

Notation and Definitions

For the set $\{1, \dots, n\}$, we write $\langle n \rangle$.

Let A be an $n \times n$ real or complex matrix. The multiset

$$\sigma(A) = \{ \lambda \mid \lambda \text{ is an eigenvalue of } A \},$$

where each eigenvalue is listed the number of times it occurs as a root of the characteristic polynomial of A , is referred to as the *spectrum* of A .

$$\rho(A) = \max_{\lambda \in \sigma(A)} \{ |\lambda| \}$$

is the *spectral radius* of A . The multiset

$$\pi(A) = \{ \lambda \in \sigma(A) \mid |\lambda| = \rho(A) \}$$

is referred to as the *peripheral spectrum* of A . The peripheral eigenvalues of A are also referred to as *dominant* eigenvalues.

By $\text{index}_0(A)$ we denote the degree of 0 as a root of the minimal polynomial of A . Consequently, when we say $\text{index}_0(A) \leq 1$, we mean that either A is invertible or that the size of the largest nilpotent Jordan block in the Jordan canonical form of A is 1×1 .

Let $t = \text{index}_0(A)$, and for each $i \in \langle t \rangle$, set $\eta_i(A) = \text{nullity}(A^i) - \text{nullity}(A^{i-1})$. The sequence $\eta(A) = (\eta_1(A), \eta_2(A), \dots, \eta_t(A))$ is referred to as the *height* or *Weyr characteristic* of A .

The *generalized eigenspace*, $E_\lambda(A)$, of the matrix A corresponding to λ is the nullspace of $(A - \lambda I)^r$, where $r = \text{index}_\lambda(A)$.

A *Jordan chain* corresponding to $E_\lambda(A)$ is a set of nonzero vectors $\{x, (\lambda I - A)x, \dots, (\lambda I - A)^{r-1}x\}$ where $(\lambda I - A)^r x = 0$. A *Jordan basis* for the generalized eigenspace of A is a basis of $E_\lambda(A)$ consisting of the union of Jordan chains.

If we have a multiset of complex numbers, σ , we let $\bar{\sigma}$ denote the multiset containing all the complex conjugates of the elements in sigma, we let $\rho(\sigma) = \max\{|\lambda| \mid \lambda \in \sigma\}$ and $\pi(\sigma) = \{ \lambda \in \sigma \mid |\lambda| = \rho(\sigma) \}$.

The *nonnegative orthant* in \mathbb{R}^n , i.e., the set of all nonnegative vectors in \mathbb{R}^n , is denoted by \mathbb{R}_+^n . For $x \in \mathbb{R}^n$, we use the notation $x \geq 0$ interchangeably with $x \in \mathbb{R}_+^n$.

An $n \times n$ matrix A is called *reducible* if there exists a permutation matrix P such that

$$PAP^T = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

where A_{11} and A_{22} are square, non-vacuous matrices. Otherwise, A is called *irreducible*.

An $n \times n$ matrix $A = [a_{ij}]$ is called:

- *nonnegative (positive)*, denoted by $A \geq 0$ ($A > 0$), if $a_{ij} \geq 0$ (> 0) for all i and j .
- *essentially nonnegative (positive)*, denoted by $A \stackrel{s}{\geq} 0$ ($A \stackrel{s}{>} 0$), if $a_{ij} \geq 0$ ($a_{ij} > 0$) for all $i \neq j$.
- *eventually nonnegative (positive)*, denoted by $A \stackrel{v}{\geq} 0$ ($A \stackrel{v}{>} 0$), if there exists positive integer k_0 such that $A^k \geq 0$ ($A^k > 0$) for all $k \geq k_0$. We denote the smallest such positive integer by $k_0 = k_0(A)$ and refer to it as the *power index* of A .
- *exponentially nonnegative (positive)* if $\forall t \geq 0, e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} \geq 0$ ($e^{tA} > 0$).
- *eventually exponentially nonnegative (positive)* if $\exists t_0 \in [0, \infty)$ such that $\forall t \geq t_0, e^{tA} \geq 0$ ($e^{tA} > 0$). We denote the smallest such nonnegative number by $t_0 = t_0(A)$ and refer to it as the *exponential index* of A .

We say that $A \in \mathbb{R}^{n \times n}$ has

- the *Perron-Frobenius property* if $\rho(A) > 0$, $\rho(A) \in \sigma(A)$ and there exists a nonnegative eigenvector corresponding to $\rho(A)$;
- the *strong Perron-Frobenius property* if, in addition to having the Perron-Frobenius property, $\rho(A)$ is a simple eigenvalue such that

$$\rho(A) > |\lambda| \quad \text{for all } \lambda \in \sigma(A), \lambda \neq \rho(A),$$

as well as there is a strictly positive eigenvector corresponding to $\rho(A)$.