

# SPECTRUM LOCALIZATION FOR ALMOST SKEW-SYMMETRIC MATRICES

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## INTRODUCTION

- For  $A \in \mathcal{M}_n(\mathbb{R})$ , write  $A = S(A) + K(A)$ , where

$$S(A) = \frac{A + A^t}{2} \quad \text{and} \quad K(A) = \frac{A - A^t}{2}$$

An *almost skew-symmetric matrix*  $A \in \mathcal{M}_n(\mathbb{R})$  ( $n \geq 2$ ) is one that satisfies  $\text{rank } S(A) = 1$

- The sole nonzero eigenvalue of  $S(A)$  is denoted by  $\delta(A)$
- We assume that  $\delta(A) > 0$  (otherwise work with  $-A$ )
- Note that  $S(A) = ww^t$  for some  $w \in \mathbb{R}^n$
- Goal: provide tight regions that contain the eigenvalues of  $A$

## MOTIVATION

- $T \in \mathcal{M}_n(\mathbb{R})$  is a *pseudo-tournament* ( $\text{rank}(T + T^t + I) = 1$ ) if and only if  $T + \frac{1}{2}I$  or its negative is almost skew-symmetric.
- If  $T \in \mathcal{M}_n(\mathbb{R})$  is a *hypertournament* (zero diagonal entries and  $T + T^t = ww^t - I$  for some nonzero  $w \in \mathbb{R}^n$ ), then  $T + \frac{1}{2}I$  is almost skew-symmetric.
- The spectra of pseudo-tournaments were initially studied by Maybee and Pullman.
- The spectra of almost skew-symmetric matrices and compact operators were considered by Friedland: Two half-planes were provided, one containing exactly one eigenvalue and the other containing all remaining eigenvalues of an almost skew-symmetric matrix.
- Many of the results can be adapted to *almost skew-Hermitian matrices* (complex matrices whose Hermitian part has rank one).
- Possibility of generalizing results & techniques to higher rank perturbations.

## VARIANCE

- For an almost skew-symmetric matrix  $A \in \mathcal{M}_n(\mathbb{R})$  with symmetric part  $S(A) = ww^t$ , the *variance* of  $A$  is

$$v(A) = \frac{\|K(A)w\|_2^2}{\|w\|_2^2}$$

- Notice that  $w/\|w\|_2$  is the unit eigenvector of  $S(A)$  corresponding to the simple eigenvalue  $\delta(A) = w^t w$ .
- If  $v(A) = 0$ , then  $(\delta(A), w)$  is an eigenpair for both  $A$  and  $A^t$ . That is,  $v(A)$  is a measure of how close  $\delta(A)$  is to being a normal eigenvalue of  $A$ .

## MAIN RESULT

Under the assumption that  $\delta(A) > 0$ , an almost skew-symmetric matrix  $A$  is *weakly positive stable* ( $\operatorname{Re} \lambda \geq 0$ ).

Another important inequality satisfied by every eigenvalue of  $A$  holds:

**Theorem** Let  $A \in \mathcal{M}_n(\mathbb{R})$  be an almost skew-symmetric matrix. Let  $\lambda$  be an eigenvalue of  $A$ . Then

$$\boxed{(\operatorname{Im} \lambda)^2 \operatorname{Re} \lambda \leq (\delta(A) - \operatorname{Re} \lambda) [\nu(A) + \operatorname{Re} \lambda (\operatorname{Re} \lambda - \delta(A))]}$$

PROOF OUTLINE

If  $\lambda = \delta(A)$  or  $\operatorname{Re}\lambda = 0$ , result holds.

Assume that  $\lambda \neq \delta(A)$  and  $\operatorname{Re}\lambda \neq 0$ . Let  $y = w/\|w\|_2 \in \mathbb{R}^n$  be the unit eigenvector of  $S(A) = ww^t$  corresponding to the simple eigenvalue  $\delta(A)$ . Then, there exists a unitary  $U \in \mathcal{M}_n(\mathbb{R})$ , whose first column is  $y$ , such that

$$U^t S(A) U = \operatorname{diag}\{\delta(A), 0, \dots, 0\}.$$

Moreover, as  $U^t K(A) U$  is real skew-symmetric, we have

$$U^t K(A) U = \begin{bmatrix} 0 & -u^t \\ u & K_1 \end{bmatrix}, \tag{1}$$

where  $K_1 \in \mathcal{M}_{n-1}(\mathbb{R})$  is skew-symmetric and  $u \in \mathbb{R}^{n-1}$ . Consequently,

$$U^t A U = \begin{bmatrix} \delta(A) & -u^t \\ u & K_1 \end{bmatrix}$$

and the matrix

$$U^t (A - \lambda I) U = \begin{bmatrix} \delta(A) - \lambda & -u^t \\ u & K_1 - \lambda I \end{bmatrix}$$

is singular. Thus, its Schur complement of the leading entry is singular, i.e.,  $0 \in \sigma(E)$ , where

$$E = K_1 - \lambda I + \frac{1}{\delta(A) - \lambda} uu^t.$$

The symmetric and skew-symmetric parts of  $E$  are

$$M = \frac{(\delta(A) - \operatorname{Re}\lambda)}{(\delta(A) - \operatorname{Re}\lambda)^2 + (\operatorname{Im}\lambda)^2} uu^t - \operatorname{Re}\lambda I$$

and

$$N = K_1 - i \operatorname{Im}\lambda I + \frac{i \operatorname{Im}\lambda}{(\delta(A) - \operatorname{Re}\lambda)^2 + (\operatorname{Im}\lambda)^2} uu^t,$$

respectively. Since  $\sigma(E) \subseteq F(E)$  and  $F(M) = \operatorname{Re}F(E)$ , it follows that  $0 \in F(M)$ , which, in turn, implies

$$\operatorname{Re}\lambda \in \frac{(\delta(A) - \operatorname{Re}\lambda)}{(\delta(A) - \operatorname{Re}\lambda)^2 + (\operatorname{Im}\lambda)^2} F(uu^t).$$

Since  $F(uu^t)$  coincides with the interval  $[0, u^t u]$  and since  $\delta(A) > 0$ , we have that

$$\operatorname{Re}\lambda \leq \frac{(\delta(A) - \operatorname{Re}\lambda)(u^t u)}{(\delta(A) - \operatorname{Re}\lambda)^2 + (\operatorname{Im}\lambda)^2}$$

or, equivalently,

$$(\operatorname{Im}\lambda)^2 \leq (\delta(A) - \operatorname{Re}\lambda) \left( \frac{u^t u}{\operatorname{Re}\lambda} - (\delta(A) - \operatorname{Re}\lambda) \right).$$

Denoting by  $e_1$  the first standard basis vector in  $\mathbb{R}^n$ , observe that

$$\begin{aligned} u^t u &= \left\| \begin{bmatrix} 0 \\ u \end{bmatrix} \right\|_2^2 = \left\| U \begin{bmatrix} 0 & -u^t \\ u & K_1 \end{bmatrix} e_1 \right\|_2^2 \\ &= \left\| U \begin{bmatrix} 0 & -u^t \\ u & K_1 \end{bmatrix} U^t U e_1 \right\|_2^2 = \|K(A)y\|_2^2 = v(A) \quad \square \end{aligned}$$

## THE SHELL

The *shell* of an almost skew-symmetric matrix  $A \in \mathcal{M}_n(\mathbb{R})$  is the curve  $\Gamma(A)$  in the complex plane given by:

$$\left\{ x + iy \in \mathbb{C} : y^2 = (\delta(A) - x) \left( \frac{v(A)}{x} + x - \delta(A) \right) \right\}$$

- $\Gamma(A)$  depends only on the variance  $v(A)$  and the nonzero eigenvalue  $\delta(A)$  of  $S(A)$ .
- It is symmetric with respect to the real axis which it intercepts at  $\delta(A)$ .
- If  $v(A) \geq \delta^2(A)/4$ , then  $\Gamma(A)$  consists of an unbounded branch.
- If  $v(A) < \delta^2(A)/4$ , then  $\Gamma(A)$  consists of an unbounded branch and a bounded branch, and it also intercepts the real axis at the points

$$\frac{\delta(A) \pm \sqrt{\delta^2(A) - 4v(A)}}{2}$$

- $\Gamma(A)$  yields a localization of the spectrum of  $A \longrightarrow$

## SHELL ILLUSTRATION

We have taken  $5 \times 5$  almost skew-symmetric matrices  $A$ ,  $B$  and  $C$  with variances  $v(A) = 1.75$ ,  $v(B) = 4.5$  and  $v(C) = 1$ . Also  $\delta(A) = 4$ ,  $\delta(B) = 4$  and  $\delta(C) = 2$ . The eigenvalues of each matrix are marked with '+'s. The shell  $\Gamma(A)$  consists of one bounded and one unbounded branch. The bounded branch surrounds a real eigenvalue of  $A$  and the unbounded branch isolates the rest of the spectrum. The shell  $\Gamma(B)$  is connected and all the eigenvalues of  $B$  are located in the region between  $\Gamma(B)$  and the imaginary axis. The shell  $\Gamma(C)$  can be loosely described as an inverted  $\alpha$ -curve.

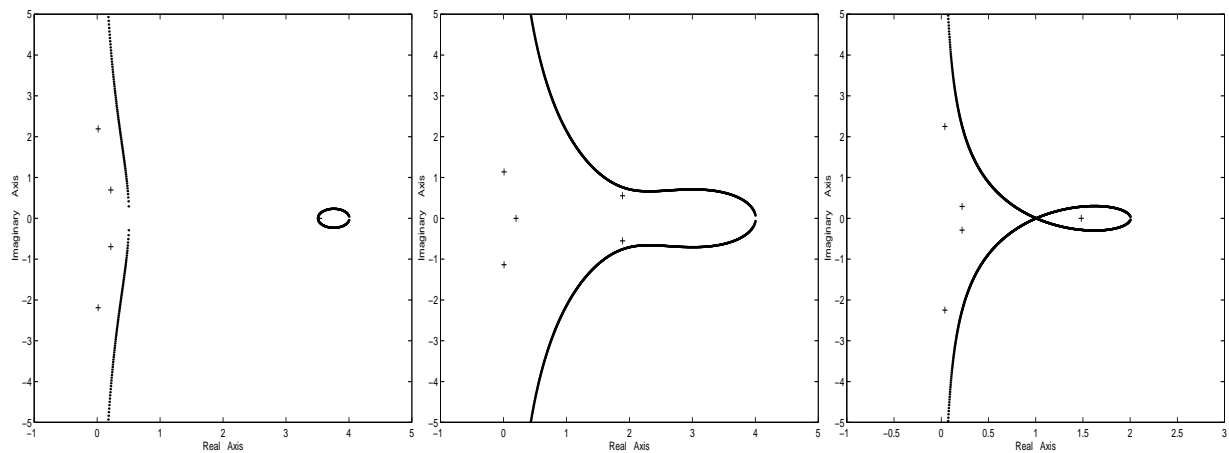


Figure 1: The shells  $\Gamma(A)$ ,  $\Gamma(B)$  and  $\Gamma(C)$ .

## TRANSFORMATIONS THAT PRESERVE ALMOST SKEW-SYMMETRY

- If  $S(A) = ww^t$  and  $Cw \neq 0$  for some  $C \in \mathcal{M}_n(\mathbb{R})$ , then the congruence  $CAC^t$  is also an almost skew-symmetric matrix.
- *Levinger's transformation* of  $A \in \mathcal{M}_n(\mathbb{R})$  is the parametrized family of matrices

$$\mathcal{L}(A, a) = (1 - a)A + aA^t, \quad a \in \mathbb{R} .$$

$\mathcal{L}(A, a)$  is almost skew-symmetric,  $S(\mathcal{L}(A, a)) = ww^t$  and

$$v(\mathcal{L}(A, a)) = (1 - 2a)^2 v(A).$$

- Let  $A \in \mathcal{M}_n(\mathbb{R})$  be an invertible almost skew-symmetric matrix. Then  $A^{-1}$  is also almost skew-symmetric and  $K(A^{-1}) = K(A^{-t}AA^{-t})$ .

**But more holds:**

## PRINCIPAL PIVOT TRANSFORMS

Suppose  $A$  is partitioned as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where  $A_{11}$  is an invertible submatrix. The *principal pivot transform* (or *exchange*) of  $A$  relative to  $A_{11}$  is

$$\text{ppt}(A, A_{11}) = \begin{pmatrix} (A_{11})^{-1} & -(A_{11})^{-1}A_{12} \\ A_{21}(A_{11})^{-1} & A_{22} - A_{21}(A_{11})^{-1}A_{12} \end{pmatrix}.$$

**Theorem** Let  $A \in \mathcal{M}_n(\mathbb{R})$  be partitioned as above,  $A_{11}$  be invertible and  $\text{rank } S(A) = 1$ . Then  $\text{rank } S(\text{ppt}(A, A_{11})) = 1$ . In particular, if  $A$  is an almost skew-symmetric matrix, then so is  $\text{ppt}(A, A_{11})$ . Moreover,  $S(A/A_{11})$  has rank at most one.

## A FEATURE OF THE NUMERICAL RANGE

Let  $A \in \mathcal{M}_n(\mathbb{R})$  be an almost skew-symmetric matrix with numerical range  $F(A)$ . Then  $\partial F(A) \cap \{z \in \mathbb{C} : \operatorname{Re} z = 0\}$  is a line segment.

**Example** Let

$$A_1 = \begin{pmatrix} 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 2 & 1 \\ 0 & 2 & 0 & 1 & 1 \\ -1 & 2 & 1 & 1 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & -1 & 1 & 0 \\ 1 & 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix},$$

which have  $v(A_1) = 3.5$ ,  $\delta(A_1) = 4$  and  $v(A_2) = 1$ ,  $\delta(A_2) = 2$ .

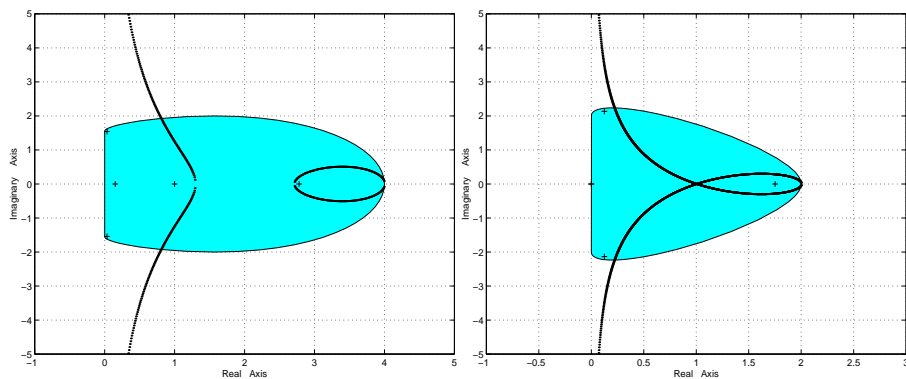


Figure 2: The numerical ranges (shaded) and shells of  $A_1$  and  $A_2$ .

## REFERENCES

- J.J. McDonald, P.J. Psarrakos and M.J. Tsatsomeros. Almost Skew-Symmetric Matrices. *Rocky Mountain Journal of Mathematics*, to appear.
- S. Kirkland, P.J. Psarrakos, and M.J. Tsatsomeros. On the Location of the Spectrum of Hypertournament Matrices. *Linear Algebra and Its Applications*, 323 : 37-49, 2001.
- P.J. Psarrakos and M.J. Tsatsomeros. The Perron Eigenspace of Nonnegative Almost Skew-Symmetric Matrices and Levinger's Transformation. *Linear Algebra and Its Applications*, 360:43-57, 2003.

(P)reprints & Matlab code can be found at:

[www.math.wsu.edu/math/faculty/tsat/](http://www.math.wsu.edu/math/faculty/tsat/)