset GL_n(\mathcal{C})$.*

Theorem 4.2 *Let $D \in GL_n(\mathcal{R})$ be any diagonal matrix and let $P \in \mathcal{P}_n(\mathcal{C})$. Then, $r(D, DP) \subset GL_n(\mathcal{C})$. Moreover, if D has positive diagonal entries, then $r(D, DP) \subset \mathcal{P}_n(\mathcal{C})$. Similar assertions hold for $c(D, PD)$.*

The converse of each of the above theorems is not true as is seen by the following example.

Example 4.3 Let

$$A = \begin{bmatrix} 1 & i \\ 1 & 1 \end{bmatrix}.$$

Since $detA = 1 - i$, $A \notin P_2(\mathcal{C})$. But $r(I, A)$ consists of matrices of the type

$$\begin{bmatrix} 1 & t_1 i \\ t_2 & 1 \end{bmatrix}, \quad t_1, t_2 \in [0, 1],$$

whose determinant is $1 - t_1 t_2 i \neq 0$, and hence $r(I, A) \subset GL_2(\mathcal{C})$.

Theorem 4.2 leads to the following characterization of P–matrices.

Theorem 4.4 *Let $A \in \mathcal{P}_n(\mathcal{C})$ be partitioned as*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \tag{4.3}$$

where A_{11}, A_{22} are $k \times k$ and $(n-k) \times (n-k)$ respectively. Then

$$\begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1} A_{12} \\ A_{21} A_{11}^{-1} & A_{22} - A_{21} A_{11}^{-1} A_{12} \end{bmatrix} \in \mathcal{P}_n(\mathcal{C}). \tag{4.4}$$

Conversely, if for some integer $0 \leq k \leq n$, $A \in M_n(\mathcal{C})$ has a block partition as in (4.3) such that (4.4) holds, then $A \in \mathcal{P}_n(\mathcal{C})$.

Proof:

By Theorem 4.2, if $A \in \mathcal{P}_n(\mathcal{C})$ then $r(I, A) \subset \mathcal{P}_n(\mathcal{C})$. Letting $I_m, 0_m$ denote the identity and the zero matrix in $M_m(\mathcal{R})$, respectively, define $T_1 = I_k \oplus 0_{n-k}$ and $T_2 = 0_k \oplus I_{n-k}$, for some $k \leq n$. Let

$$C_i = T_i + (I - T_i)A \ \in r(I, A) \ \subset \mathcal{P}_n(\mathcal{C}), \quad i = 1, 2.$$

Since $r(C_1, C_2) \subseteq r(I, A) \subset GL_n(\mathcal{C})$ and since

$$C_2^{-1} = \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12} \\ 0 & I_{n-k} \end{bmatrix},$$

by Theorem 3.3 applied to $r(C_1, C_2)$ we obtain

$$C_1C_2^{-1} = \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12} \\ A_{21}A_{11}^{-1} & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix} \in \mathcal{P}_n(\mathcal{C}),$$

proving one direction of the theorem.

Conversely, suppose that A has a block partition as in (4.3) for which (4.4) holds. Let now B be the matrix partitioned conformally to A with $B_{11} = A_{11}^{-1}$, $B_{12} = -A_{11}^{-1}A_{12}$, $B_{21} = A_{21}A_{11}^{-1}$, $B_{22} = A_{22} - A_{21}A_{11}^{-1}A_{12}$. A direct calculation shows that

$$\begin{bmatrix} B_{11}^{-1} & -B_{11}^{-1}B_{12} \\ B_{21}B_{11}^{-1} & B_{22} - B_{21}B_{11}^{-1}B_{12} \end{bmatrix} = A.$$

Thus, by the first part of the theorem applied to B, we have $A \in \mathcal{P}_n(\mathcal{C})$, completing the proof of the theorem. □

Notice that the $(2,2)$ block in (4.4) is the Schur complement of A with respect to A_{11}. Since all principal submatrices of a P–matrix are P–matrices, the well known fact that the Schur complements of a P–matrix are P–matrices follows from Theorem 4.4.

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