1 Definitions

Let $\eta = (\eta_1, \eta_2, \ldots, \eta_t)$ and $\nu = (\nu_1, \nu_2, \ldots, \nu_t)$ be two sequences of nonnegative integers. (Append zeros if necessary to the end of the shorter sequence so that they are the same length.) We say that ν is *majorized* by η if $\sum_{l=1}^{j} \nu_l \leq \sum_{l=1}^{j} \eta_l$ for all $1 \leq j \leq t$ and $\sum_{l=1}^{t} \nu_l = \sum_{l=1}^{t} \eta_l$. We write $\nu \preceq \eta$.

Let $\Gamma = (V, E)$ be a graph where V is a finite vertex set and $E \subseteq V \times V$ is an edge set. A *path* from j to m is a sequence of vertices $j = v_1, v_2, ..., v_t = m$ with $(v_l, v_{l+1}) \in E$ for $l = 1, ..., t-1$. A *simple path* is a path where the vertices are pairwise distinct. The empty path will be considered to be a simple path linking every vertex to itself. The path $v_1, v_2, ..., v_t$ is a *cycle* if $v_1 = v_t$ and $v_1, v_2, \ldots, v_{t-1}$ is a simple path.

Let $\Gamma = (V, E)$ be a graph. We say a vertex l has *access* to a vertex j if there is a path from l to j in Γ. A vertex is *final* if it does not access any other vertex in Γ. A vertex is *initial* if it is not accessed by another vertex. We define the *transitive closure* of Γ by $\overline{\Gamma} = (V, \overline{E})$ where $\overline{E} = \{(j, l) | j \text{ has access to } l \text{ in } \Gamma\}$. If l has access to j and j has access to l, we say j and l *communicate*. The communication relation is an equivalence relation. Thus we can partition V into equivalence classes which we will refer to as the *classes* of Γ.

We define the graph of a matrix A by $G(A) = (V, E)$ where $V = \langle n \rangle$ and $(l, j) \in E$ whenever $a_{lj} \neq 0$. Irreducibility is equivalent to the property that every two vertices in $G(A)$ communicate. The classes of $G(A)$ are also referred to as the irreducible classes of A.

Let $K, L \subseteq \langle n \rangle$. The matrix A_{KL} is the submatrix of A whose rows are indexed by K and whose columns are indexed by L. The sets $K_1, K_2, ..., K_k$ *partition* a set K if they are pairwise disjoint and $\bigcup_{j=1}^{k} K_j = K$. We allow $K_j = \emptyset$ for ease of notation in some places. If $\kappa = (K_1, ..., K_k)$ is an ordered partition of a subset of $\langle n \rangle$, we write:

$$A_\kappa = \begin{bmatrix} A_{K_1K_1} & A_{K_1K_2} & \ldots & A_{K_1K_k} \\ A_{K_2K_1} & A_{K_2K_2} & \ldots & A_{K_2K_k} \\ \vdots & \vdots & & \vdots \\ A_{K_kK_1} & A_{K_kK_2} & \ldots & A_{K_kK_k} \end{bmatrix}.$$

We say A_κ is *block lower triangular* if $A_{K_lK_j} = 0$ whenever $l < j$.

Given a matrix A, there exists an ordered partition $\kappa = (K_1, K_2, ..., K_k)$ of $\langle n \rangle$ so that each K_i corresponds to a class of $G(A)$ and A_κ is block lower triangular. The matrix A_κ is referred to as the *Frobenius normal form* of A. A class K_j is said to be *singular* if $A_{K_jK_j}$ is singular and *nonsingular* otherwise. The classes for which $\rho(A_{K_j,K_j}) = \rho(A)$ are referred to as the *basic classes* of A.

We define the *reduced graph of* A by $R(A) = (V, E)$ where $V = \{ \ K \mid K$ is an irreducible class of $A \ \}$, and $E = \{ \ (K, L) \mid$ there is edge from a vertex $j \in K$ to a vertex $l \in L$ in $G(A) \ \}$.

The *singular length* of a simple path in $R(A)$ is the sum of the indexes of zero of each of the singular vertices it contains. The *level* of a vertex K is the maximum singular length over all the simple paths in $R(A)$ which terminate at K.

Let $\nu_i(A)$ be the number of singular vertices with level i in $R(A)$ and let m be the largest number for which $\nu_i(A) \neq 0$. Then $\nu(A) = (\nu_1(A), \ldots, \nu_m(A))$ is referred to as the *level characteristic* of A.

2 Combinatorial Properties of Nonnegative Matrices

The relationship between the combinatorial structure of a nonnegative matrix and its spectrum, eigenvectors, and Jordan structure is surprisingly elegant and beautiful, as well as useful. Surveys of results of this type can be found in Berman and Plemmons [1], Hershkowitz [5], and Schneider [14]. We provide some of the results listed in these papers here, as well as some additional results.

We offer an example to illustrate some of the theorems we present here as well as the some of the definitions given in the definitions section.

Example 2.1 Let

$$A = \left[\begin{array}{c|c|c|c|cc} 4 & 0 & 0 & 0 & 0 & 0 \\ \hline 5 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 2 & 2 & 0 & 0 & 0 \\ \hline 1 & 1 & 0 & 4 & 0 & 0 \\ \hline 1 & 0 & 0 & 1 & 1 & 2 \\ 2 & 0 & 0 & 0 & 3 & 2 \end{array} \right].$$

Notice that A is in Frobenius Normal Form and has 5 classes, 3 basic classes, and $\rho(A) = 4$. Consider the reduced graph of A :

where circles denote basic classes and squares nonbasic classes. From the graph, we easily determine the level of each class, summarized below:

The Classes of A and Their Levels in $R(\rho(A)I - A)$		
Class	Level	$\rho(A_{K_i})$
$K_1 = \{1\}$	3	4 (Basic)
$K_2 = \{2\}$	2	1 (Nonbasic)
$K_3 = \{3\}$	0	2 (Nonbasic)
$K_4 = \{4\}$	2	4 (Basic)
$K_5 = \{5, 6\}$	1	4 (Basic)

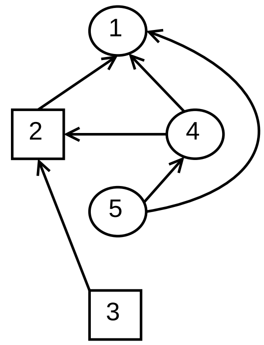

Figure 1: Reduced Graph of A

We can find generalized eigenvectors of the form

$$x^1 = \begin{bmatrix} * \\ * \\ * \\ * \\ * \\ * \end{bmatrix}, x^2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ * \\ * \\ * \end{bmatrix}, x^3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ * \\ * \end{bmatrix}$$

where $*$ denotes a positive entry, and the set $\{x^1, x^2, x^3\}$ will be a basis for the algebraic eigenspace of A with respect to $\rho(A)$. We may obtain an eigenvector of A with zero-nonzero pattern given by x^3 by attaching the unique (up to scalar multiplication) positive eigenvector of A_{K_5} to the zero vector of length 4. We find that

$$x^1 = \begin{bmatrix} 1 \\ 5/3 \\ 5/3 \\ 1 \\ 52/75 \\ 1 \end{bmatrix}, x^2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 5 \\ 3 \\ 3 \end{bmatrix}, x^3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2 \\ 3 \end{bmatrix}$$

are of the prescribed form and satisfy $x^3 \in null(A - 4I)$, $x^2 \in null[(A - 4I)^2]$, $x^1 \in null[(A - 4I)^3]$ and so form a basis for the algebraic eigenspace of A with respect to the eigenvalue 4. Then we also have that x^1 is the generalized eigenvector having the most positive entries of all generalized eigenvectors of A.

Also, notice that, if we let $\kappa = (K_1, K_2 \cup K_4, K_5, K_3)$, then

$$A_\kappa = \left[\begin{array}{c|cc|cc|c} 4 & 0 & 0 & 0 & 0 & 0 \\ \hline 5 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 4 & 0 & 0 & 0 \\ \hline 1 & 0 & 1 & 1 & 2 & 0 \\ 2 & 0 & 0 & 3 & 2 & 0 \\ \hline 0 & 2 & 0 & 0 & 0 & 2 \end{array}\right]$$

is in level form.

We now proceed with a series of theorems illustrating the relationship between the combinatorial structure of a nonnegative matrix, and its spectral properties.

Theorem 2.2 (Rothblum [12]) *Let A be an entrywise nonnegative matrix. Let $\rho = \rho(A)$. Then the $\mathrm{index}_\rho(A)$ is equal to the maximum level of a vertex in $R(\rho I - A)$.*

In [6], Hershkowitz and Schneider show that for a nonnegative matrix, the level characteristic, reordered into nonincreasing order, is majorized by the height characteristic.

Theorem 2.3 (Hershkowitz and Schneider [6]) *Let A be an entrywise nonnegative matrix. Let $\rho = \rho(A)$ and $\hat{\nu}$ be the ordered sequence formed by listing the elements of $\nu(\rho I - A)$ in nonincreasing order. Then $\hat{\nu} \preceq \eta(\rho I - A)$.*

Our next result shows there is a nonnegative basis for the generalized eigenspace of a nonnegative A, associated with $\rho(A)$, and the positive entries are combinatorially determined.

Theorem 2.4 (Rothblum [12]) *Let A be a nonnegative matrix. Let $\rho = \rho(A)$. Let $\kappa = (K_1, K_2, \ldots, K_k)$ be such that A_κ is in Frobenius normal form. Let $p = \mathrm{mult}_\rho(A)$ and $i_1, \ldots, i_p$ be chosen so that $\rho(A_{K_{i_j}K_{i_j}}) = \rho$. Then there exists a basis $\{x^{(1)}, x^{(2)}, \ldots, x^{(p)}\}$ of $E_\rho(A)$ such that*

$$x^{(j)}_{K_l} \begin{cases} \gg 0 & \text{if } K_l \text{ has access to } K_{i_j} \text{ in } R(A). \\ = 0 & \text{if } K_l \text{ does not have access to } K_{i_j} \text{ in } R(A). \end{cases}$$

Next we characterize the nonnegative matrices A that have a positive right eigenvector associated with $\rho(A)$.

Theorem 2.5 (Rothblum [13]) *Let A be a nonnegative matrix. Let $\rho = \rho(A)$. The following are equivalent:*

(i) *There exists an* $x \gg 0$ *so that* $(\rho I - A)x = 0$.

(ii) *The set of singular vertices of* $R(\rho I - A)$ *is equal to the set of final vertices of* $R(\rho I - A)$.

In the next two theorems we look at nonnegative eigenvectors associated with eigenvalues in addition to the spectral radius.

Theorem 2.6 (Victory [16]) *Let* A *be a nonnegative matrix in in Frobenius normal form. Let* λ *be a real number. The following are equivalent:*

(i) *There exists an eigenvector* x *so that* $Ax = \lambda x$ *and* $x > 0$.

(ii) *There is a vertex* K_l *of* $R(A)$ *so that whenever* K_j *has access to* K_l *in* $R(A)$*, then*

$$\lambda = \rho(A_{K_l K_l}) > \rho(A_{K_j K_j}).$$

Theorem 2.7 (Victory [16]) *Let* A *be an nonnegative matrix in Frobenius normal form. Let* λ *be a real number. If there is a vertex* K_l *of* $R(A)$ *so that whenever* K_j *has access to* K_l *in* $R(A)$*, then* $\lambda = \rho(A_{K_l K_l}) > \rho(A_{K_j K_j})$*, there is a (up to scalar multiples) unique vector* x *that satisfies* $Ax = \lambda x$ *and*

$$x_{K_j} \begin{cases} \gg 0 & \text{if } K_j \text{ has access to } K_l \text{ in } R(A). \\ = 0 & \text{if } K_j \text{ does not have access to } K_l \text{ in } R(A). \end{cases}$$

We conclude our list of combinatorial properties of nonnegative matrices with one final theorem. The combinatorial spectral properties of nonnegative matrices is a very rich area. We recognize that our list is by no means exhaustive and encourage readers to send us their favorites to add to the list.

Theorem 2.8 *Let* A *be a nonnegative matrix. Let* z *be a vector. Let* $\rho = \rho(A)$*. Then the following are equivalent:*

(i) $(\rho I - A)z \geq 0$ *implies that* $(\rho I - A)z = 0$.

(ii) *The set of initial vertices of* $R(\rho I - A)$ *is equal to the set of singular vertices of* $R(\rho I - A)$.

3 Combinatorial Properties of Eventually Nonnegative Matrices

The nonnegative matrices exhibit many combinatorial properties which do not appear to carry over to the eventually nonnegative matrices. Here we show how to construct an irreducible eventually nonnegative matrix for which the spectral radius is a multiple eigenvalue. It is the nilpotent part of an eventually nonnegative matrix is the major contributor to the apparent lack of combinatorial consistency between nonnegative and eventually nonnegative matrices.

Example 3.1 We take a reducible nonnegative matrix and turn it into an irreducible eventually nonnegative matrix by adding an an appropriate nilpotent matrix.

Let $B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$. Then B is a reducible nonnegative matrix with $\rho(B) = 2$, a double root. Notice that $x = [1, -1, 0, 0]^T$ is a right nullvector of A and $y^T = [0, 0, 1, -1]$ is a left nullvector A. Let

$$C = \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then $BC = CB = C^2 = 0$. Let $A = B + C$. Then $A^j = B^j$ for all $j \geq 2$, so A is eventually nonnegative. On the other hand,

$$A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

is irreducible with $\rho(A) = 2$ appearing as an eigenvalue with multiplicity two. Notice $\text{index}_0(A) = 2$.

Now consider

$$D = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \end{bmatrix}.$$

Then $D+C$ is an irreducible eventually nonnegative matrix with the spectral radius 4 as a simple root. The associated eigenvector is $z = [0, 0, 1, 1]^T$, which is not a positive vector.

Using this technique, we can create reducible eventually nonnegative matrices for which various combinatorial properties of nonnegative matrices fail, and we encourage interested readers to experiment on their own. We also present a few more examples at the end of our paper

In Carnochan Naqvi and McDonald [2], they show that Theorems 2.2 -2.8 hold for eventually nonnegative matrices A with $\text{index}_0(A) \leq 1$. Also in this paper the following set of interesting examples are discussed:

Example 3.2 We now look at two eventually nonnegative matrices which appear in [17]. The matrices

$$A^{(1)} = \left[\begin{array}{cc|cc|cc} 2 & 2 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & -1 & 2 & 2 \\ 0 & 0 & -1 & 1 & 2 & 2 \end{array}\right], \textit{ and } A^{(2)} = \left[\begin{array}{cc|cccc} 2 & 2 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 2 & 0 & -1 & 1 \\ 1 & 1 & 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & -1 & 1 & 3 \\ 0 & 0 & -1 & 1 & 3 & 1 \end{array}\right]$$

have $\text{index}_0(A^{(j)}) = 2$. Both matrices have vertices in $R(4I - A^{(j)})$ with level 2 even though $\text{index}_4(A^{(j)}) = 1$. If, however, we raise each matrix to the power of the $\text{index}_0(A^{(j)})$ we see that

$$(A^{(1)})^2 = (A^{(2)})^2 = \left[\begin{array}{cc|cc|cc} 8 & 8 & 0 & 0 & 0 & 0 \\ 8 & 8 & 0 & 0 & 0 & 0 \\ \hline 6 & 6 & 2 & 2 & 0 & 0 \\ 6 & 6 & 2 & 2 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 8 & 8 \\ 0 & 0 & 0 & 0 & 8 & 8 \end{array}\right]$$

which obviously exhibits the standard combinatorial structure of a nonnegative matrix.

Example 3.3 Consider the matrix A below which is 1-cyclic eventually nonnegative and A^6 is the direct sum of 6 positive matrices. We will be defining A in terms of the sum of two matrices B and C so that $\text{index}_0(B) \leq 1$, C is nilpotent and $BC = CB = 0$.

Let T be the 2×2 matrix of all twos. Let

$$R = \left[\begin{array}{rr} 1 & -1 \\ -1 & 1 \end{array}\right]$$

We define B and C as follows:

$$B = \begin{bmatrix} 0 & T & 0 & 0 & 0 & 0 \\ T & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & T & 0 & 0 \\ 0 & 0 & 0 & 0 & T & 0 \\ 0 & 0 & T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & T \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 0 & R & R & R & -R \\ 0 & 0 & -R & -R & R & -R \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ R & -R & 0 & 0 & 0 & 0 \\ R & -R & 0 & 0 & 0 & 0 \end{bmatrix}$$

with $A = B + C$.

The matrix A is an irreducible eventually nonnegative matrix. Once A is raised to power $3 = \text{index}_0(A)$, A^3 can be partitioned into the direct sum of nonnegative matrices. In particular, any prime power $g > 3$ is not an element of $D_A = \{\pm 1, \pm 2, \pm 3\}$ and then the Frobenius normal form of A^g is the same as the Frobenius normal form of B in the decomposition of A.

For example, seven is not an element in D_A. Consider

$$A^7 = B^7 = \left[\begin{array}{cccc|cccccc|cc} 0 & 0 & 2^7 & 2^7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2^7 & 2^7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2^7 & 2^7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2^7 & 2^7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 2^7 & 2^7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2^7 & 2^7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2^7 & 2^7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2^7 & 2^7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2^7 & 2^7 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2^7 & 2^7 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2^7 & 2^7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2^7 & 2^7 \end{array}\right].$$

This is the same reducible structure as the matrix B in the decomposition of A.

In our last example, we show how to construct nonsingular irreducible eventually positive matrices A so that the smallest g for which $A^g \gg 0$, depends on quantitative rather than qualitative properties of A. Thus results on exponents of primitive matrices (see [1]) will not carry over to eventually nonnegative matrices with $\text{index}_0 \leq 1$.

Example 3.4 Given any odd integer $N > 0$, we construct an $n \times n$ matrix A so that A^N is not positive, but $A^g \gg 0$ for all $g > N$. We use a method due to Soules [15], and generalized in [3], to illustrate how to construct a matrix A with prescribed eigenvalues. Let

$$\mathcal{T}_n = \{(\lambda_1, \lambda_2, \ldots, \lambda_n) | 1 = \lambda_1 \geq \lambda_2, \ldots, \lambda_n \geq -1\}.$$

be a subset of $\mathbb{R}^n$. Let D be the diagonal matrix

$$D = \begin{bmatrix} \lambda_1 & 0 & \dots & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & \lambda_{n-1} & 0 \\ 0 & \dots & \dots & 0 & \lambda_n \end{bmatrix}.$$

Then from the work in [**?**], we know that for each Soules matrix R, there is a closed convex polytope $\mathcal{S}(R) \subseteq \mathcal{T}_n$, for which $A = RDR^T \geq 0$ for $(\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathcal{T}_n$ if and only if $(\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathcal{S}(R)$. The point $(1, 0, \dots, 0) \in \mathcal{S}(R)$, so if $|\lambda_j| < 1$ for all $1 < j \leq n$, then the points $(1, \lambda_2^g, \dots, \lambda_n^g)$ will converge to points in $\mathcal{S}(R)$, even when we start with points which are not in $\mathcal{S}(R)$. In this manner we can easily construct symmetric eventually nonnegative matrices which become nonnegative matrices at a prescribed exponent. We now pick a particular Soules matrix R, and choose our eigenvalues for A so that A^N is nonnegative, and this is the first odd exponent for which this happens. For this particular example, we will go on to show that $A^g \gg 0$, for $g \geq N+1$, but the diagonal elements of A^N are zero. Consider the Soules matrix

$$R = \begin{bmatrix} \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{(n-1)(n-2)}} & \dots & \dots & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{(n-1)(n-2)}} & \dots & \dots & -\frac{1}{\sqrt{2}} \\ \vdots & \vdots & \vdots & & & 0 \\ \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{(n-1)(n-2)}} & & & \vdots \\ \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n(n-1)}} & \frac{2-n}{\sqrt{(n-1)(n-2)}} & & & \vdots \\ \frac{1}{\sqrt{n}} & \frac{1-n}{\sqrt{n(n-1)}} & 0 & \dots & \dots & 0 \end{bmatrix}.$$

Let $1 \geq \delta \geq 0$ be a positive parameter and let

$$D = \begin{bmatrix} 1 & 0 & \dots & \dots & 0 \\ 0 & -\delta & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & -\delta & 0 \\ 0 & \dots & \dots & 0 & -\delta \end{bmatrix}.$$

Let $A = RDR^T$. Then $A^g = RD^gR^T$

$$= \begin{bmatrix} \frac{1+(n-1)(-\delta)^g}{n} & \frac{1-(-\delta)^g}{n} & \cdots & \cdots & \frac{1-(-\delta)^g}{n} \\ \frac{1-\delta^g}{n} & \frac{1+(n-1)(-\delta)^g}{n} & \frac{1-(-\delta)^g}{n} & \cdots & \frac{1-(-\delta)^g}{n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \frac{1-(-\delta)^g}{n} & \cdots & \frac{1-(-\delta)^g}{n} & \frac{1+(n-1)(-\delta)^g}{n} & \frac{1-(-\delta)^g}{n} \\ \frac{1-(-\delta)^g}{n} & \cdots & \cdots & \frac{1-(-\delta)^g}{n} & \frac{1+(n-1)(-\delta)^g}{n} \end{bmatrix}.$$

Setting $\delta = (\frac{1}{n-1})^{\frac{1}{N}}$, we see that the diagonal elements of A^N are zero. Notice $A^N \geq 0$ and $A^g \gg 0$, for $g \geq N+1$. The eigenvalues of A^g are $\{1, (-\delta)^g, \ldots, (-\delta)^g\}$.

4 The Peripheral Jordan Form of a Nonnegative (or Eventually Nonnegative) Matrix

In this section, we investigate the Jordan form corresponding to the peripheral spectrum of a nonnegative matrix. As mentioned in Section 2, it is known that if ν and η are two sequences of nonnegative integers, then there exists a nonnegative matrix A with height characteristic η, corresponding to the spectral radius, and level characteristic ν, corresponding to the spectral radius (with entries rewritten in decreasing order), if and only if ν is majorized by η ([7] Theorem 3.3 and [11] Corollary 4.5). In this section, we extend this property to all eigenvalues in the peripheral spectrum and offer necessary and sufficient conditions, based on the level and height characteristics, for a multiset $\mathcal{J}$ of Jordan blocks to correspond to the peripheral spectrum of a nonnegative matrix.

It follows easily from the Perron Frobenius Theorem that the peripheral spectrum of a nonnegative matrix A is a union of complete sets of roots of unity (multiplied by $\rho(A)$). Hence, if a multiset $\mathcal{J}$, all of whose eigenvalues have modulus 1, corresponds to the peripheral spectrum of a nonnegative matrix, then $\sigma(\mathcal{J})$ must partition into complete sets of roots of unity. This partition can be determined based on the following well known lemma.

Lemma 4.1 If a multiset S can be partitioned into complete sets of roots of unity, then the partition is unique.

We extend the notion of level characteristic to all of the peripheral eigenvalues.

Definition 4.2 Suppose $A \geq 0$. Let $\mu = (M_1, M_2, \ldots, M_m, M_{m+1})$ be the level partition of A with respect to $\rho = \rho(A)$. For each $\lambda \in \pi(A)$, let $\nu_i(\lambda)$ be the number of times λ occurs as an eigenvalue of $A_{M_iM_i}$. Then the $\lambda-$*level characteristic of* A *(with respect to* ρ*)* is the sequence $\nu_{\lambda,\rho}(A) = (\nu_1(\lambda), \ldots, \nu_m(\lambda))$ (note that $\nu_{m+1}(\lambda) = 0$ so it need not be included).

Lemma 4.3 (McDonald [8]) *Let A be a nonnegative matrix. Set $m = index_{\rho(A)}(A)$ and let $(L_1, L_2, \ldots, L_{2m+1})$ be the split–level partition of A with respect to $\rho(A)$. Set*

$$P_j = \bigcup_{q=2(m+1-j)}^{2m+1} L_q$$

and let $\lambda \in \pi(A)$. Then for $j = 2, \ldots, m$, the Jordan form of λ for $A_{P_jP_j}$ can be produced from the Jordan form of λ for $A_{P_{j-1}P_{j-1}}$ by increasing the size of a select number of Jordan blocks by one, and adding copies of $J_1(\lambda)$.

Corollary 4.4 (McDonald and Morris [9]) *Let $A \geq 0$ with $\rho(A) = \rho$. Then, for each $\lambda \in \pi(A)$, $\hat{\nu}_{\lambda,\rho}(A) \preceq \eta_\lambda(A)$.*

As stated in the above definition and corollary, the peripheral eigenvalues of a nonnegative matrix A must be distributed among the levels of A such that the majorization condition in Corollary 4.4 is satisfied. We are interested in the question of whether or not there exists a nonnegative matrix with peripheral spectrum corresponding to a given multiset $\mathcal{J}$ of Jordan blocks. Corollary 4.4 asserts that the eigenvalues of $\mathcal{J}$ must partition into level sets satisfying the majorization criterion. The definition below will allow us to gather information about a partition of a multiset of eigenvalues.

Definition 4.5 Suppose $L_1, \ldots, L_k$ are multisets of eigenvalues. For each $\lambda \in \bigcup_{i=1}^{k} L_i$, let $\nu_\lambda(L_i)$ be the number of times λ is listed in L_i. Then $\nu_\lambda(L_1, L_2, \ldots, L_k)$ is defined to be the sequence $(\nu_\lambda(L_1), \nu_\lambda(L_2), \ldots, \nu_\lambda(L_k))$.

We are now ready to state a necessary and sufficient condition on the Jordan form of a nonnegative (or eventually nonnegative) matrix. Note that we state our theorem for the case where $\rho(A) = 1$. If $\rho(A) \neq 1$, consider the matrix $\frac{1}{\rho(A)}A$.

Theorem 4.6 ([9]) Let $\mathcal{J}$ be a self conjugate multiset of Jordan blocks all of whose eigenvalues have modulus 1. Let m be the size of the largest Jordan block in $\mathcal{J}$ with eigenvalue 1. Then the following are equivalent.

(i) $\mathcal{J}$ corresponds to the peripheral Jordan form of a nonnegative matrix.

(ii) $\sigma(\mathcal{J}) = L_1 \bigcup L_2 \bigcup \cdots \bigcup L_m$ where $\bigcup$ represents a multiset union and

- **(a)** Each L_i partitions into complete sets of roots of unity and
- **(b)** For each $\lambda \in \sigma(\mathcal{J})$, $\hat{\nu}_\lambda(L_1, \ldots, L_m) \preceq \eta_\lambda(\mathcal{J})$.

Remark 4.7 Part (ii) (b) of Theorem 4.6 is equivalent to saying that no two eigenvalues in an L_i come from the same Jordan block in $\mathcal{J}$.

Remark 4.8 The exact same result holds for the peripheral spectrum of an eventually nonnegative matrix.

Remark 4.9 Suppose $\mathcal{J}$ satisfies condition (ii) of Theorem 4.6. Then there exists a nonnegative matrix

$$A = \begin{bmatrix} A_{11} & 0 & \dots & 0 \\ A_{21} & A_{22} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ A_{m1} & A_{m2} & \dots & A_{mm} \end{bmatrix}$$

in level form with Jordan form corresponding to $\mathcal{J}$. Moreover, from the proof of Theorem 3.5 in [8], we may construct A so that each subdiagonal block A_{ij}, $i > j$, is positive and has arbitrarily large entries.

The level blocks $L_1, L_2, \dots, L_m$ in the above theorem can be put in any order, as illustrated by the following corollary.

Corollary 4.10 (McDonald and Morris [9]) Let

$$A = \begin{bmatrix} A_{11} & 0 & \dots & 0 \\ A_{21} & A_{22} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ A_{m1} & A_{m2} & \dots & A_{mm} \end{bmatrix}$$

be a nonnegative matrix in level form, all of whose eigenvalues have modulus 1. Then, for any permutation τ of $\langle m \rangle$, there exists a $B \geq 0$ similar to A such that the level form of B is

$$B = \begin{bmatrix} B_{11} & 0 & \dots & 0 \\ B_{21} & B_{22} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ B_{m1} & B_{m2} & \dots & B_{mm} \end{bmatrix}.$$

and $\sigma(B_{ii}) = \sigma(A_{\tau(i)\tau(i)})$ for $i = 1, \dots, m$.

We will refer to the condition stated in Theorem 4.6 (ii)(b) as the majorization condition. Suppose $\mathcal{J}$ is a multiset of Jordan blocks (all of whose eigenvalues have modulus 1) and m is the size of the largest Jordan block in $\mathcal{J}$. Note that it is possible to find a partition $L_1, \dots, L_m$ of $\sigma(\mathcal{J})$ such that the majorization condition is satisfied. As the next example illustrates, it may also be possible to partition the eigenvalues of $\mathcal{J}$ into m sets, each of which is a union of complete sets of roots of unity. However, it may not be possible to achieve both requirements with the *same* partition of $\sigma(\mathcal{J})$, and hence $\mathcal{J}$ does not correspond to the peripheral Jordan form of a nonnegative matrix.

Example 4.11 Consider the following multiset of Jordan blocks whose spectrum partitions into $\mathbb{Z}_6 \cup \mathbb{Z}_{10} \cup \mathbb{Z}_{15}$.

$$\mathcal{J} = \{J_2(1), J_2(e^{\frac{2\pi i}{2}}), J_2(e^{\frac{2\pi i}{3}}), J_2(e^{\frac{4\pi i}{3}}), J_2(e^{\frac{2\pi i}{5}}), J_2(e^{\frac{8\pi i}{5}}), J_1(1), J_1(e^{\frac{2\pi i}{6}}), J_1(e^{\frac{10\pi i}{6}}),$$
$$J_1(e^{\frac{2\pi i}{10}}), J_1(e^{\frac{6\pi i}{10}}), J_1(e^{\frac{8\pi i}{10}}), J_1(e^{\frac{12\pi i}{10}}), J_1(e^{\frac{14\pi i}{10}}), J_1(e^{\frac{18\pi i}{10}}), J_1(e^{\frac{2\pi i}{15}}), J_1(e^{\frac{4\pi i}{15}}),$$
$$J_1(e^{\frac{8\pi i}{15}}), J_1(e^{\frac{12\pi i}{15}}), J_1(e^{\frac{14\pi i}{15}}), J_1(e^{\frac{16\pi i}{15}}), J_1(e^{\frac{18\pi i}{15}}), J_1(e^{\frac{22\pi i}{15}}), J_1(e^{\frac{26\pi i}{15}}), J_1(e^{\frac{28\pi i}{15}})\}.$$

It was shown in [8] Example 3.7 that $\mathcal{J}$ does not correspond to the peripheral spectrum of a nonnegative matrix. We see that the eigenvalues partition into complete sets of roots of unity, but these sets of roots cannot be partitioned into $m = 2$ levels so that the majorization condition is satisfied. Note that the partitions of $\sigma(\mathcal{J})$ into 2 sets, each of which is a union of complete sets of roots of unity are as follows:

(a) $L_1 = \mathbb{Z}_{15}$, $L_2 = \mathbb{Z}_{10} \bigcup \mathbb{Z}_6$ or

(b) $L_1 = \mathbb{Z}_{10}$, $L_2 = \mathbb{Z}_{15} \bigcup \mathbb{Z}_6$ or

(c) $L_1 = \mathbb{Z}_6$, $L_2 = \mathbb{Z}_{15} \bigcup \mathbb{Z}_{10}$.

Then $\eta_\lambda(\mathcal{J}) = (1, 1)$ does not majorize $\hat{\nu}_\lambda(L_1, L_2) = (2, 0)$ for $\lambda = e^{\frac{2\pi i}{2}}$, $e^{\frac{2\pi i}{3}}$, and $e^{\frac{2\pi i}{5}}$, respectively.

Note that it is possible to partition $\sigma(\mathcal{J})$ into 2 sets, L_1 and L_2 such that, for each $\lambda \in \sigma(\mathcal{J})$, $\hat{\nu}_\lambda(L_1, L_2) \preceq \eta_\lambda$. For example, consider $L_1 = \mathbb{Z}_{15} \bigcup \{e^{\frac{2\pi i}{2}}\}$ and $L_2 = \mathbb{Z}_{10} \bigcup \mathbb{Z}_6 \setminus \{e^{\frac{2\pi i}{2}}\}$. However, each L_i is not a union of complete sets of roots of unity.

Note that Morris has a matlab program to test if a set of Jordan blocks of a matrix correspond to the peripheral spectrum of a nonnegative matrix. It is presented in [9].

5 The Jordan Form of Nonnegative and Eventually Nonnegative Matrices

The original ideas for looking at the Jordan form of an eventually nonnegative matrix are due to Zaslavsky (see [18] and [17]).

A multiset of Jordan blocks $\mathcal{J}$ is said to be a *Frobenius multiset* if for some positive integer h the following conditions are satisfied:

(i) there is exactly one block in $\mathcal{J}$ with eigenvalue $\rho(\mathcal{J})$ and this block is 1×1,

(i) $\pi(\mathcal{J}) = \rho(\mathcal{J})\mathbb{Z}_h$, and

(ii) $e^{\frac{2}{h}}\mathcal{J} = \mathcal{J}$.

We note that our definition differs from that of [18] and [**?**] in that we do not require $\rho(\mathcal{J}) > 0$ so we do consider $\{[0]\}$ to be a Frobenius multiset.

In this section, we offer necessary and sufficient conditions on the Jordan form of an eventually nonnegative matrix that are stated as Theorem 5.5. In keeping with the approach taken in the previous section, we attempt to construct the Jordan form of the component levels of the desired eventually nonnegative matrix, starting with the eigenvalues that have the largest magnitude. The steps taken to do this are summarized briefly below.

1. Start with a self-conjugate multiset $\mathcal{J}$ of Jordan blocks with $P(\mathcal{J}) = \{\rho_1, \rho_2, \ldots, \rho_t\}$ where $\rho_1 > \rho_2 > \cdots > \rho_t$.

2. Check to see that $\mathcal{J}^{\rho_1}$ satisfies the conditions of Theorem 4.6. If so, we obtain multisets which correspond to the peripheral Jordan form of the levels of the desired eventually nonnegative matrix.

3. Consider $\mathcal{J}^{\rho_2}$. We must be able to assign Jordan blocks to the multisets obtained in the previous step (the "existing" multisets) and/or to "new" multisets (which will correspond to the peripheral Jordan form of upper levels of the desired matrix) so that the updated existing multisets partition into self-conjugate Frobenius multisets with spectral radius ρ_1 and each new multiset satisfies the the conditions of Theorem 4.6. We must also have that the Jordan form assigned to each multiset is compatible with our original Jordan form.

4. We continue in this manner, beginning with the multisets resulting from the previous step, we add to these multisets or create new multisets between existing multisets so that new multisets satisfy the conditions of Theorem 4.6 and existing multisets remain unions of self-cojugate Frobenius collections.

We illustrate this process with an example.

Example 5.1 Let

$$\mathcal{J} = \left\{ [3], \begin{bmatrix} 3 & 0 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} -3 & 0 \\ 1 & -3 \end{bmatrix}, \left[3e^{\frac{2\pi i}{3}}\right], \left[3e^{\frac{4\pi i}{3}}\right], \right.$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}, [2i], [-2i], \left[2e^{\frac{2\pi i}{5}}\right], \left[2e^{\frac{4\pi i}{5}}\right], \left[2e^{\frac{6\pi i}{5}}\right], \left[2e^{\frac{8\pi i}{5}}\right],$$

$$\left. [1], \left[e^{\frac{2\pi i}{6}}\right], \left[e^{\frac{10\pi i}{6}}\right], \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} i & 0 \\ 1 & i \end{bmatrix}, \begin{bmatrix} -i & 0 \\ 1 & -i \end{bmatrix} \right\}$$

Then $P(\mathcal{J}) = \{3, 2, 1\}$ and we first check to see that $\mathcal{J}^3$ satisfies the extended Tam Schneider condition. Indeed it does, as can be seen by taking

$$L_2 = \{[3], [-3], [3], [3e^{\frac{2\pi i}{3}}], [3e^{\frac{4\pi i}{3}}]\} \text{ and}$$

$$L_1 = \{[3], [-3]\}$$

Notice that L_2 partitions into two self-conjugate Frobenius collections, one with index of cyclicity 2 and the other with index of cyclicity 3. This must remain the case as we update L_2 in subsequent steps by adding blocks with eigenvalues of smaller magnitudes. Likewise, L_1 must remain 2-cyclic when blocks are added. At this point, if $\mathcal{J}$ does correspond to the Jordan form of an eventually nonnegative matrix A, we know that A must have two levels (perhaps in addition to a zero level). L_1 and L_2 give us the peripheral Jordan form of the levels of A. Note that, in this example, L_1 and L_2 are unique up to a reordering of the multisets.

Now, we must consider $\mathcal{J}^2$. We may create three new upper levels at this point (interlaced with the existing levels) and/or update the existing L_1 and L_2. The "new" upper levels created at this point must satisfy the extended Tam-Schneider condition (and will perhaps be broken into several new component levels as we shall see). Notice that $[2i]$ and $[-2i]$ must be placed in either L_1 or L_2 since $[-2] \notin \mathcal{J}^2$ so it would be impossible for a multiset containing $[2i]$ or $[-2i]$ to satisfy the extended Tam-Schneider condition. Also, we may place $[2i]$ and $[-2i]$ in either L_1 or L_2 and we choose one of the multisets arbitrarily at this point. Notice that $2\mathbb{Z}_5 \subset \sigma(\mathcal{J}^2)$, but 5 is not an allowable index of cyclicity for either L_1 or L_2 (since neither 2 nor 3 divides 5) and hence, these fifth roots must be represented in a new upper level. Note that one arrangement that is allowable for this step is given by

$$L_2 = \{[3], [-3], [3], [3e^{\frac{2\pi i}{3}}], [3e^{\frac{4\pi i}{3}}]\},$$

$$U_1 = \left\{ \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}, [2e^{\frac{2\pi i}{5}}], [2e^{\frac{4\pi i}{5}}], [2e^{\frac{6\pi i}{5}}], [2e^{\frac{8\pi i}{5}}] \right\},$$

$$L_1 = \{[3], [-3], [2i], [-2i]\}.$$

Then U_1 satisfies the extended Tam-Schneider condition as required; take

$$U_1^{(2)} = \{[2], [2e^{\frac{2\pi i}{5}}], [2e^{\frac{4\pi i}{5}}], [2e^{\frac{6\pi i}{5}}], [2e^{\frac{8\pi i}{5}}]\} \text{ and}$$

$$U_1^{(1)} = \{[2]\}.$$

This results in a current assignment of Jordan blocks to component levels given by:

$$C_4 = L_2 = \{[3], [-3], [3], [3e^{\frac{2\pi i}{3}}], [3e^{\frac{4\pi i}{3}}]\},$$

$$C_3 = U_1^{(2)} = \{[2], [2e^{\frac{2\pi i}{5}}], [2e^{\frac{4\pi i}{5}}], [2e^{\frac{6\pi i}{5}}], [2e^{\frac{8\pi i}{5}}]\},$$

$$C_2 = U_1^{(1)} = \{[2]\}, \text{ and}$$

$C_1 = L_1 = \{[3], [-3], [2i], [-2i]\}.$

Notice that the multisets have simply been relabeled to avoid the use of further subscripting as we move to the next step. We now consider $\mathcal{J}^1$. We may create up to five new multisets, each satisfying the extended Tam Schneider condition and/or update the existing component levels. There are numerous options for the placement of Jordan blocks in the existing component levels and/or new levels. Before giving a few of the options, we would like to make some observations. Since 1 only occurs once as an eigenvalue in $\sigma(\mathcal{J}^1)$, we can only create one new component level at this step. Notice that $\mathbb{Z}_2$ and $\mathbb{Z}_4$ are each subsets of $\sigma(\mathcal{J}^1)$, and we may create a new component level with spectrum equal to either $\mathbb{Z}_2$ or $\mathbb{Z}_4$. Also notice that $\mathbb{Z}_2$ and $\mathbb{Z}_4$ are both 2-cyclic, so could be incorporated into any of the three component levels C_1, C_2, or C_4, but not C_3 as it is 5-cyclic. Also note that C_2 has index of cyclicity 1, so we may assign all Jordan blocks from $\mathcal{J}^1$ to C_2 and the resulting multiset would still be a self-conjugate Frobenius multiset with index of cyclicity 1. However, we give a different allowable assignment for this last step below:

$$C_4 = \{[3], [-3], [3], [3e^{\frac{2\pi i}{3}}], [3e^{\frac{4\pi i}{3}}], [e^{\frac{2\pi i}{6}}], [-1], [e^{\frac{10\pi i}{6}}]\},$$

$$C_3 = \{[2], [2e^{\frac{2\pi i}{5}}], [2e^{\frac{4\pi i}{5}}], [2e^{\frac{6\pi i}{5}}], [2e^{\frac{8\pi i}{5}}]\},$$

$$C_2 = \{[2]\}, \text{ and}$$

$$C_1 = \left\{ [3], [-3], [2i], [-2i], [1], [-1], \begin{bmatrix} i & 0 \\ 1 & i \end{bmatrix}, \begin{bmatrix} -i & 0 \\ 1 & -i \end{bmatrix}, \right\}.$$

In this case, we were able to construct self-conjugate Frobenius multisets from the given $\mathcal{J}$ and so $\mathcal{J}$ does correspond to the Jordan form of an eventually nonnegative matrix. We note that the eventually nonnegative matrix that could be formed from the multisets specified above would have two levels (since there are two C_j's with spectral radius $\rho(\mathcal{J}) = 3$) and one nonempty upper level. If we consider this one nonempty upper level in its own right, it has two levels. So we have a total of four component levels. We note that it would also be possible to construct an eventually nonnegative matrix with six component levels had we used

$$C_6 = \{[1], [-1], [i], [-i]\},$$

$$C_5 = \{[2]\},$$

$$C_4 = \{[3], [-3], [3], [3e^{\frac{2\pi i}{3}}], [3e^{\frac{4\pi i}{3}}], [e^{\frac{2\pi i}{6}}], [-1], [e^{\frac{10\pi i}{6}}]\},$$

$$C_3 = \{[2]\},$$

$$C_2 = \{[3], [-3], [2i], [-2i], [i], [-i]\}, \text{ and}$$

$$C_1 = \{[2], [2e^{\frac{2\pi i}{5}}], [2e^{\frac{4\pi i}{5}}], [2e^{\frac{6\pi i}{5}}], [2e^{\frac{8\pi i}{5}}]\},$$

for example. We could also construct five component levels, but there is no way to use fewer than four component levels. We would like to note that there are many more options for the choice of component levels than the two given above in this example. We see that the choices made at each step are not unique.

Another way to view the construction of the Jordan form of component levels in the previous example is that we simply applied a permutation similarity to $\oplus\mathcal{J}$ and block partitioned the resulting matrix so that each diagonal block corresponded to a self-conjugate Frobenius multiset (in addition to other conditions pertaining to the number of diagonal blocks and placement with respect to each other). We will see in the next example that only allowing permutation similarities is not enough and we may need to allow a more general class of similarity transformations. The more general transformations that must be considered are given in Definition 5.3.

Example 5.2 Suppose $\alpha = e^{\theta i}$ for some $\theta \in (0, \frac{\pi}{2})$. Let

$$\mathcal{J} = \{J_2(2), J_2(-2), J_3(\alpha), J_1(\alpha), J_2(-\alpha), J_2(-\alpha), J_3(\overline{\alpha}), J_1(\overline{\alpha}), J_2(-\overline{\alpha}), J_2(-\overline{\alpha})\}.$$

Note that $P(\mathcal{J}) = \{1, 2\}$ and $\mathcal{J}^2$ satisfies the extended Tam-Schneider condition (take $L_1 = \{[2], [-2]\} = L_2$). Also notice $1 \notin \mathcal{J}^1$. Hence, if $\mathcal{J}$ corresponds to the Jordan form of an eventually nonnegative matrix A, then A has two irreducible classes and the Jordan form of each class must correspond to a self-conjugate Frobenius collection with index of cyclicity 2.

Let $J = \oplus\mathcal{J}$. Notice that J is 20×20, but there is no partition $\kappa = (K_1, K_2)$ of 20 for which each $J_{K_jK_j}$ corresponds to a self-conjugate Frobenius collection. However, let

$$C = \begin{bmatrix} C_{11} & 0 \\ C_{21} & C_{22} \end{bmatrix}$$

where

$$C_{11} = C_{22} = J_1(2) \oplus J_1(-2) \oplus J_2(\alpha) \oplus J_2(-\alpha) \oplus J_2(\overline{\alpha}) \oplus J_2(-\overline{\alpha})$$

and $C_{21} = [c_{ij}]$ where $c_{11} = c_{22} = c_{33} = c_{77} = 1$ and all other $c_{ij} = 0$. Then C is similar to J and each C_{jj} is in Jordan form and has Jordan form corresponding to a self-conjugate Frobenius collection. Hence, $\mathcal{J}$ does correspond to the Jordan form of an eventually nonnegative matrix.

The following definition gives the class of matrices we must consider when constructing the Jordan form of component levels given a Jordan matrix.

Definition 5.3 Let J be an $n \times n$ matrix in Jordan form and $\kappa = (K_1, \ldots, K_k)$ a partition of $\langle n \rangle$. We say that the matrix C *is a reorganization of J with respect to κ* provided

(i) The matrix C is similar to J.

(ii) Each C_{K_j} is in Jordan form.

(iii) C_κ is lower triangular.

(iv) $c_{ii} \neq c_{jj}$ implies $c_{ij} = c_{ji} = 0$ for all $i, j \in \langle n \rangle$.

Since we are interested in considering multisets of Jordan blocks as opposed to the direct sum of Jordan blocks, we offer the following definition of a multiset reorganization.

Definition 5.4 Let $\mathcal{J}$ be a multiset of Jordan blocks and let m be a natural number. We say that $\mathcal{J}$ *can be multiset reorganized into m multisets denoted by* $(S_1, S_2, \ldots, S_m)$ provided each S_j is a multiset of Jordan blocks and there exists a reorganization

$$C = \begin{bmatrix} C_{11} & 0 & \cdots & 0 \\ C_{21} & C_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C_{m1} & C_{m2} & \cdots & C_{mm} \end{bmatrix} = [c_{ij}]$$

of $\oplus\mathcal{J}$ such that for each $j \in \langle m \rangle$, $C_{jj} = \oplus S_j$.

We are now ready to state necessary and sufficient conditions for a multiset $\mathcal{J}$ of Jordan blocks to correspond to the Jordan form of an eventually nonnegative matrix.

Theorem 5.5 (McDonald and Morris [10]) Let $\mathcal{J}$ be a multiset of Jordan blocks with $P(\mathcal{J}) = \{\rho_1, \rho_2, \ldots, \rho_t\}$ where $\rho_1 > \rho_2 > \ldots > \rho_t$. Then $\mathcal{J}$ corresponds to the Jordan form of an eventually nonnegative matrix if and only if

(a) $\mathcal{J}^{\rho_1}$ can be leveled into $m_1 = \text{index}_{\rho_1}(\mathcal{J}^{\rho_1})$ multisets denoted by $(L_1^{(1)}, L_2^{(1)}, \ldots, L_{m_1}^{(1)})$ and

(b) for $j = 2 \ldots t$, $\mathcal{J}^{\rho_j}$ can be multiset reorganized into $2m_{j-1} + 1$ multisets, some possibly empty, denoted by

$$(S_1^{(j)}, S_2^{(j)}, \ldots, S_{2m_{j-1}+1}^{(j)})$$

where, for each $k \in \langle 2m_{j-1} + 1 \rangle$,

(i) if k is even, then $L_{k/2}^{(j-1)} \bigcup S_k^{(j)}$ partitions into self-conjugate Frobenius multisets, each with spectral radius greater than ρ_j, and

(ii) if k is odd, then $S_k^{(j)}$ can be leveled into $m_{k,j}$ $=\text{index}_{\rho_j}(S_k^{(j)})$ multisets denoted by $(T_{k,1}^{(j)}, T_{k,2}^{(j)}, \ldots, T_{k,m_{k,j}}^{(j)})$.

Set $L^{(j)*} = (T_{1,1}^{(j)}, \ldots, T_{1,m_{1,j}}^{(j)}, L_1^{(j-1)} \cup S_2^{(j)}, T_{3,1}^{(j)}, \ldots, T_{3,m_{3,j}}^{(j)}, L_2^{(j-1)} \cup S_4^{(j)}, \ldots, L_{2m_{j-1}}^{(j-1)} \cup S_{m_{j-1}}^{(j)}, T_{2m_{j-1}+1,1}, \ldots, T_{2m_{j-1}+1,m_{2m_{j-1}+1,j}})$. Delete all empty sets listed in $L^{(j)*}$ and relabel the remaining multisets to obtain $L^{(j)} = (L_1^{(j)}, L_2^{(j)}, \ldots, L_{m_j}^{(j)})$. Note that m_j is the number of multisets listed in $L^{(j)}$.

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