

1 Introduction

The inverse eigenvalue problem for $n \times n$ nonnegative matrices can be stated as follows:

Find necessary and sufficient conditions for a multiset of complex numbers $\sigma = \{\lambda_1, \dots, \lambda_n\}$ to be the eigenvalues of an $n \times n$ nonnegative matrix.

This problem remains unsolved for $n \geq 4$, and has several variations including the inverse eigenvalue problem for nonnegative matrices with real spectra and the inverse eigenvalue problem for nonnegative symmetric matrices.

Clearly the Perron Frobenius theorem gives us a set of necessary conditions on the multiset σ . Additionally, let

$$S_k = \sum_{j=1}^n \lambda_j^k.$$

Since the trace of a nonnegative matrix is nonnegative and equal to the sum of its eigenvalues, and the same is true for its powers, it follows that $S_k \geq 0$, for all positive integers k . London and Loewy [4], and independently Johnson [2], discovered a more intricate set of necessary relationships between the values of S_k (listed below as the JLL condition).

Observation 1.1 *Necessary conditions for a multiset σ to be the eigenvalues of an irreducible, $n \times n$ nonnegative matrix:*

There exists a positive integer h such that:

- (a) (Perron-Frobenius) $\pi(\sigma) = \rho(\sigma)\mathbb{Z}_h$,
- (b) (Perron-Frobenius) $\sigma = e^{\frac{2\pi i}{h}} \sigma$,
- (c) (Nonnegative trace) $S_k \geq 0$,
- (d) (JLL Condition) $S_k^m \leq n^{m-1} S_{km}$ for all positive integers k and m .

Although these conditions are necessary for a multiset σ to correspond to the eigenvalues of a nonnegative irreducible matrix, they are not, in general, sufficient for $n \geq 3$.

For $n = 4$, Reams [7] was able to show that σ corresponds to the eigenvalues of a 4×4 (possibly reducible) nonnegative matrix whose **trace is zero** if and only if $s_2 \geq 0, s_3 \geq 0$ and $s_2^2 \leq 4s_4$. Moreover, this spectrum can be realized using the matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{s_2}{4} & 0 & 1 & 0 \\ \frac{s_3}{4} & 0 & 0 & 1 \\ \frac{4s_4 - s_2^2}{16} & \frac{s_3}{12} & \frac{s_2}{4} & 0 \end{bmatrix}$$

Laffey and Mehan have solved the 4×4 case where $s_1 > 0$.

For $n = 5$, Laffey and Mehan [5], have shown that σ corresponds to the eigenvalues of a 5×5 (possibly reducible) nonnegative matrix whose **trace is zero** if and only if s_2, s_3, s_4 and s_5 are all nonnegative, $s_2^2 \leq 4s_4$, and $12s_5 - 5s_2s_3 + 5s_3\sqrt{4s_4 - s_2^2} \geq 0$.

Boyle and Handleman [1] were able to show that a multiset σ of nonzero complex numbers satisfies $\rho(\sigma) \in \sigma$, s_k is positive for all positive integers k , and all the coefficients of

$$\prod_{j=1}^n (x - \lambda_j)$$

are real if and only if σ corresponds to the nonzero spectrum of a positive matrix. The number of zeros needed in the spectrum is not determined.

In the case of the inverse eigenvalue problem for nonnegative matrices with real spectra and symmetric nonnegative matrices, it is known that for $n \leq 4$ that the realizable spectra are the same for both types of matrices, and that if σ can be partitioned into sets satisfying Observation 1.1, then it is realizable. The two sets differ when $n = 5$.

In [6], McDonald and Neumann use Soules matrices to construct nonnegative symmetric matrices. In particular, they show how to create a nonnegative symmetric matrix for any realizable spectrum with $n \leq 4$. For $n = 5$, they establish the boundaries of the region that can be realized using Soules matrices.

Note: There are many interesting recent results that need to be added to this section.

References

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