Generating and Detecting Matrices with Positive Principal Minors

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DEFINITION & ORIGINS

$A \in \mathcal{M}_n(\mathbb{C})$ is a **P-matrix** ($A \in \mathbb{P}$) if

all principal minors are positive:

$$\boxed{\det A[\alpha] > 0 \quad \forall \alpha \subseteq \{1, \ldots, n\}}$$

Origins

- Gantmacher-Krein, Kotelyanski, 1950.
- Taussky 1958: unification problem.
- Fiedler & Ptak 1962, 1966.
- Tucker 1963.
- Gale & Nikaido 1965: Jacobian matrix, global univalence.

Characterizing properties of (real) P-matrices

- Every real eigenvalue of any principal submatrix is positive.
- $\forall x \neq 0,\ \exists\, k \ :\ x_k(Ax)_k > 0$.
- $\forall D$ positive diagonal and $\forall \alpha$, $\det(A + D)[\alpha] \neq 0$.

Examples

- Real/Hermitian positive definite matrices.
- M-matrices: $A = sI - B,\ B \geq 0,\ s > \rho(B)$.
- Totally positive matrices (all minors positive).
- Real diagonally dominant matrices with + diagonal entries.

APPLICATIONS

1. Linear complementarity problems/complexity

[LCP] Given M, q, find $x, y \geq 0$ such that

$$y = Mx + q \quad \text{and} \quad x^T y = 0.$$

▷ LCP has a unique solution *iff* $M \in \mathbb{P}$.
▷ Can be found by Lemke's algorithm in exponential time.
▷ Interior point methods, smoothing methods developed.
▷ Numerical validation for existence of solutions desired.

Three problems:

[P-LCP] LCP given that $M \in \mathbb{P}$.

[P-LCP*] LCP or exhibit a nonpositive principal minor of M.

[P-PROBLEM] Determine whether $M \in \mathbb{P}$.

Facts

- If P-LCP or P-LCP* is NP-hard, then NP = coNP (Megiddo & Papadimitriou).
- The P-PROBLEM is co-NP-complete (G.E. Coxson).

Aside…

▷ P-LCP belongs to a complexity class with some other nice problems: [TFNP] - problems for Total Functions from NP.

Knuth

Given $m \times n$ matrix A $(m < n)$, find $n \times n$ submatrix B such that $B^{-1}A$ contains elements with absolute value ≤ 1.

[tantamount to finding a local maximum: cost = | det |]

Nash-equilibrium/bimatrix game

For each strategy i of player A and each strategy j of player B, their payoffs are a_{ij} and b_{ij}. The expected payoffs for strategies (vectors) x, y $(x, y \geq 0,\ x^T e = y^T e = 1)$ for the two players are

$$x^T A y \quad \text{and} \quad x^T B y\ .$$

The pair (x, y) is a Nash-equilibrium if

$$x^T A y \geq (x')^T A y \quad \text{and} \quad x^T B y \geq x^T A y'$$

for all strategies x', y' : Always exists !

[The challenge is finding it in polynomial time]

APPLICATIONS CONTINUED...

2. Interval matrices

$$[\,\underline{A},\overline{A}\,] = \{A \;:\; \underline{A} \le A \le \overline{A}\,\}$$

- Used in theory of linear interval equations/interval analysis.
- Question of whether all matrices in $[\,\underline{A},\overline{A}\,]$ are invertible arises (also discussed are stability, positive definiteness etc.)

  2^n *extremal* matrices in $[\,\underline{A},\overline{A}\,]$ are defined and certain matrices are checked for being in $\mathbb{P}$. If so, $[\,\underline{A},\overline{A}\,]$ comprises invertible matrices (regular).

- Interval P-matrices: [Rohn & Rex, SIMAX 1996]

$$[\,\underline{A},\overline{A}\,] \subset \mathbb{P}$$

iff

$$\forall x \neq 0,\ \exists\, k \ :\ \forall A \in [\,\underline{A},\overline{A}\,],\ x_k (Ax)_k > 0.$$

APPLICATIONS CONTINUED...

3. Linear Differential Inclusion (LDI) problems

$$\boxed{\dot{x} \in \Omega\, x, \quad x(0) = x_0, \quad \Omega \subset \mathcal{M}_n(\mathbb{R})}$$

DNLDI (Diagonal norm-bound LDI): A linear differential system, uncertain, time-varying (bounded) feedback gains. It leads to:

$$\Omega = \{A + B\Delta(I - E\Delta)^{-1}C \ : \ \|\Delta\| \leq 1,\ \Delta = \text{diagonal}\}.$$

Theorem [L. El Ghaoui, Ph.D. Thesis]
DNLDI is well-posed *iff* $(I + E)^{-1}(I - E) \in \mathbb{P}$.

EMERGING THEMES

In essence, the problems evolve around the following (**related**) themes on P-matrices:

- **Construction/Generation** (what is the fibre of $\mathbb{P}$?)
- **Detection** (how does one check if $A \in \mathbb{P}$?)
- **Invariance** (which transformations preserve $\mathbb{P}$?)

IN PURSUIT OF WHAT 'MAKES' OF A P-MATRIX

Theorem [**row hull**] [C.R. Johnson & T., LaMA 1995]
$BC^{-1} \in \mathbb{P}$ *iff* $TB + (I - T)C$ is invertible for all $T \in [0, I]$.

- ▷ [Rohn, LaMA 1991] explicitly states a weaker form of the 'if' part.
- ▷ This *characterization* of $\mathbb{P}$ is used/rediscovered implicitly and explicitly in a lot of recent work (Jansson, Gabriel, More, Shogenji, Yamasaki, Rump etc.)
- ▷ It is the premise of **block** P-matrices (Elsner & Szulc).

Corollary $A \in \mathbb{P}$ *iff* $TA - T + I$ is invertible (in fact, a P-matrix) for all $T \in [0, I]$.

Corollary If B and C are strictly diagonally dominant with positive diagonal entries, then so is $TB + (I - T)C$, for all $T \in [0, I]$. Thus $BC^{-1} \in \mathbb{P}$.

C.R. Johnson's question (Oberwolfach 2000):

Can every real P-matrix be factored into BC^{-1}, where B, C are strictly (row) diagonally dominant with positive diagonal entries ?

CAYLEY TRANSFORMS

An idea toward Johnson's question: In [Fallat & T., 2002] the **Cayley transform** F of a P-matrix A is examined:

$$\boxed{F := (I-A)(I+A)^{-1}}; \quad A = (I-F)(I+F)^{-1}.$$

Theorem If $A \in \mathbb{P}$, then $A = (I-F)(I+F)^{-1}$ is a factorization into P-matrices.

Theorem $A \in \mathbb{P}$ *iff* $I - FD$ is invertible $\forall D \in [-I,\ I]$.

Similar scenario unfolds in the work of [Rump, LAA 2002] :

$$G := (A-I)^{-1}(A+I); \quad A = (I+G)(G-I)^{-1}.$$

Theorem $A \in \mathbb{P}$ *iff* $[G-I,\ G+I]$ is a regular (nonsingular) matrix interval.

Speculation $I-F$ and $I+F$ don't necessarily have the properties in Johnson's question. But maybe some other (linear) fractional transformation of A could work.

DETECTION OF P-MATRICES

- Checking whether $A \in \mathbb{P}$ or not is NP-hard.

- Checking all principal minors (using G.E.) is O($n^3\ 2^n$).

- [T. & Lei Li, BIT 2000] A recursive algorithm using Schur complements is provided requiring about $7 \cdot 2^n$ operations:

$$|\alpha| = 1; \quad A \in \mathbb{P} \quad \textit{iff} \quad A[\alpha],\ A[\alpha^{\mathrm{c}}],\ A/A[\alpha]\ \in\ \mathbb{P}.$$

```
function [r]=ptest(A)
% r=1 if A is a P-matrix
n = length(A);
if ~(A(1,1)>0), r=0;
elseif n==1, r=1;
else
  B=A(2:n,2:n);
  D=A(2:n,1)/A(1,1);
  C=B-D*A(1,2:n);
  % Call function recursively
  r=ptest(B) & ptest(C);
end
```

DETECTION CONTINUED...

Features of the previous algorithm:

- Parallelized nature (large space complexity).
- Can deal with complex matrices.
- It is an exponential algorithm in the worst case. But when $A \notin \mathbb{P}$, the algorithm is 'probably' a lot faster. This touches upon fundamental probabilistic, methodological and other issues of complexity theory.

Another approach to the P-PROBLEM (Rump):

Recall: $\boxed{G := (A-I)^{-1}(A+I)}$ and $A = (I+G)(G-I)^{-1}$.

Then: $A \in \mathbb{P}$ *iff* $[G-I,\ G+I]$ comprises invertible matrices.

Idea: Test $[G-I, G+I]$ based on an algorithm that is not a priori exponential (Jansson & Rohn): $\Sigma =$ solutions to $\tilde{G}x = b$.

- Check if Σ is unbounded ($\notin \mathbb{P}$) or not ($\in \mathbb{P}$).
- Do so by checking $\Sigma \cap$ [orthants in $\mathbb{R}^n$].
- Choice of b is crucial to keep the number of orthants small (exponential in n is *still* the worst case).

HOW ABOUT P_0-MATRICES?

Apart from checking all principal minors, how does one detect an $n \times n$ P_0-matrix ($\mathbb{P}_0$), i.e.,

$$\det A\,[\alpha] \geq 0 \quad \forall \alpha \subseteq \{1, \ldots, n\}\ ?$$

We know that $A \in \mathbb{P}_0$ *iff* $A + \varepsilon I \in \mathbb{P} \quad \forall \varepsilon > 0$.

How can we turn the recursive test for P-matrices into a test for P_0-matrices that is

(1) **practical** (choose an $\varepsilon = \varepsilon(A)$)

and

(2) **robust** ($A \in \mathbb{P}_0$ *iff* $A + \varepsilon I \in \mathbb{P}$) ?

INVARIANCE OF $\mathbb{P}$

Recurring transform: **principal pivot transform** (ppt)

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad A_{11} \text{ is invertible,}$$

$$B = \operatorname{ppt}(A, A_{11}) := \begin{bmatrix} (A_{11})^{-1} & -(A_{11})^{-1}A_{12} \\ A_{21}(A_{11})^{-1} & A_{22} - A_{21}(A_{11})^{-1}A_{12} \end{bmatrix}.$$

The matrices A and B are related as follows:

$$A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad \textit{iff} \quad B \begin{bmatrix} y_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_2 \end{bmatrix}.$$

▷ [Tucker, SIAM Rev. 1963] asserts that if a real matrix $A \in \mathbb{P}$, then $\operatorname{ppt}(A, A_{11}) \in \mathbb{P}$.

▷ Ingredient of Principal Pivoting Algorithm for LCP [Cottle & Dantzig, LAA **1** 1968].

▷ A proof for the complex case is presented in [T., LAA 2000].

WHAT IS INTERESTING ABOUT PPT'S:

- ▷ Applying a ppt to a P-matrix gives you a P-matrix not of the 'usual' type.
- ▷ Same can be said about Johnson's (conjectured) factorization of P-matrices: it could be used to get 'generic' P-matrices.
- ▷ ppt's expand (constructively) the world of M-matrices and their inverses:
  - o There is class that generalizes both: **Mimes**.
  - o Pang introduced them as 'hidden Minkowski' matrices.
  - o Extend notions introduced by Mangasarian and Dantzig.

‘M-matrix and Inverse M-matrix Extension’

$$\boxed{A = (s_1 I - P_1)\,(s_2 I - P_2)^{-1},}$$

where

$$s_1,\ s_2 \in \mathbb{R},\quad P_1,\ P_2 \geq 0$$

$$\exists u \geq 0 \ \text{ such that } \ P_1 u < s_1 u \ \text{ and } \ P_2 u < s_2 u.$$

- Mimes contain the **M-matrices** & **their inverses**.
- A is a mime $\implies A \in \mathbb{P}$.
- If $B \geq 0$, $\rho(B) < 1$, and $0 \leq a_{k+1} \leq a_k \leq 1$, then
$$A \;=\; I \;+\; \sum_{k=1}^{m} a_k B^k \ \ \underline{\text{is a mime.}}$$
- Let $B \geq 0$. Then e^{tB} $\underline{\text{is a mime}}$ $\forall\, t \in [0,\ 1/\rho(B))$.
- Let $A \in \mathcal{M}_n(\mathbb{R})$ be an **H-matrix with positive diagonal entries**. Then A $\underline{\text{is a mime}}$.
- Mimes are **closed under ppt**.
- Mimes satisfy Johnson’s factorization.

In conclusion

Motivated by applications, we looked at a few P-matrix issues:

- Construction/Generation (row hull theorem, factorization).
- Detection of P-matrices (two algorithms).
- Invariance of $\mathbb{P}$ (principal pivot transforms).

Not discussed but important in this context: Eigenvalues and generalizations of P-matrices.

Report (construction techniques & survey of invariance):

`www.math.wsu.edu/math/faculty/tsat/files/pcon.pdf`

This talk :

`www.math.wsu.edu/math/faculty/tsat/files/ilas02.pdf`