

# Noncirculant Toeplitz Matrices All of Whose Powers are Toeplitz

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## Abstract

Let  $a, b$  and  $c$  be fixed complex numbers. Let  $M_n(a, b, c)$  be the  $n \times n$  Toeplitz matrix all of whose entries above the diagonal are  $a$ , all of whose entries below the diagonal are  $b$ , and all of whose entries on the diagonal are  $c$ . For  $1 \leq k \leq n$ , each  $k \times k$  principal minor of  $M_n(a, b, c)$  has the same value. We find explicit and recursive formulae for the principal minors and characteristic polynomial of  $M_n(a, b, c)$ . We also show that all complex polynomials in  $M_n(a, b, c)$  are Toeplitz matrices. In particular, the inverse of  $M_n(a, b, c)$  is a Toeplitz matrix when it exists.

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## 1 Introduction

For each positive integer  $n$  and for all  $a, b, c \in \mathbb{C}$ , let  $M_n(a, b, c)$  denote the  $n \times n$  Toeplitz matrix with all entries above the diagonal equal to  $a$ , all entries below the diagonal equal to  $b$ , and all entries on the diagonal equal to  $c$ . Thus, for

example,

$$M_3(a, b, c) = \begin{bmatrix} c & a & a \\ b & c & a \\ b & b & c \end{bmatrix}.$$

Using the observation that each  $k \times k$  principal minor of  $M_n(a, b, c)$  is just  $M_k(a, b, c)$ , in **Section 2**, we show that  $\det(M_n(a, b, c))$  satisfies a linear recurrence relation. We solve that relation to obtain a simple formula for the determinant of  $M_n(a, b, c)$  and to obtain the characteristic polynomial of  $M_n(a, b, c)$ . We also study the sequence of principal minors for  $M_n(a, b, c)$  for special choices of  $a$ ,  $b$  and  $c$ . In **Section 3**, we show that every positive integer power of  $M_n(a, b, c)$  is a Toeplitz matrix, and consequently, that every complex polynomial in  $M_n(a, b, c)$  is a Toeplitz matrix. In particular, when  $M_n(a, b, c)$  is invertible, its inverse is a Toeplitz matrix.

## 2 The principal minor sequence and the characteristic polynomial for $M_n(a, b, c)$

For a matrix  $A$ ,  $A(1|1)$  will denote the column vector obtained by deleting the first entry of the first column of  $A$ .  $A[1|1]$  will denote the row vector obtained by deleting the first entry from the first row of  $A$ .  $A(1)$  will denote the principal submatrix obtained from  $A$  by deleting the first row and the first column of  $A$ .

**Lemma 1** *Let  $a, b, c \in \mathbb{C}$ . For each positive integer  $n$ , let  $M_n = M_n(a, b, c)$ . Then  $\det(M_1) = c$ ,  $\det(M_2) = c^2 - ab$ , and for  $n \geq 3$ ,*

$$\det(M_n) = (2c - a - b) \det(M_{n-1}) - (a - c)(b - c) \det(M_{n-2}).$$

**Proof.** Let  $n \geq 3$ . Let  $H$  be obtained from  $M_n$  by performing two elementary operations: Subtract the second row of  $M_n$  from the first row of  $M_n$ , and subtract the second column of the resulting matrix from the first column of the resulting matrix. Thus

$$H = \left[ \begin{array}{c|cccc} 2c - a - b & a - c & 0 & 0 & \cdots & 0 \\ \hline b - c & & & & & \\ 0 & & & & & \\ 0 & & & & & \\ \vdots & & & & & \\ 0 & & & & & \end{array} \right]$$

and  $\det(M_n) = \det(H)$ . Apply minor-cofactor expansion to the first row of  $H$  and then to the first column of the  $H(1|2)$ . ■

Solving the second order linear recursion for the determinant in the previous lemma yields

**Theorem 2** Let  $a, b, c \in \mathbb{C}$ . For each positive integer  $n$ , let  $M_n = M_n(a, b, c)$ . If  $a = b = c$ , then  $\det(M_1) = c$ , and  $\det(M_n) = 0$  for  $n \geq 2$ . If  $a = b \neq c$ , then for  $n \geq 1$ ,

$$\det(M_n) = [c + a(n-1)](c-a)^{n-1},$$

and  $M_n$  is nonsingular unless  $n = 1 - c/a$ , in which case,  $M_n$  is singular. If  $a \neq b$ , then for  $n \geq 1$ ,

$$\det(M_n) = \frac{b}{b-a}(c-a)^n - \frac{a}{b-a}(c-b)^n,$$

and  $M_n$  is nonsingular unless

$$b(c-a)^n = a(c-b)^n.$$

**Proof.** It is well-known that the second order linear recurrence  $a_k = pa_{k-1} + qa_{k-2}$  for  $k \geq 3$ , where  $p$  and  $q$  are constants, with initial conditions  $a_1$  and  $a_2$  specified, has a unique solution. The solution is obtained as follows. Let  $r_1$  and  $r_2$  be the roots of the quadratic  $x^2 - px - q = 0$ . When  $r_1 \neq r_2$ , the general solution is  $a_k = s_1(r_1)^{k-1} + s_2(r_2)^{k-1}$  where  $s_1$  and  $s_2$  are constants chosen so that  $a_k$  has the specified initial values  $a_1$  and  $a_2$ . When  $r_1 = r_2$ , let  $r$  denote the common root. If  $r \neq 0$ , then the general solution is  $a_k = [a_1 + s(k-1)]r^{k-1}$  where  $s = \frac{a_2}{r} - a_1$ . When  $r = 0$ , it follows that  $p = q = 0$ , and we have  $a_k = 0$  for  $k \geq 3$ .

From Lemma 1, we have  $p = 2c - a - b = (c-a) + (c-b)$  and  $q = -(c-a)(c-b)$ . Thus the quadratic is

$$x^2 - ((c-a) + (c-b))x + (c-a)(c-b) = 0.$$

Clearly, the roots are  $c-a$  and  $c-b$ , so the roots are distinct exactly when  $a \neq b$ . When  $a = b$ , the common value for the roots is  $r = c-a$ . It remains to examine the initial conditions. Direct substitution shows that  $a_1 = \det(M_1) = c$ , and  $a_2 = \det(M_2) = c^2 - ab$ . Using these initial conditions leads to the specified values of  $s_1$  and  $s_2$ .

The singularity conditions follow from simple algebra. ■

**Theorem 3** Let  $a, b, c \in \mathbb{C}$ . For  $n \geq 1$ , let  $p_n(x)$  denote the characteristic polynomial of  $M_n(a, b, c)$ . Then  $p_n(x)$  satisfies the recursion relationship

$$p_n(x) = (2x - 2c + a + b)p_{n-1}(x) - (x - a + c)(x - b + c)p_{n-2}(x)$$

with  $p_1(x) = x - c$  and  $p_2(x) = c^2 - ab$ . Alternatively,  $p_n(x)$  can be expressed as

$$p_n(x) = x^n - \sum_{k=1}^n (-1)^k \binom{n}{k} [\det(M_k(a, b, c))] x^{n-k}.$$

When  $a = b$ ,

$$p_n(x) = [x - c - a(n-1)](x - c + a)^{n-1}.$$

When  $a \neq b$ ,

$$p_n(x) = \frac{b}{b-a}(x+a-c)^n - \frac{a}{b-a}(x+b-c)^n.$$

**Proof.** Since  $p_n(x) = \det(xI_n - A) = \det(M_n(-a, -b, x-c))$ , apply Theorem 2 and simplify. The recurrence relationship is obtained from Lemma 1. Finally, the coefficients in the sum of powers of  $x$  come from the well known fact that the coefficient of  $x^{n-k}$  in the characteristic polynomial for the  $n \times n$  matrix  $A$  is, up to a factor of  $(-1)^k$ , the sum of all  $k \times k$  principal minors of  $A$ . Since each of the  $k \times k$  principal minors of  $M_n(a, b, c)$  has value  $\det(M_k(a, b, c))$ , and since there are  $\binom{n}{k}$  such minors, the result follows. ■

The following result is an immediate consequence of the well-known Gershgorin Circles Theorem:

**Theorem 4** *Let  $a, b, c \in \mathbb{C}$ . Let  $n$  be a positive integer. If  $\lambda$  is an eigenvalue of  $M_n(a, b, c)$ , then*

$$|\lambda - c| \leq (n-1) \max\{|a|, |b|\}.$$

*In particular, if  $|c| > (n-1) \max\{|a|, |b|\}$ , then  $M_n(a, b, c)$  is nonsingular.*

What can be said about the rank of  $M_n(a, b, c)$  when the matrix is singular? Observe that  $\text{rank}(M_n(a, b, c))$  must be  $n-1$  unless  $M_{n-1}(a, b, c)$  is also singular. This leads to the following result:

**Theorem 5** *Let  $a, b, c \in \mathbb{C}$ . For each positive integer  $n$ , let  $M_n = M_n(a, b, c)$ .*

- (i) If  $a = b = c$ , then  $\text{rank}(M_n) = 1$  if  $c \neq 0$ , and  $\text{rank}(M_n) = 0$  if  $c = 0$ .*
- (ii) If  $a = b \neq c$ , then  $\text{rank}(M_n) = n$  except when  $n = 1 - c/a$ , in which case,  $\text{rank}(M_n) = n - 1$ .*
- (iii) If  $a \neq b$ , then  $\text{rank}(M_n) = n$  unless*

$$b(c-a)^n = a(c-b)^n. \tag{1}$$

*If the equality holds, then  $\text{rank}(M_n) = n - 1$ .*

**Proof.** All but the last part of (iii) follow immediately from Theorem 2.

Suppose that equality (1) holds, that  $a \neq b$ , and that  $ab = 0$ . Then equality (1) forces  $c = 0$ , and the result follows from the fact that  $M_n$  is strictly triangular with either all entries below the diagonal or all entries above the diagonal nonzero.

Suppose that equality (1) holds, that  $a \neq b$ , and that  $ab \neq 0$ . Since  $c - a$  and  $c - b$  are distinct, it follows from equality (1) that  $c - a \neq 0$  and  $c - b \neq 0$ . Thus

$$\left(\frac{c-a}{c-b}\right)^n = \frac{b}{a} \neq 0.$$

If  $\text{rank}(M_n) < n - 1$ , then  $M_{n-1}(a, b, c)$  is singular, and hence,

$$\left(\frac{c-a}{c-b}\right)^{n-1} = \frac{b}{a} \neq 0.$$

Then

$$\left(\frac{c-a}{c-b}\right) = 1,$$

which implies  $a = b$ , a contradiction. ■

When are  $a$  and  $b$  themselves the roots of the recursion relationship for the determinant? Exactly when  $\{a, b\} = \{c-a, c-b\}$ . This is equivalent to  $a+b = c$ . We note several interesting cases when  $a + b = c$ .

**Lemma 6** *Let  $a \in \mathbb{C}$ . For each positive integer  $n$ , let  $N_n = M_n(a, -a, 0)$ . If  $a = 0$ , then  $\det(N_n) = 0$  for all  $n \geq 1$ . If  $a \neq 0$ , then*

$$\det(N_k) = \begin{cases} 0 & \text{if } k \text{ is odd} \\ a^k & \text{if } k \text{ is even.} \end{cases}$$

Finally, when  $a \neq 0$  and  $k$  is odd,  $\text{rank}(N_k) = k - 1$ , and the null space of  $N_k$  is spanned by the vector  $v = [1 \ -1 \ 1 \ -1 \ \dots \ 1 \ -1 \ 1]^T$ .

**Proof.** Applying Theorems 2 and 4 with  $b = -a$  and  $c = 0$  yields the formulae for  $\det(N_k)$  and the rank result. When  $k$  is odd, each odd numbered row of  $N_k$  contains an even number of  $-1$  entries, followed by 0, followed by an even number of 1 entries. Consequently, the alternating sum in the dot product of the row with  $v$  is zero. When  $k$  is even, the first and last entry of each even numbered row of  $N_k$  have opposite signs in the dot product with  $v$ , and hence cancel each other, leaving an even number of consecutive  $-1$  entries and an even number of consecutive 1 entries; thus the remaining terms produce an alternating sum summing to zero. ■

**Lemma 7** *Let  $\phi = \frac{1+\sqrt{5}}{2}$ , the golden ratio. For each positive integer  $n$ , let  $P_n = M_n(\phi, 1 - \phi, 1)$  and let  $Q_n = M_n(-\phi, \phi - 1, 0)$ . Then  $\det(P_n)$  is the  $(n+1)^{\text{st}}$  Fibonacci number  $F_{n+1}$ , and  $\det(Q_n)$  is the  $(n-1)^{\text{st}}$  Fibonacci number  $F_{n-1}$ , where the Fibonacci sequence is given its classical indexing starting with  $F_0 = 0$  and  $F_1 = F_2 = 1$ .*

**Proof.** For  $P_n$ , the choice of  $a, b$  and  $c$  yields  $p = q = 1$  in the proof of Theorem 2. So the recursion for  $\det(P_n)$  is

$$\det(P_n) = \det(Q_{n-1}) + \det(Q_{n-2}), \quad n \geq 3$$

with the initial conditions

$$\det(M_1) = 1 \quad \text{and} \quad \det(M_2) = 1 - \phi(1 - \phi) = 2.$$

For  $Q_n$ , the choice of  $a, b$  and  $c$  again yields  $p = q = 1$ , so we again get the Fibonacci recursion. This time the initial conditions are

$$\det(M_1) = 0 \quad \text{and} \quad \det(M_2) = 0 - (-\phi)(\phi - 1) = 1.$$

■

The well-known matrix generator for the Fibonacci numbers is

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^k = \begin{bmatrix} F_k & F_{k-1} \\ F_{k-1} & F_{k-2} \end{bmatrix}$$

where  $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$  is a Hessenberg Toeplitz matrix whose eigenvalues are  $\phi$  and  $-\phi$ . Thus the matrices  $M_n(\phi, 1 - \phi, 1)$  and  $M_n(-\phi, \phi - 1, 0)$  provide another connection between matrices, the Fibonacci sequence, and the golden ratio.

Which principal minor sequences  $s = (s_1, s_2, \dots, s_n)$  can be obtained from a matrix of the form  $M_n(a, b, c)$ ? Clearly, we must have  $s_1 = c$  and  $s_2 = c^2 - ab$ , and  $s_k = ps_{k-1} + qs_{k-2}$  for  $2 \leq k \leq n$  where  $p = 2c - a - b$  and  $q = -(a - c)(b - c)$ . Since the initial conditions together with  $p$  and  $q$  completely determine the sequences, what we are really asking is which 4-tuples  $(s_1, s_2, p, q)$  can be realized by appropriate choices of  $a, b$  and  $c$ . Since  $s_1 = c$  and  $s_2 = c^2 - ab$ , we must have  $ab = s_1^2 - s_2$ . Given  $p$ , we must have  $a + b = 2c - p$ . Finally, since

$$q = -(c - a)(c - b) = (c^2 - ab) - (2c - a - b)c = s_2 - ps_1,$$

the value for a realizable  $q$  is dependent on the choices for  $s_1, s_2$  and  $p$ . Specifically, we have shown that:

**Theorem 8** Given  $a_1, a_2, p, q \in \mathbb{C}$ , the linear recursion  $a_k = pa_{k-1} + qa_{k-2}$  for  $k \geq 2$  with initial conditions  $a_1$  and  $a_2$  can be realized as the sequence of principal minors for a matrix  $M_n(a, b, c)$  exactly when  $q = a_2 - pa_1$ . In this case, the linear recursion and the initial conditions are achieved by setting  $c = a_1$ , and by setting  $a$  and  $b$  to be the roots of  $x^2 + (p - 2a_1)x + (a_1^2 - a_2) = 0$ .

As a special but interesting case, we determine matrices all of whose principal minors of every order have the value  $x$  where  $x$  is an arbitrary complex number.

**Theorem 9** For all positive integers  $n$  and for all  $x \in \mathbb{C}$ , all of the principal minors of  $R_n = M_n(x, x - 1, x)$  are equal to  $x$ .

**Proof.** By Lemma 1, for  $n \geq 3$ ,

$$\det(R_n) = (2x - x - (x - 1)) \det(R_{n-1}) - (x - x)((x - 1) - x) \det(R_{n-2}) = \det(R_{n-1})$$

with  $\det(R_1) = x$  and  $\det(R_2) = x^2 - x(x - 1) = x$ . ■

**Remark 10** In [3] the inverse problem of constructing a matrix from its principal minors is considered. Under certain conditions, this problem has a solution that is produced by the algorithm pm2mat. When  $x \notin \{0, 1\}$ , the matrix  $M_n(x, x - 1, x)$  in Theorem 9 is (up to diagonal similarity and transposition) the output of the algorithm pm2mat in [3] when all principal minors are required to equal  $x$ .

Moreover, in agreement with the above comment,  $M_n(x, x - 1, x)$  and  $M_n(x - 1, x, x)$  are the only choices of matrices of the form  $M_n(a, b, c)$  with the property

that all principal minors equal  $x$ . Indeed, it must be that  $c = x$ ; enforcing the  $2 \times 2$  and  $3 \times 3$  principal minors be equal to  $x$  imposes that

$$ab = x(x - 1) \quad \text{and} \quad a + b = 2x - 1$$

whose only solutions are  $(a = x, b = x - 1)$  or  $(a = x - 1, b = x)$ .

Finally note, that by Theorem 8, the Fibonacci sequence cannot be obtained as  $F_n = M_n(a, b, c)$  for any  $a, b, c \in \mathbb{C}$ , since this indexing corresponds to the 4-tuple  $(1, 1, 1, 1)$ , and  $q \neq 1 - (1)(1)$ .

### 3 Powers of $M_n(a, b, c)$ are Toeplitz matrices

We begin this section by recalling some definitions and by stating several elementary results.

The  $n \times n$  matrix  $A$  is said to be *persymmetric* if  $J_n A^T J_n = A$  where  $J_n$  is the  $n \times n$  permutation matrix with ones on the cross-diagonal.

Observe that  $J_n = J_n^T = J_n^{-1}$ , and that if  $e_n$  denotes the  $n \times 1$  vector of ones, then  $J_n e_n = e_n$  and  $e_n^T J_n = e_n^T$ .

The  $n \times n$  matrix  $A = [a_{ij}]$  is said to be *Toeplitz* if there exist  $2n - 1$  scalars

$$a_{-n+1}, a_{-n+2}, \dots, a_{-1}, a_0, a_1, \dots, a_{n-2}, a_{n-1}$$

such that  $a_{ij} = a_{i-j}$ . That is, the entries on each diagonal of a Toeplitz matrix descending from left to right have a common value.

**Lemma 11** *Let  $A$  be a persymmetric matrix. Then  $A^k$  is persymmetric for every positive integer  $k$ . If  $A^{-1}$  exists, then  $A^k$  is persymmetric for every negative integer  $k$ .*

Note that every Toeplitz matrix is persymmetric. The following result is a partial converse.

**Lemma 12** [4, Lemma 1] *Let  $A$  be an  $n \times n$  persymmetric matrix with  $n \geq 2$ . Then  $A$  is a Toeplitz matrix if and only if  $A(1)$  is persymmetric.*

**Theorem 13** *For all positive integers  $n$ , for all polynomials  $p(x)$  in  $\mathbb{C}[x]$ , and for all  $a, b, c \in \mathbb{C}$ , the matrix  $p(M_n(a, b, c))$  is a Toeplitz matrix. In particular, all positive integer powers of  $M_n(a, b, c)$  are Toeplitz matrices. Further, if  $M_n(a, b, c)$  is invertible, then its inverse is a Toeplitz matrix.*

**Proof.** Since the set of  $n \times n$  Toeplitz matrices is a subspace of the set of  $n \times n$  complex matrices, it suffices to prove that each positive integer power of  $M_n(a, b, c)$  is a Toeplitz matrix in order to prove the result for polynomials in  $M_n(a, b, c)$ . Since the inverse of a matrix, when it exists, is a polynomial in the matrix, the result on inverses is clear. Since  $cI_n$  is a Toeplitz matrix,  $M_n(a, b, c)$  is a Toeplitz matrix if and only if  $M_n(a, b, c) - cI_n = M_n(a, b, 0)$

is a Toeplitz matrix. Since the  $k^{\text{th}}$  power of  $M_n(a, b, c)$  is a polynomial in  $I_n$  and positive integer powers of  $M_n(a, b, 0)$ , it suffices to prove that all positive integers powers of  $M_n(a, b, 0)$  are Toeplitz matrices. If  $a \neq 0$ , then  $M_n(a, b, 0) = aM_n(1, b/a, 0)$ , and consequently, the  $k^{\text{th}}$  power of  $M_n(a, b, c)$  is a Toeplitz matrix if and only if the  $k^{\text{th}}$  power of  $M_n(1, b/a, 0)$  is a Toeplitz matrix. If  $a = 0$ , then  $M_n(0, b, 0) = bM_n(0, 1, 0)$ , and all powers of the nilpotent matrix  $M_n(0, 1, 0)$  are known to be Toeplitz matrices. Thus it suffices to prove that an arbitrary positive integer power of  $N = M_n(1, b, 0)$  is a Toeplitz matrix when  $b \neq 0$ .

Since  $A$  is a Toeplitz matrix,  $A$  and  $A(1)$  are persymmetric, and  $A^k$  is persymmetric for every positive integer  $k$ . We will use induction on  $k$  to prove that  $A^k$  is a Toeplitz matrix. Specifically, for each  $k$ , we will prove that  $A^k(1)$  is persymmetric and that  $bJ_{n-1}(A^k[1|1])^T = A^k(1|1)$ . Clearly, when  $k = 1$ ,  $A(1)$  is persymmetric by Lemma 11, and  $bJ_{n-1}(A[1|1])^T = bJ_{n-1}(e_{n-1}^T)^T = be_{n-1} = A(1|1)$ . Suppose that the induction hypothesis holds for  $k$ . Observe that

$$A = \begin{bmatrix} 0 & e_{n-1}^T \\ be_{n-1} & A(1) \end{bmatrix}$$

and that we can write  $A^k$  as

$$A^k = \begin{bmatrix} \alpha & u^T \\ v & M \end{bmatrix},$$

where  $\alpha \in \mathbb{C}$ ,  $u$  and  $v$  are  $(n-1) \times 1$  vectors and  $M = A^k(1)$ . By the induction hypothesis,  $bJ_{n-1}u = v$ , and  $M$  is persymmetric. Writing  $A^{k+1} = AA^k = A^kA$  gives

$$A^{k+1} = \begin{bmatrix} e^T v & e^T M \\ \alpha be + A(1)M & beu^T + A(1)M \end{bmatrix} = \begin{bmatrix} bu^T e & \alpha e^T + u^T A(1) \\ bMe & ve^T + MA(1) \end{bmatrix}.$$

Since  $M$  is persymmetric,  $JM^T = MJ$ . Thus

$$bJ_{n-1}(A^{k+1}[1|1])^T = bJ(e^T M)^T = bJM^T e = bMJe = bMe = A^{k+1}(1|1).$$

Next,

$$\begin{aligned} J(A^{k+1}(1))^T J &= J(beu^T + A(1)M)^T J \\ &= bJue^T J + JM^T(A(1))^T J \\ &= (bJu)e^T + MJ(A(1))^T J. \end{aligned}$$

Since  $A(1)$  is persymmetric and since, by the induction hypothesis,  $bJu = v$ ,

$$J(A^{k+1}(1))^T J = ve^T + MA(1) = A^{k+1}(1).$$

Thus  $A^{k+1}(1)$  is persymmetric. Thus the induction hypothesis holds for  $k+1$ . By the principle of induction, we have the desired result, that  $A^k(1)$  is persymmetric for all positive integers  $k$ . By applying Lemma 11, we conclude that  $A^k$  is a Toeplitz matrix for all positive integers  $k$ . ■

Note added just prior to publication: Theorem 13 also follows from Theorem 1.3 of [5].

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