On Matrices with Common Invariant Cones with Applications in Neural and Gene Networks

Roderick Edwards[1], Judith J. McDonald[2]
and Michael J. Tsatsomeros[2]

June 4, 2004

Abstract

Motivated by a differential continuous-time switching model for gene and neural networks, we investigate matrix theoretic problems regarding the relative location and topology of the dominant eigenvectors of words constructed multiplicatively from two matrices A and B. These problems are naturally associated with the existence of common invariant subspaces and common invariant proper cones of A and B. The commuting case and the two dimensional case are rich and considered analytically. We also analyze and recast the problem of the existence of a common invariant polyhedral cone in a multilinear framework, as well as present necessary conditions for the existence of low dimensional common invariant cones.

Keywords: Invariant cone; proper cone; invariant subspace; matrix word; nonnegative matrix; Perron-Frobenius; tensor product; compound matrix; exterior product; decomposable vector; dominant eigenvector; Glass network; gene network; neural network; Cantor set.

AMS Subject Classifications: 15A48, 15A75, 15A69, 37N25, 47A15, 92B20, 92C45.

---

[1] Department of Mathematics and Statistics, Univ. of Victoria, Victoria, British Columbia, Canada V8W 3P4 (`edwards@math.uvic.ca`).

[2] Mathematics Department, Washington State University, Pullman, Washington 99164-3113, U.S.A. (`jmcdonald@math.wsu.edu, tsat@math.wsu.edu`).

1 Introduction

In this paper, we describe and investigate matrix theoretic problems that arise in the study of the dynamics of Glass networks. These are continuous-time switching networks in the form of systems of differential equations that are used to model gene regulatory networks as well as neural networks. The emerging questions regard the relative location of the dominant eigenvectors (*i.e.*, of the eigenvectors corresponding to eigenvalues of maximal modulus) of matrix words constructed multiplicatively from two matrices A and B. Naturally, the fundamental theoretical questions of whether A, B have a nontrivial common invariant set or a nontrivial common invariant (proper) cone are of particular interest.

Glass networks, as well as how the problems alluded to above arise, are described in detail in Section 2. Sections 3 and 4 contain notation, definitions and preliminary observations regarding invariant sets and cones. In Section 5 we consider the topological properties of the set of all words of finite length constructed from two matrices, as well as other sets of interest to our analysis, associated with the dominant eigenspaces of words. In Section 6, we consider the case of commuting matrices. The problem of common invariant proper cones is examined for 2×2 matrices in Section 7. Finally, the case of common invariant polyhedral cones is analyzed and recast in a multilinear framework in Section 8.

2 Glass networks

Glass networks are a class of differential equation systems in which the interactions between variables depends only on whether they are above or below a threshold. Originally proposed as a simplified model for gene regulatory networks, they have been used in a more general context of chemical kinetics as well as neural networks [6, 7, 13]. Although a simplified approach to modeling concentrations of gene products in a cell, genes are commonly described in binary terms as 'active' or 'inactive', their activity dependent on the presence or absence of protein products of other genes' activity. Chemical concentrations are of course continuous variables evolving continuously in time, so the Glass network equations are differential equations of the following form:

$$\dot{y}_i = -y_i + F_i(\tilde{y})\,, \quad \tilde{y} = (\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_n)^T\,, \quad i = 1, \ldots, n\,, \tag{2.1}$$

where

$$\tilde{y}_i = \begin{cases} 0 & \text{if } y_i < 0 \\ 1 & \text{if } y_i > 0 \end{cases}\,, \tag{2.2}$$

and $\cdot^T$ denotes matrix transposition. The functions F_i map $\{0,1\}^n$ to $\mathbb{R}$. In vector notation, we can write this as

$$\dot{y} = -y + F(\tilde{y})$$

with $y \in \mathbb{R}^n$. Non-zero thresholds and values of the thresholded variables other than 0 and 1 can be handled without difficulty in the same theoretical framework, but we will assume (2.1) and (2.2) here. Of course, the binary-valued interaction function is an idealization. A sigmoidal interaction would be more realistic, though the system of equations is then much less amenable to analysis. Sigmoidal interactions also occur in many neural network models, such as Hopfield-Cohen-Grossberg networks, which can be thought of as a subclass of (2.1) if the sigmoids are replaced by binary step functions. The step-function approximation is reasonable when the threshold-dependence is strong, *i.e.*, when the sigmoid functions are steep. The main advantage is the analytic leverage obtained from the approximation, as we outline below. For a more thorough treatment, see [3].

We will restrict ourselves for present purposes to a subclass of such networks in which

$$\operatorname{sgn}(F_i(\tilde{y}_1, \ldots, \tilde{y}_i = 0, \ldots, \tilde{y}_n)) = \operatorname{sgn}(F_i(\tilde{y}_1, \ldots, \tilde{y}_i = 1, \ldots, \tilde{y}_n)), \quad \forall i, \forall \tilde{y} \tag{2.3}$$

which states that the sign of F_i does not depend on $\tilde{y}_i$, for any variable i. This is true, for example, when there is no self-input in the network at all, *i.e.*, when F_i does not depend on $\tilde{y}_i$. We also assume that

$$F_i(\tilde{y}) \neq 0, \quad \forall i, \forall \tilde{y}. \tag{2.4}$$

Under these conditions, trajectories have an unambiguous direction across any boundary between orthants of phase space and always cross them transversally. These assumptions are again not crucial, but the situation can potentially be more complicated without them.

Solutions to the Glass network (2.1) and (2.2) take the form of straight line segments between orthant boundaries in phase space, each orthant corresponding to a sign structure of the state vector, y, where the solution follows each segment through an orthant, converging exponentially in time toward a 'focal point,' $f = (f_1, \ldots, f_n) = F(\tilde{y}) = (F_1, \ldots, F_n)(\tilde{y})$, defined by the values of $F_i, i = 1, \ldots, n$ for that orthant. If a focal point for an orthant lies in that orthant, then trajectories converge to that point, which is then an asymptotically stable fixed point for the network. If no focal point is in its own orthant, then solution trajectories never reach any focal points, as a switching takes place first, after which the trajectory approaches some other focal point, possibly making a sharp corner at the boundary crossing. Although the vector field is not well defined at orthant boundary crossings by (2.1) and (2.2), solution trajectories can be unambiguously extended from each side to include the boundary crossing under condition (2.3), at least as long as only one variable reaches zero at a time, which is the generic situation.

As long as a trajectory does not encounter a fixed point within an orthant, it consists of a sequence of orthant boundary crossings. The mapping between boundaries at the k^{th} leg of a trajectory is essentially a projection onto the next boundary, given by

$$y^{(k+1)} = M^{(k)} y^{(k)} = \frac{B^{(k)} y^{(k)}}{1 + (\psi^{(k)})^T y^{(k)}}, \tag{2.5}$$

where

$$B^{(k)} = I - \frac{f^{(k)} e_j^T}{f_j^{(k)}}, \quad \psi^{(k)} = \frac{-e_j}{f_j^{(k)}}, \tag{2.6}$$

j is the index of the variable that switches on exiting the k^{th} orthant along the trajectory, and e_j denotes the standard basis vector in $\mathbb{R}^n$. This mapping can be reduced to $n-1$ dimensions by removing the i^{th} column and j^{th} row of the matrix $B^{(k)}$ and the i^{th} element of all the vectors, where i is the index of the variable that switched on entry to the orthant.

A periodic orbit follows some cyclic sequence of orthant boundary crossings returning to the starting boundary, which we denote $\mathcal{O}$. On this $(n-1)$-dimensional boundary, we can define a Poincaré section and a return map for the periodic orbit. Fractional linear mappings in the form of (2.5) retain the same form under composition. Thus, the return map can be shown to be

$$y^{(k+1)} = My^{(k)} = \frac{Ay^{(k)}}{1 + \phi^T y^{(k)}} \,, \tag{2.7}$$

where $A \in \mathbb{R}^{(n-1)\times(n-1)}$ and $\phi \in \mathbb{R}^{n-1}$ are calculated from the mappings for each leg around the cycle. In particular, if $B = B^{(m-1)}B^{(m-2)}\dots B^{(0)}$, then $A = B|_{(i)}$, *i.e.*, B with the i^{th} row and column removed, where $y_i = 0$ on the boundary $\mathcal{O}$.

Whenever it is possible to exit from an orthant by multiple boundaries (*i.e.*, more than one variable has the possibility of reaching zero first) only part of a given entry boundary to the orthant will map to each possible exit boundary. Extrapolating, only part of a boundary will contain trajectories that follow a given sequence of orthants and return to that boundary. In general, this domain of definition for a cycle map is a subset of the starting boundary, $\mathcal{O}$, defined by a set of linear inequalities, and we call it the returning cone, C, for the map. It can be calculated as follows:

$$C = \{y \in \mathcal{O} | Ry \geq 0\} \,, \tag{2.8}$$

where R is a matrix with one row for each alternate exit variable around the cycle, being the row vector

$$R_{i,\cdot} = -\frac{e_i^T}{f_i^{(k)}} B^{(k)}B^{(k-1)}\dots B^{(0)} \tag{2.9}$$

in each case. The cone C may in fact be empty, in which case no trajectories follow this cycle, or it may be all of $\mathcal{O}$. In general, $M : C \rightarrow \mathcal{O}$. Note that M will not usually map C into C.

For the study of periodic orbits, the most important result from the literature on these networks is the following proposition (for details, see [3]).

Proposition 2.1 *Consider a Glass network and a particular cyclic sequence of orthants consistent with the direction of flow across orthant boundaries. There is a periodic orbit through this cycle of orthants if and only if the return map, M in* (2.7)*, for that cycle has both of the following properties:*

(1) *Some real eigenvector v_i of A lies in C, the returning cone for M.*
(2) *The eigenvalue corresponding to v_i is $\lambda_i > 1$.*

The periodic orbit is stable if and only if the return map M satisfies the additional condition:

(3) $\lambda_i \geq |\lambda|$ *for all eigenvalues λ of A.*

The periodic orbit is asymptotically stable if strict inequality holds in the third condition. The fixed point of the return map corresponding to the periodic orbit is in fact a particular multiple of the eigenvector v_i, given by

$$y^* = \frac{(\lambda_i - 1)v_i}{\phi^T v_i} \,. \tag{2.10}$$

The import of this result is that in order for a stable periodic orbit to exist, a dominant eigenvector of the appropriate cycle's matrix A must lie in a particular location in $\mathbb{R}^{n-1}$, namely in the returning cone, C, on which the cycle mapping is defined. This requirement is useful when trying to prove *non-existence* of stable periodic orbits in particular Glass networks. Some Glass networks, even in only four dimensions, have apparently chaotic dynamics but others have very long and complex stable limit cycles [3, 13, 15]. Proofs of the existence of chaotic attractors in these networks have been accomplished partly by showing that no stable periodic orbit can exist in some 'trapping region' [4].

In terms of the above description, this means that on a particular orthant boundary, two (or more) cycles (sequences of orthants returning to the same boundary) are possible. Let the two cycles be denoted A and B and suppose $T = T_A \cup T_B$ is the union of two polyhedral cones on this orthant boundary such that the Poincaré maps M_A and M_B are defined on T_A and T_B respectively. Suppose also that T is a trapping region in the sense that $M_A(T_A) \subseteq T$ and $M_B(T_B) \subseteq T$. Then trajectories beginning in T must follow some sequence of cycles A and B for all future time. Even if there is no stable periodic orbit for A or for B in T, as can be checked by applying Proposition 2.1, there may be some more complex periodic orbit involving multiple circuits of cycles A and B, *i.e.*, some repeating symbolic sequence of A's and B's. Since the composition of the fractional linear maps in (2.7) involve multiplication of the matrices in the maps, we can also consider the symbolic sequence of A's and B's as a matrix product: If

$$M_A(y) = \frac{Ay}{1 + \phi_A^T y} \quad \text{and} \quad M_B(y) = \frac{By}{1 + \phi_B^T y}$$

then

$$M_A(M_B(y)) = \frac{ABy}{(1 + \phi_A^T M_B(y))(1 + \phi_B^T y)} \, .$$

Thus, to prove that there is no stable periodic orbit through T, it is required to show that no matrix word W composed of A's and B's has a dominant eigenvector in T.

The question posed to matrix theorists, then, is: *What can we say about the location of (real) dominant eigenvectors of words in A and B ?* At first glance, the answer might be "not much", since there is no simple relation between the eigenvectors of A and B and their products in general. However, as eigenspaces are naturally related to invariant subspaces and cones, there are instances when such a relation can be tracked. The Perron-Frobenius theorem, for example, is a result of the right kind, saying that nonnegative matrices have an eigenvector corresponding to the spectral radius in the nonnegative orthant, and clearly products of nonnegative matrices are still nonnegative. So one approach is to use Perron-Frobenius theory and its generalization to proper cones to track common invariant cones. If such a cone can be found that is invariant for both A and B, then it is clearly invariant for every word W in A and B, so that every such word has a dominant eigenvector in this cone. In order to *preclude* the existence of such eigenvectors in other parts of $\mathbb{R}^n$, we need the stronger form of the Perron-Frobenius theorem, which guarantees that the dominant eigenvector in the invariant cone is unique. Then, the more we can restrict the region in which dominant eigenvectors of words lie, the better chance we have of concluding that they do not lie in our trapping region T.

Below, we explore several approaches to this problem, including the common invariant proper cone approach and an attempt to characterize the set of dominant eigenvectors of words itself, which we can do quite thoroughly in $\mathbb{R}^2$. The invariant cone problem can also be recast in a multilinear framework, which we outline in Section 8.

3 Notation, definitions and preliminaries

This section contains preliminary material on matrix invariant cones. For details and further bibliography, see [1, 11].

Let $S \subseteq \mathbb{R}^n$. The *convex hull* of S is denoted by $\mathrm{co}\,(S)$. The *complement* of S in $\mathbb{R}^n$ is denoted by $S^{\,\mathrm{c}}$. The topological *closure* of S is denoted by $\overline{S}$ and the topological interior of K by $\mathrm{int}\,K$.

The spectrum of $A \in \mathbb{C}^{n\times n}$ is denoted by $\sigma(A)$ and its *spectral radius* by $\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$. When $\lambda \in \sigma(A)$ and $|\lambda| = \rho(A)$, we call λ a *dominant* eigenvalue of A, and refer to the corresponding eigenvectors and generalized eigenspaces as *dominant.* We also refer to the direct sum of the dominant generalized eigenspaces of A as the *dominant subspace* of A.

Note that if the dominant subspace of A has dimension 1, then A has a unique dominant eigenvalue, which is also simple.

$\mathbb{R}^n_+$ denotes the *nonnegative orthant* in $\mathbb{R}^n$, *i.e.*, the set of all entrywise nonnegative vectors in $\mathbb{R}^n$.

A convex set $K \subseteq \mathbb{R}^n$ $(n > 1)$ is called a *cone* if $aK \subseteq K$ for all scalars $a \geq 0$. Equivalently, K is a cone if $K = S^{\,\mathrm{G}}$, where $S \subseteq \mathbb{R}^n$ and where $S^{\,\mathrm{G}}$ denotes the set of all linear combinations of elements of S with nonnegative coefficients. The set S is called a set of *generators* of K. If S is finite, K is called a *polyhedral cone.* It follows that K is a polyhedral cone if and only if

$$K = X\mathbb{R}^k_+ \quad \text{for some} \quad X \in \mathbb{R}^{n\times k}.$$

When X is square and nonsingular, we refer to $K = X\mathbb{R}^n_+$ as a *simplicial* cone. The *dimension* of a cone $K \subseteq \mathbb{R}^n$ is defined as $\dim \mathrm{span}\, K$.

When a cone $K \subseteq \mathbb{R}^n$ is *pointed* $(K \cap (-K) = \{0\})$, *solid* $(\mathrm{int}\, K \neq \emptyset)$ and *reproducing* $(\mathrm{span}\, K = \mathbb{R}^n)$, we refer to K as a *proper* cone.

Given a cone $K \in \mathbb{R}^n$ and a matrix $A \in \mathbb{R}^{n\times n}$, we say that K is A*-invariant* if $AK \subseteq K$; equivalently, we say that A leaves K invariant or that A is K*-nonnegative.* We refer to A as K*-positive* if $A(K \setminus \{0\}) \subseteq \mathrm{int}\, K$.

Entrywise ordering of arrays of the same size is indicated by $\geq$. We write $A > B$ if A, B are real and every entry of $A - B$ is positive. When $A \geq 0$ $[\,A > 0\,]$, we refer to A as *nonnegative* [*positive*].

The prototypical (polyhedral, proper) cone is $\mathbb{R}^n_+$. Notice that $\mathbb{R}^n_+$ is A-invariant for every $n \times n$ matrix $A \geq 0$ and conversely, every matrix that leaves $\mathbb{R}^n_+$ invariant is nonnegative.

The spectral consequences of K-nonnegativity when K is a proper cone are examined in the next two classical theorems. Recall that the *degree* of an eigenvalue λ of A, $\deg \lambda$, is the degree of λ as a root of the minimal polynomial of A.

Theorem 3.1 [Perron-Frobenius] *Let $A \in \mathbb{R}^{n\times n}$ and $K \subseteq \mathbb{R}^n$ be a proper cone. If $AK \subseteq K$, then*

(a) $\rho(A) \in \sigma(A)$ *and* $\deg \rho(A) \geq \deg \lambda$ *for every* $\lambda \in \sigma(A)$ *with* $|\lambda| = \rho(A)$.

(b) *K contains an eigenvector of A corresponding to $\rho(A)$.*

Moreover, if A is K-positive, then $\rho(A)$ is a simple eigenvalue of A greater in modulus than any other eigenvalue of A, and $\operatorname{int} K$ *contains a unique (up to scalar multiples) eigenvector of A, which corresponds to $\rho(A)$.*

A converse to the Perron-Frobenius theorem holds.

Theorem 3.2 [Krein-Rutman] *Let $A \in \mathbb{R}^{n\times n}$. If $\rho(A) \in \sigma(A)$ and $\deg \rho(A) \geq \deg \lambda$ for any $\lambda \in \sigma(A)$ with $|\lambda| = \rho(A)$, then there exists a proper cone $K \subseteq \mathbb{R}^n$ such that $AK \subseteq K$.*

Moreover, if $\rho(A)$ is a simple eigenvalue of A greater in modulus than any other eigenvalue of A, then there exists a proper cone $K \subseteq \mathbb{R}^n$ such that A is K-positive.

When $\rho(A) \in \sigma(A)$ we refer to $\rho(A)$ as the *Perron eigenvalue* of A. If, in addition, $\deg \rho(A) \geq \deg \lambda$ for any $\lambda \in \sigma(A)$ with $|\lambda| = \rho(A)$, we say that A satisfies the *Perron condition.*

4 Invariance alternatives and double cones

In this section, we explore some alternatives to cone invariance that suit the purposes of the motivating applications. By the Perron-Frobenius theorem, if a matrix does not have a real positive dominant eigenvalue, there is no hope of a proper invariant cone. There is, however, a possibility for an invariant set constructed from a proper cone under the following circumstances. If A has eigenvalue $-\rho(A)$, then $-A$ has eigenvalue $\rho(A)$ and so $-A$ may have an invariant proper cone K. In that case the set $K \cup (-K)$ is A-invariant. We refer to $K \cup (-K)$ as a *double cone* and its invariance is examined next.

Proposition 4.1 *Let $A \in \mathbb{R}^{n\times n}$ and $K \subseteq \mathbb{R}^n$ be a proper cone such that* $\operatorname{Null}(A) \cap K = \{0\}$. *Then $A(K \cup (-K)) \subseteq K \cup (-K)$ if and only if either $AK \subseteq K$ or $-AK \subseteq K$.*

Proof. Suppose that $A(K \cup (-K)) \subseteq K \cup (-K)$. If $AK \not\subseteq K$ and $-AK \not\subseteq K$, then there exist nonzero vectors $x \in K$ and $y \in K$ such that $Ax \in (-K)$ and $-Ay \in (-K)$. As K is closed and pointed, there must exist $t \in [0, 1]$ such that $tAx + (1 - t)Ay = 0$; that is $A(tx + (1 - t)y) = 0$. Also as K is pointed and x and y are nonzero, $tx + (1 - t)y \neq 0$, contradicting the assumption that $\operatorname{Null}(A) \cap K = \{0\}$ and thus showing that either $AK \subseteq K$ or $-AK \subseteq K$. For the converse, if $AK \subseteq K$ or $-AK \subseteq K$, then clearly $A(K \cup (-K)) \subseteq (K \cup (-K))$. □

Remarks 4.2 Note the following regarding Proposition 4.1:

(1) For a double cone $K \cup (-K)$ with $\operatorname{Null}(A) \cap K = \{0\}$ to be A-invariant either $\rho(A)$ or $-\rho(A)$ must be an eigenvalue of A. Therefore, when the dominant eigenvalues of A are non-real, no proper cone nor any double cone intersecting the nullspace of A trivially can be A-invariant.

(2) We can state generalizations of the Perron-Frobenius and Krein-Rutman theorems for matrices that leave a double cone invariant. For instance, if $A(K\cup(-K)) \subseteq K\cup(-K)$ and $\operatorname{Null}(A)\cap K = \{0\}$ for some matrix $A \in \mathbb{R}^{n\times n}$ and a proper cone $K \subseteq \mathbb{R}^n$, then

(a) there exists $\hat{\lambda} \in \{\rho(A), -\rho(A)\}$ such that $\hat{\lambda} \in \sigma(A)$ and $\deg \hat{\lambda} \geq \deg \lambda$ for any $\lambda \in \sigma(A)$ with $|\lambda| = \rho(A)$, and

(b) K contains an eigenvector of A corresponding to $\hat{\lambda}$.

Observation 4.3 *Suppose* $A \in \mathbb{R}^{n \times n}$ *is invertible. Then* $X \subseteq \mathbb{R}^n$ *is* A*-invariant if and only if* X^{c} *is* (A^{-1})*-invariant.*

Proof. It is enough to prove one direction of this statement. Suppose $AX \subseteq X$. By way of contradiction, let $A^{-1}X^{\mathrm{c}} \not\subseteq X^{\mathrm{c}}$, that is, there exists $w \in X^{\mathrm{c}}$ such that $A^{-1}w \in X$. Then $w = A(A^{-1}w) \in AX \subseteq X$, a contradiction. $\square$

5 Dominant eigenvectors of words in matrices A and B

In this section we pursue directly our goal of studying the relative location of the dominant eigenvectors of words in two matrices $A, B \in \mathbb{R}^{n \times n}$. In the process, we also examine some topological aspects of the dominant eigenvectors. Guided by the context of our motivating application, we focus on cases where the dominant eigenvalues and eigenvectors are real. Note that a real matrix whose dominant eigenspace (as defined in Section 3) has dimension 1, must possess a unique dominant eigenvalue, which is therefore real and simple.

Some more definitions and notation are first in order. We let:

- $\mathcal{W}(A, B)$ be the set of matrix *words* in A, B; namely, all finite products of the form $X_1 X_2 \ldots X_k$, where $X_j \in \{A, B\}$ $(j = 1, 2 \ldots, k)$. By convention, we do not include $A^0 B^0$ in $\mathcal{W}(A, B)$, so that $I \notin \mathcal{W}(A, B)$ unless $A^k B^\ell = I$ for some k, ℓ not both 0.

- $\mathcal{E}(A, B)$ be the set containing all the *dominant real eigenvectors* of matrices in $\mathcal{W}(A, B)$.

- $\mathcal{F}(A, B)$ be the set containing all the dominant real eigenvectors of A and B, and their images under the members of $\mathcal{W}(A, B)$. That is,

$$\mathcal{F}(A, B) = \{Wx \; : \; W = I \;\text{ or }\; W \in \mathcal{W}(A, B) \;\text{ and } x \text{ is a dominant real eigenvector of } A \text{ or } B\}.$$

The following two observations are based on the fact that the sets in question are countable unions of countable sets.

Observation 5.1 *Let* $A, B \in \mathbb{R}^{n \times n}$. *Then* $\mathcal{W}(A, B)$ *is a countable subset of* $\mathbb{R}^{n \times n}$.

Observation 5.2 *Let* $A, B \in \mathbb{R}^{n \times n}$ *and suppose that the dominant subspaces of* A *and* B *have dimension 1. Then the set of unit vectors in* $\mathcal{F}(A, B)$ *is a countable subset of* $\mathbb{R}^n$. *If the dominant eigenspace of every* $W \in \mathcal{W}(A, B)$ *has dimension 1, then the set of unit vectors in* $\mathcal{E}(A, B)$ *is also a countable subset of* $\mathbb{R}^n$.

Proposition 5.3 *Let* $A, B \in \mathbb{R}^{n\times n}$. *Then*

(i) $\mathcal{F}(A,B)$ *and* $\overline{\mathcal{F}(A,B)}$ *are* A*-invariant and* B*-invariant sets.*
(ii) *For every* $z \in \mathcal{E}(A,B)$, *either* $Az \in \mathcal{E}(A,B) \cup \{0\}$ *or* $Bz \in \mathcal{E}(A,B) \cup \{0\}$.

Proof.
(i) The claims follow readily from the definition of $\mathcal{F}(A,B)$ and continuity of matrix multiplication.

(ii) Let $z \in \mathcal{E}(A,B)$ and suppose $Wz = \lambda z$, where $W \in \mathcal{W}(A,B)$ and $|\lambda| = \rho(W)$. There are two cases to consider: First, suppose the word W ends in A, namely, $W = W_1 A$ for some $W_1 \in \mathcal{W}(A,B)$. It follows that $AW_1Az = \lambda Az$. If $Az \neq 0$, then

$$\lambda \in \sigma(AW_1) = \sigma(W_1 A) = \sigma(W)$$

and thus $|\lambda| = \rho(AW_1)$. Consequently, Az is a dominant eigenvector of AW_1 and so $Ax \in \mathcal{E}(A,B)$. Second, suppose the word W ends in B. As above, it follows that either $Bz = 0$ or $Bz \in \mathcal{E}(A,B)$, completing the proof. □

Theorem 5.4 *Let* $A, B \in \mathbb{R}^{n\times n}$. *If* $\dim(\operatorname{span}\{\mathcal{F}(A,B)\}) = n$ *and if the dominant subspace of every* $W \in \mathcal{W}(A,B)$ *has dimension 1, then* $\mathcal{E}(A,B) \subseteq \overline{\mathcal{F}(A,B)}$.

Proof. Let $z \in \mathcal{E}(A,B)$. Then $z \in \mathbb{R}^n$ is the dominant eigenvector of some word $W \in \mathcal{W}(A,B)$, corresponding to the eigenvalue λ. Since $\dim(\operatorname{span}\{\mathcal{F}(A,B)\}) = n$, we can choose $u \in \mathcal{F}(A,B)$ such that u has a nonzero component in the direction of the dominant subspace of W. Then, per the power method (see *e.g.* [17]), and since by assumption the dominant eigenspace of W is spanned by z,

$$\lim_{k\to\infty} \frac{W^k u}{\lambda^k} \in \operatorname{span}\{z\}.$$

Since $u \in \mathcal{F}(A,B)$ and since $\mathcal{F}(A,B)$ is A-invariant and B-invariant, it follows that $\frac{W^k}{\lambda^k} u \in \mathcal{F}(A,B)$; hence $z \in \overline{\mathcal{F}(A,B)}$. □

Theorem 5.5 *Let* $A, B \in \mathbb{R}^{n\times n}$ *and suppose that the dominant subspaces of* A *and* B *have dimension 1. Then,* $\dim(\operatorname{span}\{\mathcal{F}(A,B)\}) < n$ *if and only if there exists invertible matrix* T *such that*

$$T^{-1}AT = C = \begin{bmatrix} C_{11} & C_{12} \\ 0 & C_{22} \end{bmatrix} \quad \text{and} \quad T^{-1}BT = D = \begin{bmatrix} D_{11} & D_{12} \\ 0 & D_{22} \end{bmatrix},$$

where C *and* D *are partitioned conformally,* $\rho(A) = \rho(C_{11})$, *and* $\rho(B) = \rho(D_{11})$.

Proof. Suppose $\dim(\operatorname{span}\{\mathcal{F}(A,B)\}) < n$. Let $z^{(1)} \in \mathbb{R}^n$ be a dominant eigenvector of A. If B possesses a dominant eigenvector $y \in \mathbb{R}^n$ that is linearly independent of $z^{(1)}$, set $z^{(2)} = y$. Consider a basis $\{z^{(1)}, z^{(2)}, \ldots, z^{(k)}\}$ for $\operatorname{span}\{\mathcal{F}(A,B)\}$. Extend this basis to a basis

$$\{z^{(1)}, z^{(2)}, \ldots, z^{(k)}, z^{(k+1)}, \ldots, z^{(n)}\}$$

for $\mathbb{R}^n$ and consider the $n \times n$ matrix

$$T = [z^{(1)}|z^{(2)}|\dots|z^{(k)}|z^{(k+1)}|\dots|z^{(n)}].$$

Since $\{z^{(1)}, z^{(2)}, \dots, z^{(k)}\}$ spans an A-invariant subspace and it contains the dominant eigenvector of A, $T^{-1}AT$ has the claimed partition, where in fact C_{11} is $k \times k$. The claim for $T^{-1}BT$ follows similarly.

Conversely, suppose there exists invertible matrix T such that $T^{-1}AT$ is partitioned as prescribed in the theorem. Assume that C_{11} and D_{11} are $k \times k$. Since T is nonsingular, its columns span all of $\mathbb{R}^n$. Label them as $z^{(1)}, z^{(2)}, \dots, z^{(k)}, z^{(k+1)}, \dots, z^{(n)}$. Let u be a dominant eigenvector of C_{11} so that $\begin{bmatrix} u \\ 0 \end{bmatrix}$ is an eigenvector for C. Since A and C are similar, $\rho(C_{11}) = \rho(A) = \rho(C)$. Thus, without loss of generality, we can assume

$$x^{(1)} = T \begin{bmatrix} u \\ 0 \end{bmatrix}.$$

Notice next that if $W \in \mathcal{W}(A,B)$, then

$$T^{-1}WT = V = \begin{bmatrix} V_{11} & V_{12} \\ 0 & V_{22} \end{bmatrix},$$

where V is partitioned conformally with C and D. But then

$$Wx^{(1)} = TVT^{-1}T \begin{bmatrix} u \\ 0 \end{bmatrix} = TV \begin{bmatrix} u \\ 0 \end{bmatrix} \in \text{span}\,\{z^{(1)}, z^{(2)}, \dots, z^{(k)}\}.$$

Similarly, $Wy^{(1)} \in \text{span}\,\{z^{(1)}, z^{(2)}, \dots, z^{(k)}\}$. Thus

$$\mathcal{F}(A,B) \subseteq \text{span}\,\{z^{(1)}, z^{(2)}, \dots, z^{(k)}\}$$

and hence $\dim(\text{span}\,\{\mathcal{F}(A,B)\}) \leq k < n$. □

Definition 5.6 Let $A, B \in \mathbb{R}^{n \times n}$. We call A and B *simultaneously conformally reducible* if there exists invertible matrix T such that

$$T^{-1}AT = C = \begin{bmatrix} C_{11} & C_{12} \\ 0 & C_{22} \end{bmatrix} \quad \text{and} \quad T^{-1}BT = D = \begin{bmatrix} D_{11} & D_{12} \\ 0 & D_{22} \end{bmatrix},$$

where C, D are partitioned conformally, namely, $C_{11}, D_{11} \in \mathbb{R}^{k \times k}$ for a positive integer $k < n$.

Remark 5.7 Note that A and B are simultaneously conformally reducible if and only if A, B have a nontrivial common invariant subspace of dimension $k < n$.

Theorem 5.8 *Let $A, B \in \mathbb{R}^{n \times n}$. If the dominant subspaces of A and B have dimension 1 and if A and B are not simultaneously conformally reducible , then $\mathcal{F}(A,B) \subseteq \overline{\mathcal{E}(A,B)}$.*

Proof. Let $z \in \mathcal{F}(A,B)$. Without loss of generality, $z = Wx$, where $W \in \mathcal{W}(A,B)$ and x is a dominant eigenvector of A corresponding to λ. Let $\mathcal{S} = \{VWx \ : \ V \in \mathcal{W}(A,B)\}$. Notice that $\mathcal{S}$ is A-invariant and B-invariant. Note also that $\dim(\operatorname{span}\mathcal{S}) = n$; otherwise, by Remark 5.7, A and B would be simultaneously conformally reducible . Hence, there exists $v = VWx \in \mathcal{S}$ such that v has a nonzero component in the direction of the dominant subspace of A. Therefore

$$\lim_{k\to\infty} \frac{A^k v}{\lambda^k} = \lim_{k\to\infty} \frac{A^k VWx}{\lambda^k} \in \operatorname{span}\{x\}.$$

But then, the dominant eigenvectors of the words WA^kVW $(k = 1, 2, \ldots)$ approach $Wx = z$ as k approaches infinity. Thus $\mathcal{F}(A,B) \subseteq \overline{\mathcal{E}(A,B)}$. $\square$

The following results are consequences of Theorems 5.4, 5.5 and 5.8.

Corollary 5.9 *Let $A, B \in \mathbb{R}^{n\times n}$. If A, B are not simultaneously conformally reducible and if the dominant subspace of every $W \in \mathcal{W}(A,B)$ has dimension 1, then $\overline{\mathcal{E}(A,B)} = \overline{\mathcal{F}(A,B)}$.*

Corollary 5.10 *Let $A, B \in \mathbb{R}^{n\times n}$. If A, B are not simultaneously conformally reducible and if the dominant subspace of every $W \in \mathcal{W}(A,B)$ has dimension 1, then $\overline{\mathcal{E}(A,B)}$ is A-invariant and B-invariant.*

One way to show that dominant eigenvectors of words in A and B are localized is to find a common invariant proper cone, K, for $\pm A$ and $\pm B$. We have found some properties of the set of dominant eigenvectors itself, $\mathcal{E}(A,B)$, particularly in relation to the set $\mathcal{F}(A,B)$ of images of the dominant eigenvectors of A and B themselves. Now we show a relationship between these sets and $K \cup (-K)$, by considering images of K under multiplication by A and B.

Proposition 5.11 *Suppose $A, B \in \mathbb{R}^{n\times n}$ have a common invariant proper cone, K. Let*

$$F_0 = K\,, \quad F_j = AF_{j-1} \cup BF_{j-1}\,, \quad \text{for } j \geq 1 \quad \text{and}$$

$$F_\infty = \bigcap_{j=0}^{\infty} F_j\,. \tag{5.1}$$

Then $\mathcal{F}(A,B) \subseteq F_\infty \cup (-F_\infty)$.

Proof. First we show that $F_{j+1} \subseteq F_j$, by induction. $AK \subseteq K$ and $BK \subseteq K$, so $F_1 \subseteq F_0$. Suppose $F_j \subseteq F_{j-1}$. Then $F_{j+1} = AF_j \cup BF_j \subseteq AF_{j-1} \cup BF_{j-1} = F_j$.

Now, consider an arbitrary element of $\mathcal{F}(A,B)$, say $Wx^{(1)}$, where $x^{(1)}$ is a dominant real eigenvector of A. Since by the Perron-Frobenius theorem and without loss of generality $x^{(1)} \in K$, $Wx^{(1)} \in WK$; this is to say that $Wx^{(1)} \in F_{|W|}$, where $|W|$ is the length of the word W, because F_j is by definition the image of K under words of length j. By the chain of inclusion established above, $Wx^{(1)} \in F_j\,, \forall j \leq |W|$. But also, $Wx^{(1)} \in WA^mK\,, \forall m \geq 0$ since $A^m x^{(1)} = \lambda_1^m x^{(1)}$ so that $Wx^{(1)} \in F_j\,, \forall j \geq |W|$. Thus, $Wx^{(1)} \in F_j\,, \forall j \geq 0$ and $Wx^{(1)} \in F_\infty$. This is true for any member

$Wx^{(1)}$ or $Wy^{(1)}$ of $\mathcal{F}(A,B)$, where $x^{(1)}$ or $y^{(1)}$ (a dominant real eigenvector of B) may be in K or $-K$. Hence $\mathcal{F}(A,B) \subseteq F_\infty \cup (-F_\infty)$. □

In Section 7 we show that, at least in $\mathbb{R}^2$, $\overline{\mathcal{F}(A,B)} = F_\infty \cup (-F_\infty)$ and this is either all of $K \cup (-K)$ or a Cantor set contained in $K \cup (-K)$, though similar results should hold in $\mathbb{R}^n$, $n > 2$.

6 The commuting case

In this section, we consider $n \times n$ real commuting matrices A, B. Recall that this is equivalent to saying that A and B are simultaneously diagonalizable (and hence A and B are "highly" simultaneously conformally reducible). In this instance, there exists a matrix P such that $C = P^{-1}AP$ and $D = P^{-1}BP$ are diagonal matrices.

Theorem 6.1 *Let $A, B \in \mathbb{R}^{n\times n}$ be commuting matrices such that their dominant subspaces have dimension 1. If A and B have the same dominant eigenvector, say x, then $\mathcal{E}(A,B) = \mathcal{F}(A,B) =$* span $\{x\} \setminus \{0\}$.

Proof. We can always choose P so that the first column of P is x, $P^{-1}AP = C$ and $P^{-1}BP = D$, where $c_{11} > c_{ii}$ and $d_{11} > d_{ii}$ for all $1 < i \leq n$. But then every word in A and B can be written as $PC^kD^\ell P^{-1}$ for appropriate choices of k and ℓ. The largest entry in magnitude of C^kD^ℓ is the (1,1)-entry, and the corresponding eigenvector for the word in A and B is the first column of P, which is x. Thus $\mathcal{E}(A,B) = \text{span}\,\{x\} \setminus \{0\}$. Since $Ax = c_{11}x$ and $Bx = d_{11}x$, clearly $\mathcal{F}(A,B) = \text{span}\,\{x\} \setminus \{0\}$. □

Theorem 6.2 *Let $A, B \in \mathbb{R}^{n\times n}$ commuting matrices such that their dominant subspaces have dimension 1. Let $z_1, z_2, \ldots z_n$ be the common eigenvectors of A and B. If A and B have linearly independent eigenvectors associated with their dominant eigenvalues (say, z_1 and z_n, respectively), then $\mathcal{F}(A,B) = (\text{span}\,\{z_1\} \cup \text{span}\,\{z_n\}) \setminus \{0\}$, and there exist $J_1, J_2, \ldots, J_\ell \subseteq \{1,2,\ldots,n\}$ such that*

$$\mathcal{E}(A,B) = \bigcup_{i=1}^{\ell} \left\{ \text{span} \bigcup_{j\in J_i} \{z_j\} \right\} \setminus \{0\}.$$

Proof. Let $P = [z_1|z_2|\ldots|z_n]$, $C = P^{-1}AP$ and $D = P^{-1}BP$. Since $Az_i = c_{ii}z_i$ and $Bz_i = d_{ii}z_i$, it follows that

$$\mathcal{F} = (\text{span}\,\{z_1\} \cup \text{span}\,\{z_n\}) \setminus \{0\}.$$

Notice that the eigenvalues of any word in A and B are $c_{11}^k d_{11}^\ell, c_{22}^k d_{22}^\ell, \ldots, c_{nn}^k d_{nn}^\ell$ for some k and ℓ. Hence the eigenspace associated with the dominant eigenvalue(s) is of the form

$$\text{span} \bigcup_{j\in J} \{z_j\}$$

for some $J \subseteq \{1,2,\ldots,n\}$. Hence $\mathcal{E}(A,B)$ must be as claimed. □

Next we illustrate some of the variety of possibilities for $\mathcal{E}(A,B)$, when A and B are both diagonal matrices. These examples show that when A and B are simultaneously conformally reducible, then the theorems from the previous section need *not* hold.

In all three examples to follow, we let $e_1, e_2, \ldots, e_n$ be the standard basis for $\mathbb{R}^n$ and set A to be the diagonal matrix with $a_{ii} = 1/i$ for $i = 1, 2, \ldots, n$.

Example 6.3 First, consider the diagonal matrix B with $b_{11} = 1$, $b_{ii} = 1/i$ for $2 \leq i \leq n-1$. If n is a power of 2, set $b_{nn} = 3n$; otherwise set $b_{nn} = 2n$. Then the dominant eigenvalue of any word in A and B is either the (1,1)-entry or the (n,n)-entry (but not both) of that word; hence the associated eigenvector is either z_1 or z_n. Thus $\mathcal{E}(A,B) = (\operatorname{span}\{e_1\} \cup \operatorname{span}\{e_n\}) \setminus \{0\}$. In particular, $\mathcal{E}(A,B) = \mathcal{F}(A,B)$.

Example 6.4 Let $\{1,n\} \subseteq J \subseteq \{1,2,\ldots,n\}$ and consider the diagonal matrix B with $b_{ii} = i$ for $i \in J$ and $b_{ii} = 1/i$ for $i \notin J$. Then the dominant eigenvalue of AB is 1 and it occurs on the diagonal in each position listed in J. Hence

$$(\operatorname{span} \bigcup_{j \in J} \{e_j\}) \setminus \{0\} \subseteq \mathcal{E}(A,B).$$

Since the $(1,1)$-entry of every word in A and B equals 1 and the (k,k)-entry for $k \notin J$ is always less than 1, we see that the eigenvectors z_k associated with $k \notin J$ can never correspond to dominant eigenvalues. Thus

$$\mathcal{E}(A,B) = (\operatorname{span} \bigcup_{j \in J} \{e_j\}) \setminus \{0\}.$$

Notice that in this instance the dominant eigenspaces of the words in A and B need not have dimension 1, and $\mathcal{E}(A,B) \not\subseteq \overline{\mathcal{F}(A,B)}$. When $n = 2$, we even have that $\dim(\operatorname{span}\{\mathcal{F}(A,B)\}) = n$.

Example 6.5 Lastly, let $\{2,n\} \subset J \subseteq \{2,\ldots,n\}$. Consider the diagonal matrix B with $b_{ii} = 2i$ for $i \in J$, and $b_{ii} = 1/i$ for $i \notin J$. Then the dominant eigenvalue of AB is 2 and it occurs on the diagonal in each position listed in J. Hence

$$(\operatorname{span} \bigcup_{j \in J} \{e_j\}) \setminus \{0\} \subseteq \mathcal{E}(A,B).$$

For A^2B, however, the dominant eigenvalue is 1 and it occurs in the $(1,1)$ and the $(2,2)$ positions. Since the $(1,1)$-entry of every word in A and B is 1 and the (k,k)-entry for every other $k \notin J$ is always less than 1, we see that the eigenvectors e_k associated with these $k \notin J$ can never correspond to dominant eigenvalues. Thus

$$\mathcal{E}(A,B) = \left(\operatorname{span}\{e_1, e_2\} \bigcup \operatorname{span} \bigcup_{j \in J} \{e_j\} \right) \setminus \{0\}.$$

Notice that $\mathcal{E}(A,B) \cup \{0\}$ is the union of two subspaces with nontrivial intersection.

7 The 2-dimensional case

In this section, we let A and B be 2×2 matrices whose eigenvalues are real and distinct, and the dominant eigenvalue of each matrix is positive. We denote the eigenvectors and eigenvalues of A and B as follows:

$$Ax^{(1)} = \lambda_1 x^{(1)},\ Ax^{(2)} = \lambda_2 x^{(2)},\quad \lambda_1 > 0,\quad |\lambda_1| > |\lambda_2|.$$

$$By^{(1)} = \mu_1 y^{(1)},\ By^{(2)} = \mu_2 y^{(2)},\quad \mu_1 > 0,\quad |\mu_1| > |\mu_2|.$$

Let $P = [x^{(1)}|x^{(2)}]$ and let $Q = [y^{(1)}|y^{(2)}]$. Then

$$P^{-1}AP = C = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad \text{and} \quad Q^{-1}BQ = D = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix}.$$

Observation 7.1 *The matrices A and $B \in \mathbb{R}^{2\times 2}$ are not simultaneously conformally reducible if and only if any two vectors from the set $\{x^{(1)}, x^{(2)}, y^{(1)}, y^{(2)}\}$ are linearly independent.*

Proof. We prove the equivalence of the contrapositives. Notice the following equivalences:

There exists an invertible matrix T such that

$$T^{-1}AT = U = \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix} \quad \text{and} \quad T^{-1}BT = V = \begin{bmatrix} v_{11} & v_{12} \\ 0 & v_{22} \end{bmatrix}$$

$\iff$ u_{11} is an eigenvalue of A whose eigenvector corresponds to the first column of T and v_{11} is an eigenvalue of B whose eigenvector corresponds to the first column of T

$\iff$ A and B have a common eigenvector

$\iff$ two vectors in the set $\{x^{(1)}, x^{(2)}, y^{(1)}, y^{(2)}\}$ are linearly dependent. $\square$

Let $T = Q^{-1}P$ and $R = P^{-1}Q$. Then $P = QT$ and $Q = PR$, hence

$$\begin{aligned} y^{(1)} &= r_{11}x^{(1)} + r_{21}x^{(2)}, \\ y^{(2)} &= r_{12}x^{(1)} + r_{22}x^{(2)}, \\ x^{(1)} &= t_{11}y^{(1)} + t_{21}y^{(2)}, \\ x^{(2)} &= t_{12}y^{(1)} + t_{22}y^{(2)}. \end{aligned}$$

Assuming $x^{(1)}$ and $y^{(1)}$ are linearly independent, we can also write any vector z as $z = \alpha x^{(1)} + \beta y^{(1)}$. Then we see that

$$\begin{aligned} Az &= \alpha\lambda_1 x^{(1)} + \beta A(r_{11}x^{(1)} + r_{21}x^{(2)}) \\ &= \alpha\lambda_1 x^{(1)} + \beta(r_{11}\lambda_1 x^{(1)} + r_{21}\lambda_2 x^{(2)}) \\ &= \alpha\lambda_1 x^{(1)} + \beta(r_{11}(\lambda_1 - \lambda_2)x^{(1)} + \lambda_2(r_{11}x^{(1)} + r_{21}x^{(2)}) \end{aligned}$$

$$\begin{aligned} &= \alpha\lambda_1 x^{(1)} + \beta(r_{11}(\lambda_1 - \lambda_2)x^{(1)} + \lambda_2 y^{(1)}) \\ &= (\alpha\lambda_1 + \beta r_{11}(\lambda_1 - \lambda_2))x^{(1)} + \beta\lambda_2 y^{(1)}) \end{aligned}$$

and similarly $Bz = \alpha\mu_2 x^{(1)} + (\alpha t_{11}(\mu_1 - \mu_2) + \beta\mu_1)y^{(1)}$. We can now easily write down the result of applying simple words in A and B to our dominant eigenvectors:

$$\begin{aligned} A^k x^{(1)} &= \lambda_1^k x^{(1)} \ \ (k \geq 1) \\ B^k x^{(1)} &= \mu_2^k x^{(1)} + t_{11}(\mu_1^k - \mu_2^k)y^{(1)} \ \ (k \geq 1) \\ A^k y^{(1)} &= r_{11}(\lambda_1^k - \lambda_2^k)x^{(1)} + \lambda_2^k y^{(1)} \ \ (k \geq 1) \\ B^k y^{(1)} &= \mu_1^k y^{(1)} \ \ (k \geq 1) \qquad (7.1) \\ BAx^{(1)} &= \lambda_1 Bx^{(1)} = \lambda_1\mu_2 x^{(1)} + t_{11}\lambda_1(\mu_1 - \mu_2)y^{(1)} \\ ABx^{(1)} &= (\lambda_1\mu_2 + r_{11}t_{11}(\lambda_1 - \lambda_2)(\mu_1 - \mu_2))x^{(1)} + t_{11}\lambda_2(\mu_1 - \mu_2)y^{(1)} \\ BAy^{(1)} &= r_{11}(\lambda_1 - \lambda_2)\mu_2 x^{(1)} + (r_{11}t_{11}(\lambda_1 - \lambda_2)(\mu_1 - \mu_2) + \lambda_2\mu_1)y^{(1)} \\ ABy^{(1)} &= \mu_1 Ay^{(1)} = r_{11}(\lambda_1 - \lambda_2)\mu_1 x^{(1)} + \lambda_2\mu_1 y^{(1)}. \end{aligned}$$

We now determine conditions for the existence of a common proper invariant cone, depending on the signs of the non-dominant eigenvalues. We proceed using case analysis, recalling that under our current assumptions,

$$\lambda_1 > 0, \quad \lambda_1^k - \lambda_2^k > 0 \ \text{ for } \ k \geq 1, \quad \mu_1 > 0, \quad \mu_1^k - \mu_2^k > 0 \ \text{ for } \ k \geq 1.$$

Theorem 7.2 *If* $\text{sign}(P^{-1}Q)_{11} \neq \text{sign}(Q^{-1}P)_{11}$, *then there is no common invariant proper cone of* A *and* $B \in \mathbb{R}^{2\times 2}$.

Proof. Suppose K is a common invariant proper cone for A and B, and that $\text{sign}(P^{-1}Q)_{11} \neq \text{sign}(Q^{-1}P)_{11}$. Then $t_{11}r_{11} < 0$ and hence either $t_{11} < 0$ or $r_{11} < 0$, but not both. We establish the case where $t_{11} < 0$ and $r_{11} > 0$. The other case is analogous. Assume, without loss of generality, that $x^{(1)} \in K$. Then

$$(\frac{1}{\mu_1}B)^k x^{(1)} = (\frac{\mu_2}{\mu_1})^k x^{(1)} + t_{11}(1 - (\frac{\mu_2}{\mu_1})^k)y^{(1)},$$

which converges to $t_{11}y^{(1)}$ as k approaches infinity; hence $-y^{(1)} \in K$. But then

$$(\frac{1}{\lambda_1}A)^k(-y^{(1)}) = r_{11}(1 - (\frac{\lambda_2}{\lambda_1})^k)(-x^{(1)}) + (\frac{\lambda_2}{\lambda_1})^k(-y^{(1)}),$$

which converges to $-r_{11}x^{(1)}$ as k approaches infinity; hence $-x^{(1)} \in K$. But then $x^{(1)}$ and $-x^{(1)}$ are both in K, which contradicts that K is a proper (pointed) cone. □

Theorem 7.3 *Suppose that* $\lambda_2 > 0$ *and* $\mu_2 > 0$ *for* A *and* $B \in \mathbb{R}^{2\times 2}$. *Then the following are equivalent:*

(i) $\text{sign}(P^{-1}Q)_{11} = \text{sign}(Q^{-1}P)_{11}$.
(ii) $\{x^{(1)}, y^{(1)}\}^{\text{G}}$ *is a common invariant cone of* A *and* B.
(iii) *There is a common invariant proper cone of* A *and* B.

Proof.
(i)⇒(ii): If $\mathrm{sign}(P^{-1}Q)_{11} = \mathrm{sign}(Q^{-1}P)_{11}$, then we can assume without loss of generality that t_{11} and r_{11} are both positive. If they are both positive, then the result follows immediately from the formulas for $Ax^{(1)}$, $Ay^{(1)}$, $Bx^{(1)}$, $By^{(1)}$ in (7.1).
(ii)⇒(iii): Obvious.
(iii)⇒(i): Follows from Theorem 7.2. □

Theorem 7.4 *Suppose that* $\lambda_2 \geq 0$ *and* $\mu_2 \leq 0$ *for* A *and* $B \in \mathbb{R}^{2\times 2}$. *Then the following are equivalent:*

(i) $(P^{-1}Q)_{11}(Q^{-1}P)_{11} \geq \dfrac{-\mu_2}{\mu_1 - \mu_2}$.
(ii) $\{x^{(1)}, Bx^{(1)}\}^{\mathrm{G}}$ *is a common invariant cone of* A *and* B.
(iii) *There is a common invariant cone of* A *and* B.

Proof.
(i)⇒(ii): Assume that

$$(P^{-1}Q)_{11}(Q^{-1}P)_{11} \geq \frac{-\mu_2}{(\mu_1 - \mu_2)}.$$

Since $\mu_2 \leq 0$ and $\mu_1 - \mu_2 > 0$, $(P^{-1}Q)_{11}(Q^{-1}P)_{11} > 0$. We again assume, without loss of generality, that $t_{11} > 0$ and $r_{11} > 0$. Recall that

$$Bx^{(1)} = \mu_2 x^{(1)} + t_{11}(\mu_1 - \mu_2)y^{(1)}.$$

Solving for $y^{(1)}$, we have

$$y^{(1)} = \frac{1}{t_{11}(\mu_1 - \mu_2)}((-\mu_2)x^{(1)} + Bx^{(1)}).$$

Thus $\mu_2 \leq 0$ and $t_{11} > 0$ imply that $y^{(1)} \in \{x^{(1)}, Bx^{(1)}\}$. Since $Ax^{(1)} = \lambda_1 x^{(1)}$, we see that $Ax^{(1)} \in \{x^{(1)}, Bx^{(1)}\}$. Notice that

$$\begin{aligned} A(Bx^{(1)}) &= ABx^{(1)} = (\lambda_1\mu_2 + r_{11}t_{11}(\lambda_1 - \lambda_2)(\mu_1 - \mu_2))x^{(1)} + t_{11}\lambda_2(\mu_1 - \mu_2)y^{(1)} \\ &= (\lambda_1\mu_2 + r_{11}t_{11}(\lambda_1 - \lambda_2)(\mu_1 - \mu_2))x^{(1)} + \lambda_2(-\mu_2 x^{(1)} + Bx^{(1)}) \\ &= ((\lambda_1 - \lambda_2)\mu_2 + r_{11}t_{11}(\lambda_1 - \lambda_2)(\mu_1 - \mu_2))x^{(1)} + \lambda_2 Bx^{(1)} \\ &= (\lambda_1 - \lambda_2)(r_{11}t_{11}(\mu_1 - \mu_2) + \mu_2)x^{(1)} + \lambda_2 Bx^{(1)}. \end{aligned}$$

Thus $A(Bx^{(1)}) \in \{x^{(1)}, Bx^{(1)}\}^{\mathrm{G}}$ since $r_{11}t_{11}(\mu_1 - \mu_2) + \mu_2 \geq 0$. Clearly $Bx^{(1)} \in \{x^{(1)}, Bx^{(1)}\}^{\mathrm{G}}$. Lastly, we have

$$\begin{aligned} B(Bx^{(1)}) &= B^2x^{(1)} = \mu_2^2 x^{(1)} + t_{11}(\mu_1^2 - \mu_2^2)y^{(1)} \\ &= \mu_2^2 x^{(1)} + t_{11}(\mu_1^2 - \mu_2^2)(\frac{1}{t_{11}(\mu_1 - \mu_2)})((-\mu_2)x^{(1)} + Bx^{(1)}) \\ &= -\mu_1\mu_2 x^{(1)} + (\mu_1 + \mu_2)Bx^{(1)} \in K(x^{(1)}, Bx^{(1)}). \end{aligned}$$

Hence $\{x^{(1)}, Bx^{(1)}\}^{\mathrm{G}}$ is a common invariant proper cone of A and B.
(ii)⇒(iii): Obvious.

(iii)⇒(i): Let K be an invariant proper cone of A and B. Without loss of generality assume that $x^{(1)} \in K$. But then

$$\begin{aligned}(\frac{1}{\lambda_1}A)^k Bx^{(1)} &= (\frac{1}{\lambda_1})^k[(\lambda_1^k\mu_2 + r_{11}t_{11}(\lambda_1^k - \lambda_2^k)(\mu_1-\mu_2))x^{(1)} + t_{11}\lambda_2^k(\mu_1-\mu_2)y^{(1)}] \\ &= (\mu_2 + r_{11}t_{11}(1-(\frac{\lambda_2}{\lambda_1})^k)(\mu_1-\mu_2))x^{(1)} + t_{11}(\frac{\lambda_2}{\lambda_1})^k(\mu_1-\mu_2)y^{(1)}),\end{aligned}$$

which converges to $-x^{(1)}$, unless $t_{11}r_{11} \geq \frac{-\mu_2}{\mu_1-\mu_2}$. □

Theorem 7.5 *Suppose that* $\lambda_2 \leq 0$ *and* $\mu_2 \geq 0$ *for* A *and* $B \in \mathbb{R}^{2\times 2}$. *Then the following are equivalent:*
(i) $(P^{-1}Q)_{11}(Q^{-1}P)_{11} \geq \dfrac{-\lambda_2}{\lambda_1-\lambda_2}$.
(ii) $\{y^{(1)}, Ay^{(1)}\}$ *is a common invariant cone of* A *and* B.
(iii) *There is a common invariant cone of* A *and* B.

Proof.
(i)⇒(ii): Since $(P^{-1}Q)_{11}(Q^{-1}P)_{11} > 0$, be can assume, without loss of generality, that $r_{11} > 0$ and $t_{11} > 0$. Recall that

$$Ay^{(1)} = r_{11}(\lambda_1-\lambda_2)x^{(1)} + \lambda_2 y^{(1)}.$$

Solving for $x^{(1)}$, we have

$$x^{(1)} = \frac{1}{r_{11}(\lambda_1-\lambda_2)}(Ay^{(1)} - \lambda_2 y^{(1)}).$$

Thus $x^{(1)} \in \{Ay^{(1)}, y^{(1)}\}^{\mathrm{G}}$. Clearly $Ay^{(1)} \in \{y^{(1)}, Ay^{(1)}\}^{\mathrm{G}}$. Notice that

$$\begin{aligned}A(Ay^{(1)}) &= A^2y^{(1)} = r_{11}(\lambda_1^2-\lambda_2^2))x^{(1)} + \lambda_2^2 y^{(1)} \\ &= r_{11}(\lambda_1^2-\lambda_2^2)\frac{1}{r_{11}(\lambda_1-\lambda_2)}(Ay^{(1)} - \lambda_2 y^{(1)}) + \lambda_2^2 y^{(1)} \\ &= -\lambda_1\lambda_2 y^{(1)} + (\lambda_1+\lambda_2)Ay^{(1)} \in \{y^{(1)}, Ay^{(1)}\}.\end{aligned}$$

Since $By^{(1)} = \mu_1 y^{(1)}$, we see that $By^{(1)} \in \{y^{(1)}, Ay^{(1)}\}$. Lastly, consider

$$\begin{aligned}B(Ay^{(1)}) &= r_{11}(\lambda_1-\lambda_2)\mu_2 x^{(1)} + (r_{11}t_{11}(\lambda_1-\lambda_2)(\mu_1-\mu_2) + \lambda_2\mu_1)y^{(1)} \\ &= \mu_2(-\lambda_2 y^{(1)} + Ay^{(1)}) + (r_{11}t_{11}(\lambda_1-\lambda_2)(\mu_1-\mu_2) + \lambda_2\mu_1)y^{(1)} \\ &= \mu_2 Ay^{(1)} + (\mu_1-\mu_2)(r_{11}t_{11}(\lambda_1-\lambda_2) + \lambda_2)y^{(1)}.\end{aligned}$$

Thus $B(Ay^{(1)})$ is in $\{Ay^{(1)}, y^{(1)}\}^{\mathrm{G}}$, provided that $t_{11}r_{11} \geq \frac{-\lambda_2}{\lambda_1-\lambda_2} \geq 0$.
(ii)⇒(iii): Obvious.
(iii)⇒(i): Let K be an invariant proper cone of A and B. Without loss of generality, assume that $y^{(1)} \in K$. But then

$$\begin{aligned}(\frac{1}{\mu_1}B)^k Ay^{(1)} &= (\frac{1}{\mu_1})^k[r_{11}(\lambda_1-\lambda_2)(\mu_2^k x^{(1)} + t_{11}(\mu_1^k-\mu_2^k)y^{(1)}) + \mu_1^k\lambda_2 y^{(1)} \\ &= r_{11}(\lambda_1-\lambda_2)(\frac{\mu_2}{\mu_1})^k x^{(1)} + (\lambda_2 + r_{11}t_{11}(\lambda_1-\lambda_2)(1-(\frac{\mu_2}{\mu_1})^k))y^{(1)},\end{aligned}$$

which converges to a multiple of $-y^{(1)}$, unless $t_{11}r_{11} \geq \frac{-\lambda_2}{\lambda_1-\lambda_2}$. □

Theorem 7.6 *Suppose that* $\lambda_2 < 0$ *and* $\mu_2 < 0$ *for* A *and* $B \in \mathbb{R}^{2\times 2}$. *Then the following are equivalent:*
(i) *There is a common invariant proper cone of* A *and* B.
(ii) *The matrix* AB *satisfies the Perron condition.*
(iii) *The spectral radius of* AB *is an eigenvalue with associated eigenvector* u, *and* $\{u, Bu\}^{\mathrm{G}}$ *is a common invariant cone of* A *and* B.
(iv) *The matrix* BA *satisfies the Perron condition.*
(v) *The spectral radius of* BA *is an eigenvalue with associated eigenvector* v *and* $\{v, Av\}^{\mathrm{G}}$ *is a common invariant cone of* A *and* B.

Proof. We show (i)⇒(ii)⇒(iii)⇒(i). The proof for (i)⇒(iv)⇒(v)⇒(i) is analogous.
(i)⇒(ii): If A and B have a common invariant proper cone, then this cone is invariant for AB and hence AB must satisfy the Perron condition by Theorem 3.1.
(ii)⇒(iii): If AB satisfies the Perron condition, then $\rho = \rho(AB) \in \sigma(A)$. Let u be an eigenvector of AB associated with $\rho > 0$.

Since A is a 2×2 matrix, by Cayley–Hamilton, we see that $A^2 = \operatorname{trace}(A)A - \det(A)I$. Notice that $\operatorname{trace}(A) = \lambda_1 + \lambda_2 > 0$ and $\det(A) = \lambda_1\lambda_2 < 0$. Similarly, $B^2 = \operatorname{trace}(B)A - \det(B)I$, where $\operatorname{trace}(B) > 0$ and $\det(B) < 0$. Then

$$Au = \frac{1}{\rho}A(ABu) = A^2(Bu) = \operatorname{trace}(A)ABu - \det(A)Bu = \operatorname{trace}(A)\rho u - \det(A)Bu \in \{u, Bu\}^{\mathrm{G}}.$$

Clearly

$$A(Bu) = (AB)u = \rho u \in \{u, Bu\} \text{ and } Bu \in \{u, Bu\}^{\mathrm{G}}.$$

Lastly we see that

$$B(Bu) = B^2u = \operatorname{trace}(B)Bu - \det(B)u \in \{u, Bu\}.$$

Thus $\{u, Bu\}$ is a common invariant cone of A and B.
(iii)⇒(i): Obvious. □

The following observation shows that if the statements in the above theorem are all true, then $\{u, Bu\}^{\mathrm{G}} = \{v, Av\}^{\mathrm{G}}$.

Observation 7.7 *Suppose* $A, B \in \mathbb{R}^{2\times 2}$.
(i) *If* u *is an eigenvector for* AB *corresponding to the nonzero eigenvalue* λ, *then* Bu *is an eigenvector for* BA *corresponding to the eigenvalue* λ.
(ii) *If* v *is an eigenvector for* BA *corresponding to the eigenvalue* λ, *then* Av *is an eigenvector for* AB *corresponding to the nonzero eigenvalue* λ.

Proof. Suppose $ABu = \lambda u$. Then $BA(Bu) = B(ABu) = \lambda Bu$. Similarly if $BAv = \lambda v$, then $AB(Av) = A(BAv) = \lambda Av$. □

The common invariant proper cones of the four cases above can be combined into a single statement using the fact that the set of vectors used as extremals in the four cases always lie inside the invariant cone, if it exists.

Theorem 7.8 *Suppose that A and $B \in \mathbb{R}^{2\times 2}$ are not simultaneously conformally reducible, each has their spectral radius as an eigenvalue, and their dominant subspaces have dimension 1. Then the following are equivalent:*

(i) $\{x^{(1)}, y^{(1)}, Bx^{(1)}, Ay^{(1)}, u, v\}^G$ *is a common invariant cone of A and B, where u and v are eigenvectors associated with the spectral radius of AB and BA, respectively.*
(ii) *There is a common invariant proper cone of A and B.*

Proof. It is clear that $x^{(1)}$ and $y^{(1)}$ must lie in any common invariant proper cone since $A^m z \to x^{(1)}$ for any $z \in K$ and similarly for $y^{(1)}$. Thus, $Bx^{(1)}, Ay^{(1)} \in K$ also. So it only remains to show that $u, v \in K$ when they are not already extremals. Note that $(AB)^m x^{(1)} \in K$ for any $m \geq 0$, so since K is closed, $u = \lim_{m\to\infty}(AB)^m x^{(1)} \in K$. A similar argument shows $v \in K$. □

To see the relationship between the sets $\mathcal{E}(A,B)$ and $\mathcal{F}(A,B)$ of Section 5 and the invariant cones K of this section, we consider images of K under words and the set F_∞ defined in (5.1), but restricting our attention now to $\mathbb{R}^2$. Recall that $\lambda_1 = \rho(A) > 0$ and $\mu_1 = \rho(B) > 0$. Note that if a dominant eigenvalue is negative, we can apply the results below to $-A$ or $-B$. There are two cases to consider. Either $AK \cup BK = K$ or it is a proper subset of K.

Proposition 7.9 *Suppose $\lambda_2 > 0$ and $\mu_2 > 0$ for $A, B \in \mathbb{R}^{2\times 2}$ so that $K = \{x^{(1)}, y^{(1)}\}^G$ is a common invariant proper cone. If $AK \cup BK \neq K$, then $\overline{\mathcal{F}(A,B)} = F_\infty \cup (-F_\infty)$, which is a set of all scalar multiples of a Cantor set of unit vectors.*

Proof. If $AK \cup BK \neq K$, then AK and BK are disjoint (apart from the origin) and lie at either extremity of K: $x^{(1)} \in AK$ and $y^{(1)} \in BK$. The set F_∞, projected onto the unit circle, say, or a line crossing both extremals of K, is a Cantor set, *i.e.*, is closed and has no isolated or interior points (see [10, pp.97–100] or [9, p.229]). This is a result of the way F_∞ was constructed, which is similar to the construction of the generalized Cantor set of Folland [5, p.40–41]), except that the 'middle' interval removed from an interval, I, at each stage is not centred in I. The interval removed is nevertheless proportionally the same at each stage by linearity of matrix multiplication. Thus, the remaining intervals at each stage shrink in length to zero, so F_∞ has no interior points and is homeomorphic to the Cantor middle-third set [2, p.37]. Symbolically, the points of the projection of K onto a line can be represented as ternary expansions of the interval $[0, 1]$, and F_∞ consists of all points with expansions that can be represented by 0's and 2's but no 1's. Now, $\mathcal{F}(A,B)$ by definition, consists of the images of $x^{(1)}$ and $y^{(1)}$ under words in A and B, but $x^{(1)}$ and $y^{(1)}$ are the extremals of K, so $\mathcal{F}(A,B)$ consists of the endpoints of intervals at every stage of construction, which are symbolically represented by the points in $[0, 1]$ with finite ternary expansions. Since points with infinite ternary expansions can clearly be approximated arbitrarily closely by points with finite ternary expansions, $F_\infty \cup -(F_\infty) \subseteq \overline{\mathcal{F}(A,B)}$, and therefore $F_\infty \cup (-F_\infty) = \overline{\mathcal{F}(A,B)}$, by Proposition 5.11. □

We note here that for the K's corresponding to the other choices of signs of λ_2 and μ_2, the same result can be proved, but the argument is more subtle. Furthermore, if $AK \cup BK = K$, then we still have $F_\infty \cup (-F_\infty) = \overline{\mathcal{F}(A,B)}$, so that $\overline{\mathcal{F}(A,B)} = K \cup (-K)$. We do not prove these results here, but plan to develop these arguments in a future publication.

The above results lead to the following conclusion.

Corollary 7.10 *Let* $A, B \in \mathbb{R}^{2\times 2}$ *have a common invariant proper cone. If* A, B *are not simultaneously conformally reducible and if the dominant subspace of every* $W \in \mathcal{W}(A, B)$ *has dimension 1, then* $\mathcal{E}(A, B)$ *is either dense in* $K \cup (-K)$ *or consists of scalar multiples of a Cantor set of unit vectors in* K*.*

8 The multilinear approach

In this section, we use tools of multilinear algebra in order to (i) analyze and recast the problem of common invariant polyhedral (or simplicial) cones, and (ii) describe a strategy and a necessary condition for nontrivial common invariant cones based on common invariant subspaces. In both instances, we are afforded the opportunity to construct examples of matrices having common invariant cones, or to exclude the possibility of common invariant cones for certain matrix pairs.

8.1 The Kronecker product approach for polyhedral cones

We first recall some definitions and notation pertaining to Kronecker matrix products; see *e.g.*, [12] for details.

Given $Z \in \mathbb{R}^{m\times n}$, by $z = \mathrm{vec}\,(Z)$ we denote the array in $\mathbb{R}^{mn}$ obtained by stacking the columns of Z in their natural order with the first column first. The inverse of the vec operator (as a function into $\mathbb{R}^{m\times n}$) is well-defined and denoted by $Z = \mathrm{vec}^{-1}(z)$.

Given two matrices $X = [x_{ij}] \in \mathbb{R}^{m\times n}$ and $Y \in \mathbb{R}^{r\times s}$, the *Kronecker product* of X and Y is

$$X \otimes Y = \begin{bmatrix} x_{11}Y & x_{12}Y & \dots & x_{1n}Y \\ x_{21}Y & x_{22}Y & \dots & x_{2n}Y \\ \vdots & \vdots & \ddots & \vdots \\ x_{11}Y & x_{12}Y & \dots & x_{1n}Y \end{bmatrix} \in \mathbb{R}^{mr\times ns}.$$

The *Kronecker sum* of two square matrices $X \in \mathbb{R}^{n\times n}$ and $Y \in \mathbb{R}^{m\times m}$ is

$$X \oplus Y = I_m \otimes X + Y \otimes I_n \in \mathbb{R}^{mn\times mn},$$

where I_m, I_n are the $m \times m$ and $n \times n$ identity matrices, respectively. The notation $X \oplus Y$ for the Kronecker sum adopted here is not to be confused with the direct sum of two matrices. The spectrum of the Kronecker sum comprises all pairwise sums of the eigenvalues of the summands; that is, if $\sigma(X) = \{\lambda_1, \dots, \lambda_n\}$ and $\sigma(Y) = \{\mu_1, \dots, \mu_m\}$, then

$$\sigma(X \oplus Y) = \{\lambda_i + \mu_j \,:\, i = 1, \dots, n,\ j = 1, \dots, m\}.$$

With $X \in \mathbb{R}^{n\times n}$ and $Y \in \mathbb{R}^{m\times m}$, it readily follows from the definitions above that the solutions of the matrix equation

$$XW + WY = V \quad \text{in} \quad W \in \mathbb{R}^{n\times m}$$

are precisely those matrices $W \in \mathbb{R}^{n\times m}$ satisfying the equation

$$(Y^T \oplus X)\,\mathrm{vec}\,(W) \;=\; \mathrm{vec}\,(V).$$

Based on the latter observation and the following lemma, we can recast the problem of finding invariant polyhedral cones to solving a parametric matrix equation.

Lemma 8.1 *Let $K = X\mathbb{R}^k_+$, where $X \in \mathbb{R}^{n\times k}$. The matrix $A \in \mathbb{R}^{n\times n}$ leaves the polyhedral cone K invariant if and only if there exists nonnegative matrix $C \in \mathbb{R}^{k\times k}$ such that*

$$AX - XC = 0. \tag{8.1}$$

Proof. If (8.1) holds for some $C \geq 0$, then

$$AK = AX\,\mathbb{R}^k_+ \;=\; XC\,\mathbb{R}^k_+ \subseteq X\,\mathbb{R}^k_+ \;=\; K.$$

Conversely, suppose $AK \subseteq K$ and $K = X\mathbb{R}^k_+$, where $X \in \mathbb{R}^{n\times k}$. Since $AK = AX\mathbb{R}^n_+$, AK is the polyhedral cone generated by the columns of AX, that is,

$$AK = \{f_1, f_2, \ldots, f_k\}^{\mathrm{G}}, \text{ where } AX = [f_1|f_2|\ldots|f_k].$$

Since $AK \subseteq K$, we have $f_i \in K \;\; (i = 1, 2, \ldots, k)$, and since $K = X\mathbb{R}^k_+$, we have that

$$f_i = XC_i \text{ for some } C_i \in \mathbb{R}^k_+ \quad (i = 1, 2, \ldots, k).$$

Thus

$$AX = [XC_1|XC_2|\ldots|XC_k] = XC, \text{ where } \; 0 \leq C = [C_1|C_2|\ldots|C_k] \in \mathbb{R}^k_+. \quad \square$$

We can now state the following necessary and sufficient condition for the existence of a common invariant polyhedral cone.

Theorem 8.2 *Matrices $A, B \in \mathbb{R}^{n\times n}$ have a common invariant polyhedral cone $K \subseteq \mathbb{R}^n$ if and only if there exist nonnegative matrices $C, E \in \mathbb{R}^{k\times k}$ such that*

$$\mathrm{Nul}\,(A \oplus (-C^T)) \;\cap\; \mathrm{Nul}\,(B \oplus (-E^T)) \;\neq\; \{0\}.$$

Moreover, every common invariant polyhedral cone of A and B is of the form

$$K \;=\; \mathrm{vec}^{-1}(z)\,\mathbb{R}^k_+, \quad \textit{where} \quad z \in \mathrm{Nul}\,(A \oplus (-C^T)) \cap \mathrm{Nul}\,(B \oplus (-E^T)).$$

Proof. Apply Lemma 8.1 to A and recall that the solutions of the matrix equation in X in (8.1) can be found via the equivalent homogeneous system

$$[A \oplus (-C^T)]\,\mathrm{vec}(X) = 0. \tag{8.2}$$

Considering the corresponding equation from Lemma 8.1 applied to B, we obtain that $A, B \in \mathbb{R}^{n\times n}$ have a common invariant polyhedral cone $K \subseteq \mathbb{R}^n$ if and only if there exist nonnegative matrices $C, E \in \mathbb{R}^{n\times k}$ such that $A \oplus (-C^T)$ and $B \oplus (-E^T)$ have a common nullvector $z \in \mathbb{R}^{nk}$. If we let $X = \mathrm{vec}^{-1}(z) \in \mathbb{R}^{n\times k}$, it follows that $K = X\mathbb{R}^k_+$ is left invariant by both A and B. $\square$

Corollary 8.3 *Matrices $A, B \in \mathbb{R}^{n\times n}$ have a common invariant simplicial cone if and only if there exist nonnegative matrices $C, E \in \mathbb{R}^{n\times n}$ and*

$$z \in \operatorname{Nul}(A \oplus (-C^T)) \cap \operatorname{Nul}(B \oplus (-E^T))$$

such that $X = \operatorname{vec}^{-1}(z) \in \mathbb{R}^{n\times n}$ is invertible. Moreover, every common invariant simplicial cone of A and B is of the form $K = X\,\mathbb{R}^n_+$, where $X = \operatorname{vec}^{-1}(z)$ is invertible and $z \in \operatorname{Nul}(A \oplus (-C^T)) \cap \operatorname{Nul}(B \oplus (-E^T))$.

Example 8.4 To illustrate Corollary 8.3, let us consider whether the matrices

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 5 & 4 \\ -3 & -2 \end{bmatrix}$$

have a common invariant simplicial cone. Letting $C = [c_{ij}]$, $E = (e_{ij}) \in \mathbb{R}^{2\times 2}$,

$$A \oplus (-C^T) = \begin{bmatrix} 2-c_{11} & 0 & -c_{21} & 0 \\ 0 & 1-c_{11} & 0 & -c_{21} \\ -c_{12} & 0 & 2-c_{22} & 0 \\ 0 & -c_{12} & 0 & 1-c_{22} \end{bmatrix}$$

and

$$B \oplus (-E^T) = \begin{bmatrix} 5-e_{11} & 4 & -e_{21} & 0 \\ 0 & -2-e_{11} & 0 & -e_{21} \\ -e_{12} & 0 & 5-e_{22} & 4 \\ 0 & -e_{12} & -3 & -2-e_{22} \end{bmatrix}.$$

We take C and E to be nonnegative. To ensure that the nullspaces of $A \oplus (-C^T)$ and $B \oplus (-E^T)$ intersect nontrivially, the entries are chosen so that $\sigma(C) = \sigma(A)$ and $\sigma(E) = \sigma(B)$. In particular, we let

$$C = \begin{bmatrix} 2 & c_{12} \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad E = \begin{bmatrix} 1 & 0 \\ e_{21} & 2 \end{bmatrix}.$$

Notice that $\operatorname{Nul}(A \oplus (-C^T))$ is spanned by

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ c_{12} \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

and $\operatorname{Nul}(B \oplus (-E^T))$ is spanned by

$$v_1 = \begin{bmatrix} e_{21} \\ 0 \\ 4 \\ -3 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Taking $c_{12} = 4$ and $e_{21} = 1$, it follows that $u_1 - 3u_2 = v_1$. That is,

$$v_1 = [1 \quad 0 \quad 4 \quad -3]^T \in \operatorname{Nul}(A \oplus (-C^T)) \cap \operatorname{Nul}(B \oplus (-E^T)).$$

Consequently, letting

$$X = \mathrm{vec}^{-1}(v_1) = \begin{bmatrix} 1 & 4 \\ 0 & -3 \end{bmatrix},$$

we have that the simplicial cone

$$K = X\mathbb{R}^n_+ = \left\{ \begin{bmatrix} x_1 + 4x_2 \\ -3x_2 \end{bmatrix} \ : \ x_1 \geq 0, \ x_2 \geq 0 \right\}$$

is left invariant by both A and B $\square$.

8.2 A strategy and a necessary condition for low dimensional cones

Here we rely on results obtained in [18] regarding matrices with common invariant subspaces. First we describe a plausible strategy to discover common invariant cones based on knowing a common invariant subspace. Second, we obtain a necessary condition that can be used to exclude the existence of common invariant cones.

Recall that the k-th compound of $A \in \mathbb{R}^{n\times n}$, $A^{(k)}$, is the $\binom{n}{k} \times \binom{n}{k}$ matrix of all $k \times k$ minors of A arranged in lexicographic order of their row and column indices. Also recall that to every k-dimensional subspace W of $\mathbb{R}^n$ we can associate a (unique up to scalar multiples) *Grassmann representative* consisting of the exterior product of the vectors in a basis for W. For details, see [14, 18]. The following result is shown in [18].

Theorem 8.5 *Let $A, B \in \mathbb{C}^{n\times n}$ and W a subspace of $\mathbb{C}^n$ of dimension k $(1 \leq k < n)$. The following are equivalent.*

(i) *W is a common invariant subspace of A and B.*

(ii) *There exists decomposable $x \in \mathbb{C}^{\binom{n}{k}}$ such that for all $s \in \mathbb{C}$, x is a common eigenvector of $(A+sI)^{(k)}$ and $(B+sI)^{(k)}$.*

(iii) *There exist decomposable $x \in \mathbb{C}^{\binom{n}{k}}$ and $\hat{s} \in \mathbb{C}$ such that $A+\hat{s}I$ and $B+\hat{s}I$ are invertible and x is a common eigenvector of $(A+\hat{s}I)^{(k)}$ and $(B+\hat{s}I)^{(k)}$.*

The vector x in (ii) *and* (iii) *is a Grassmann representative for W.*

To apply Theorem 8.5 we need the following criterion for the existence of a common eigenvector found in [16].

Theorem 8.6 *Let $X, Y \in \mathbb{R}^{p\times p}$ and define*

$$C(X,Y) = \sum_{m,\ell=1}^{p-1} [X^m, Y^\ell]^* [X^m, Y^\ell],$$

where $[X^m, Y^\ell]$ denotes the commutator $X^m Y^\ell - Y^\ell X^m$. Then X and Y have a common eigenvector if and only if $C(X,Y)$ is not invertible.

A strategy to discover common invariant cones can now be based on the above results as follows. Find a common invariant subspace W of dimension less than n and a basis for W. Then, the cones generated by different orientations of the basis vectors of W are candidate common invariant cones because their spans are indeed equal to W. These candidate cones can be tested for invariance with the help of Lemma 8.1 as illustrated in the next example.

Example 8.7 Consider the matrices

$$A = \begin{bmatrix} 3 & -3 & 1 \\ 0 & 4 & 0 \\ -1 & -3 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & -3 & 5 \\ -2 & 6 & 2 \\ -7 & -1 & 11 \end{bmatrix}.$$

As in [18], the following application of Theorem 8.5 ensues. The eigenvalues of A are all equal to 4 and of B are 4 (double) and 8. Hence Theorem 8.5 part (iii) applies with $\hat{s} = 0$. We compute the second compounds of A and B to be

$$A^{(2)} = \begin{bmatrix} 12 & 0 & -4 \\ -12 & 16 & -12 \\ 4 & 0 & 20 \end{bmatrix}, \quad B^{(2)} = \begin{bmatrix} -12 & 8 & -36 \\ -20 & 24 & -28 \\ 44 & -8 & 68 \end{bmatrix}.$$

The matrix $C(A^{(2)}, B^{(2)})$ of Theorem 8.6 is a scalar multiple of

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

and thus it is not invertible. Thus $A^{(2)}$ and $B^{(2)}$ have a common eigenvector. In fact,

$$\text{Nul}(A^{(2)} - 16I) = \text{span}\,\{\alpha, \beta\}, \quad \text{where} \quad \alpha = [\,-1 \quad 0 \quad 1\,]^T,\ \beta = [\,0 \quad 1 \quad 0\,]^T$$

and

$$\text{Nul}(B^{(2)} - 16I) = \text{span}\,\{\gamma\} \quad \text{where} \quad \gamma = [\,1 \quad -1 \quad -1\,]^T.$$

Since $\gamma \in \text{span}\,\{\alpha, \beta\}$, we have that γ is a common eigenvector of $A^{(2)}$ and $B^{(2)}$. Moreover, γ is decomposable as $\gamma = x_1 \wedge x_2$, where

$$x_1 = [\,1 \quad 0 \quad 1\,]^T, \quad x_2 = [\,0 \quad 1 \quad -1\,]^T.$$

It follows that $W = \text{span}\,\{x_1, x_2\}$ is a common invariant subspaces of A and B.

Next we will consider whether the cone generated by some orientation of x_1 and x_2 is left invariant by both A and B. For that purpose, let

$$K = X\mathbb{R}^2_+ \subseteq \mathbb{R}^3, \quad \text{where} \quad X = [x_1 \quad -x_2] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 1 & 1 \end{bmatrix}.$$

Applying Lemma 8.1 to A and B and the cone K, we find that with the choices

$$C = \begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix} \quad \text{and} \quad E = \begin{bmatrix} 4 & 8 \\ 0 & 4 \end{bmatrix},$$

we have

$$AX - XC = 0 \quad \text{and} \quad BX - XE = 0.$$

That is, $AK \subseteq K$ and $BK \subseteq K$. We note that not all orientations of x_1 and x_2 yield common invariant cones. □

The following necessary condition for the existence of a common invariant (not necessarily polyhedral or proper) cone is also shown in [18] and is based on Theorem 8.5.

Theorem 8.8 *Let* $K \subseteq \mathbb{R}^n$ *be cone of dimension* k *and suppose that* $AK \subseteq K$ *and* $BK \subseteq K$. *Then* $A^{(k)}$ *and* $B^{(k)}$ *have a common decomposable eigenvector that is a Grassmann representative of* $\operatorname{span} K$.

Of course, the above theorem provides a meaningful necessary condition for the existence of common invariant cones in $\mathbb{R}^n$ of dimension less that n, namely, non-reproducing cones. Its use is illustrated next.

Example 8.9 Let

$$A = \begin{bmatrix} 3 & 0 & 1 \\ -5 & -2 & -1 \\ 1 & 0 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Both A and B satisfy the Perron condition and so they each have (proper) invariant cones. Let us consider whether A and B have a common invariant cone of dimension $k \leq 2$ or not. We compute the second compounds of A and B to be

$$A^{(2)} = \begin{bmatrix} -6 & 2 & 2 \\ 0 & 8 & 0 \\ 2 & -14 & -6 \end{bmatrix} \quad \text{and} \quad B^{(2)} = \begin{bmatrix} 2 & 1 & 2 \\ -1 & 1 & 2 \\ -1 & -2 & -1 \end{bmatrix}.$$

The matrices $C(A, B)$ and $C(A^{(2)}, B^{(2)})$ of Theorem 8.6 are invertible. As a consequence, A and B do not have a common invariant subspace of dimension 1 (eigenvector) nor dimension 2. Thus they cannot have a common invariant cone of dimension 1 nor dimension 2 □.

9 Discussion

We have outlined a number of approaches to the problem of identifying the location of dominant eigenvectors or regions of $\mathbb{R}^n$ occupied by dominant eigenvectors of words in matrices A and B. One approach is to look for common proper invariant cones of $\pm A$ and $\pm B$.

In $\mathbb{R}^2$, we have completely characterized the conditions under which this can be done, and the cones involved. In the application that motivated this problem, however, the problem really only arises in $\mathbb{R}^n$ with $n \geq 3$, because the chaotic dynamics we are trying to demonstrate can be shown not to exist in networks of less than four variables (and therefore Poincaré maps of less than 3 dimensions). Extensions of the ideas used in Section 7 may help in $\mathbb{R}^n$ with $n \geq 3$, though even in $\mathbb{R}^3$ things are considerably more complicated. An idea about how to proceed is given by some examples in reference [4].

An alternative approach is the multilinear one, outlined in Section 8, which casts the problem in a different framework and as we have shown by examples, can allow us to discover common proper invariant cones in $\mathbb{R}^n$.

Common invariant proper cones or double cones restrict the locations of dominant eigenvectors of words, but may not be the only way to do so. For example, we observed in Section 4 that if A^{-1} has an invariant set, then the complement of that set is invariant for A. Thus, if A^{-1} has an invariant double cone, the complement of that double-cone is invariant for A. Complements of double cones are also candidates for common invariant sets.

We can also investigate the set of dominant eigenvectors of words directly, apart from any consideration of cones, as we did in Section 5, and find some of its properties. In $\mathbb{R}^2$ at least, the generic situation is that if a common invariant proper cone or double cone exists, the set of dominant eigenvectors of words is either dense in the double cone, or is the set of scalar multiples of a Cantor set within it. We were able to specify exactly what 'generic' means in this context, and we showed counterexamples to the generic results in the most 'special' case, that of commuting matrices (Section 6).

The problem of identifying the location of dominant eigenvectors of words in matrices A and B is certainly not completely solved. The various approaches we have developed here allow us to understand the possibilities for types of sets containing these dominant eigenvectors, and in some cases to actually specify where they are. However, we cannot yet constructively solve the problem for two arbitrary matrices in $\mathbb{R}^n$ for large n. In terms of the application to abstract gene and neural network models, the problem is still not satisfactorily solved even for two 3×3 matrices. Furthermore, there are networks where more than two cycles are involved and thus more than 2 matrices in the Poincaré maps, so ultimately, it would be desirable to extend the problem to words in m matrices, $m \geq 2$. Finally, aside from its application, we believe that this is a fascinating new problem in linear algebra.

Acknowledgments

This work was partially supported by a grant from the Natural Sciences and Engineering Council of Canada (NSERC).

References

[1] A. Berman and R.J. Plemmons. *Nonnegative Matrices in the Mathematical Sciences.* SIAM, Philadelphia, 1994.

[2] R.L. Devaney. *An Introduction to Chaotic Dynamical Systems* (2nd Ed.). Addison-Wesley, New York, 1989.

[3] R. Edwards. Analysis of continuous-time switching networks. *Physica D*, 146: 165–199, 2000.

[4] R. Edwards. Chaos in neural and gene networks with hard switching. *Differential Equations and Dynamical Systems*, 9:187–220, 2001.

[5] G.B. Folland. *Real Analysis: Modern Techniques and their Applications.* Wiley, New York, 1984.

[6] L. Glass. Classification of biological networks by their qualitative dynamics. *Journal of Theoretical Biology*, 54:85–107, 1975.

[7] L. Glass. Combinatorial and topological methods in nonlinear chemical kinetics. *Journal of Chemical Physics*, 63:1325–1335, 1975.

[8] I. Gohberg, P. Lancaster and L. Rodman. *Invariant Subspaces of Matrices with Applications.* Wiley-Interscience, New York, 1986.

[9] J. Guckenheimer and P. Holmes. *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields.* Springer, New York, 1990.

[10] J.G. Hocking and G.S. Young. *Topology.* Addison-Wesley, London, 1961.

[11] R. Horn and C.R. Johnson. *Matrix Analysis.* Cambridge University Press, Cambridge 1990.

[12] R. Horn and C.R. Johnson. *Topics in Matrix Analysis.* Cambridge University Press, Cambridge 1991.

[13] J.E. Lewis and L. Glass. Nonlinear dynamics and symbolic dynamics of neural networks. *Neural Computation*, 4:621–642, 1992.

[14] M. Marcus. *Finite Dimensional Multilinear Algebra.* Two Volumes. Marcel Dekker, New York, 1973-1975.

[15] T. Mestl, C. Lemay and L. Glass. Chaos in high-dimensional neural and gene networks. *Physica D*, 98:33–52, 1996.

[16] D. Shemesh. Common eigenvectors of two matrices. *Linear Algebra and Its Applications*, 62:11-18, 1984.

[17] G.W. Stewart. *Introduction to Matrix Computations.* Academic Press, New York, 1973.

[18] M. Tsatsomeros. Matrices with a common non-trivial invariant subspace. *Linear Algebra and Its Applications*, 322:51-59, 2001.