1 Some notation and definitions

Given an $n \times n$ matrix A, the spectrum of A is denoted by $\sigma(A)$ and its spectral radius by $\rho(A) = \max\{|\lambda| \mid \lambda \in \sigma(A)\}$. An eigenvalue λ of A is said to be *dominant* if $|\lambda| = \rho(A)$. The *spectral abscissa* of A is defined and denoted by $\lambda(A) := \max\{\operatorname{Re} \lambda \mid \lambda \in \sigma(A)\}$.

2 Eventually nonnegative states and matrices

In the theory and applications of dynamical systems, one is frequently interested in qualitative information regarding state evolution. In particular, due to physical and modeling constraints arising in engineering, biological, medical and economic applications, it is commonly of interest to impose or consider conditions for nonnegativity of the states; see, e.g., [6, 17]. Such applications typically draw on the theory, or even directly take the form of a linear differential system

$$\dot{x}(t) = Ax(t), \quad A \in \mathbb{R}^{n\times n}, \quad x(0) = x_0 \in \mathbb{R}^n, \quad t \geq 0, \tag{2.1}$$

whose solution is given by $x(t) = e^{tA}x_0$. We shall refer here to the set

$$\{x(t) = e^{tA}x_0 \mid t \in [0, \infty)\}$$

as the *trajectory emanating from* x_0 and say that x_0 gives rise to this trajectory.

It is proposed that we consider conditions for the entrywise nonnegativity of the trajectories associated with (2.1). A main concern is the following 'hit and hold' problem:

When does the trajectory emanating from an initial point x_0 *become (entrywise) nonnegative and remain nonnegative for all time thereafter?*

More specifically, we propose to seek characterizations of system parameters that lead to a trajectory becoming nonnegative at a finite time (*reachability of* $\mathbb{R}^n_+$) and remaining nonnegative for all time thereafter (*holdability*[1] *of* $\mathbb{R}^n_+$). This endeavor comprises several related efforts:

(1) Study matrices $A \in \mathbb{R}^{n\times n}$ for which $\exists t_0 \in [0, \infty)$ such that $e^{tA} \geq 0$, $\forall t \geq t_0$. We call such matrices *eventually exponentially nonnegative.*

(2) Given an eventually exponentially nonnegative matrix A, study initial points $x_0 \in \mathbb{R}^n$ for which $\exists \hat{t} \in [0, \infty)$ such that $e^{tA}x_0 \geq 0$, $\forall t \geq \hat{t}$. We shall refer to such initial points as *points of nonnegative potential.*

Central to points (1) and (2) above are the *eventually nonnegative* matrices and their eigenstructure. These are matrices whose powers become and remain entrywise nonnegative, leading us to a third proposed effort:

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[1] The terms *holdability* and *reachability* are usually associated with control theory. We use them here because a homogeneous system like (2.1) is frequently the result of linear control system $\dot{x}(t) = Ax(t)+Bu(t)$ operating under a feedback law $u(t) = Fx(t)$.

(3) Study the spectra, eigenspaces and combinatorial structure of eventually nonnegative matrices and their powers.

Eventual nonnegativity of trajectories and matrices, as well as eventual exponential nonnegativity of matrices can be difficult to detect and confirm via their definitions. Consequently, we set an additional goal of ours as follows:

(4) Pursue characterizations of eventually nonnegativity of states and matrices that lend themselves to efficient numerical methods.

3 Eventually exponentially nonnegative matrices

Recall that $A \in \mathbb{R}^{n\times n}$ is exponentially nonnegative if and only if $A \overset{\text{s}}{\geq} 0$; see [7]. As a consequence, every essentially nonnegative matrix A is eventually exponentially nonnegative with exponential index $t_0 = 0$. In the following theorem, this result is extended (in the positive case) by offering a characterization of eventually exponentially positive matrices. Its validity is based on recent results from [43].

Theorem 3.1 [44] *For a matrix $A \in \mathbb{R}^{n\times n}$ the following properties are equivalent:*

(i) *There exists $a \geq 0$ such that both matrices $A + aI$ and $A^T + aI$ have the strong Perron-Frobenius property.*

(ii) *$A + aI$ is eventually positive for some $a \geq 0$.*

(iii) *$A^T + aI$ is eventually positive for some $a \geq 0$.*

(iv) *A is eventually exponentially positive.*

(v) *A^T is eventually exponentially positive.*

Remark 3.2

(a) It is important to realize the practical value of the above theorem: Eventual positivity and eventual exponential positivity are difficult to recognize and confirm via their definitions. This is due to the difficulty in computing matrix exponentials and powers, especially given the lack of *a priori* estimates of the exponential index and the power index. Condition (i) of Theorem 3.1 transforms this problem to an eigenspace problem.

(b) It is also worth noting that the equivalence of (ii) and (iv) in Theorem 3.1 is a generalization of the fact that $A \overset{\text{s}}{>} 0$ is equivalent to A being exponentially positive.

Example 3.3 Consider the matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$ and observe that

$$A^2 = \begin{bmatrix} 2 & 3 & 4 & 4 \\ 2 & 3 & 4 & 4 \\ 0 & 1 & 2 & 2 \\ 1 & 2 & 3 & 3 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 5 & 9 & 13 & 13 \\ 5 & 9 & 13 & 13 \\ 1 & 3 & 5 & 5 \\ 3 & 6 & 9 & 9 \end{bmatrix}.$$

It is easily checked that A is an eventually positive matrix with power index $k_0 = 3$, so by Theorem 3.1, A is an eventually exponentially positive matrix. Computing e^{tA} for $t = 1, 2$ we obtain, respectively,

$$\begin{bmatrix} 5.0401 & 6.3618 & 8.6836 & 8.6836 \\ 4.0401 & 7.3618 & 8.6836 & 8.6836 \\ -0.4655 & 2.7873 & 5.0401 & 4.0401 \\ 2.7873 & 3.5746 & 6.3618 & 7.3618 \end{bmatrix}, \begin{bmatrix} 71.2660 & 134.1429 & 198.0199 & 198.0199 \\ 70.2660 & 135.1429 & 198.0199 & 198.0199 \\ 18.4960 & 45.3810 & 71.2660 & 70.2660 \\ 45.3810 & 88.7620 & 134.1429 & 135.1429 \end{bmatrix}.$$

Taking into consideration the location of the nonpositive entries of A and A^2, one can infer that the exponential index of A is some $t_0 \in (1, 2)$.

Having an understanding of eventual exponential *positivity*, we must next study eventually exponentially *nonnegative* matrices and connect them to eventually *nonnegative* matrices. However, there are distinct difficulties in extending Theorem 3.1 to eventual nonnegativity, many of which stem from the richer combinatorial structure allowed within the class of eventually nonnegative matrices. As is the case with Perron-Frobenius theory for nonnegative matrices, the combinatorial structure (graph, reducibility) is intimately related to the eigenstructure and the behavior of powers.

In what follows we report some partial progress regarding the above goal from [44]: Conditions that are sufficient for eventual exponential nonnegativity, as well as certain necessary conditions.

Theorem 3.4 *Let $A \in \mathbb{R}^{n\times n}$ be an eventually nonnegative with* $\operatorname{index}_0(A) \le 1$. *Then A is an eventually exponentially nonnegative matrix.*

Corollary 3.5 *Let $A \in \mathbb{R}^{n\times n}$ such that $A + aI$ is eventually nonnegative for all $a \in [a_1, a_2]$ $(a_1 < a_2)$. Then A is an eventually exponentially nonnegative matrix.*

We illustrate the above results on eventual nonnegativity with the following examples.

Example 3.6 Consider $A = \begin{bmatrix} 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ for which

$$A^2 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 0 & 1 & 3 & 1 \\ 1 & 0 & 5 & 5 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 4 & 4 \end{bmatrix},$$

$$A^4 = \begin{bmatrix} 1 & 0 & 5 & 5 \\ 0 & 1 & 11 & 9 \\ 0 & 0 & 8 & 8 \\ 0 & 0 & 8 & 8 \end{bmatrix}, \quad A^5 = \begin{bmatrix} 0 & 1 & 11 & 9 \\ 1 & 0 & 21 & 21 \\ 0 & 0 & 16 & 16 \\ 0 & 0 & 16 & 16 \end{bmatrix}.$$

Notice that A is reducible and eventually nonnegative with $k_0 = 2$. Since $\operatorname{index}_0(A) = 1$, Theorem 3.4 implies that A is an eventually exponentially nonnegative matrix.

Example 3.7 Consider the matrix $A = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ and its sequence of powers

$$A^k = \begin{bmatrix} 2^{k-1} & 2^{k-1} & 0 & 0 \\ 2^{k-1} & 2^{k-1} & 0 & 0 \\ 0 & 0 & 2^{k-1} & 2^{k-1} \\ 0 & 0 & 2^{k-1} & 2^{k-1} \end{bmatrix} \quad (k = 2, 3, \ldots).$$

The matrix A is eventually nonnegative with $k_0 = 2$. As the $(1, 2)$ block of A^k is 0 for all $k \geq 2$, while the $(1, 2)$ block of A contains negative entries, A is not eventually exponentially nonnegative. In agreement, the assumptions of Theorem 3.4 do not hold since $\text{index}_0(A) = 2$.

The failure of eventual nonnegativity to force eventual exponential nonnegativity observed in the above example can occur even if A is irreducible, as the following example shows.

Example 3.8 Consider the matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix}$ and its sequence of powers

$$A^k = \begin{bmatrix} 2^{k-1} & 2^{k-1} & k2^{k-1} & k2^{k-1} \\ 2^{k-1} & 2^{k-1} & k2^{k-1} & k2^{k-1} \\ 0 & 0 & 2^{k-1} & 2^{k-1} \\ 0 & 0 & 2^{k-1} & 2^{k-1} \end{bmatrix} \quad (k = 2, 3, \ldots).$$

The matrix A is an eventually nonnegative matrix with $k_0 = 2$ and $\text{index}_0(A) = 2$. As the assumptions of Theorem 3.4 do not hold, we may not conclude that A is eventually exponentially nonnegative. Indeed, the $(2, 1)$ block of A^k is 0 for all $k \geq 2$, while the $(2, 1)$ block of A contains negative entries. Thus A is not eventually exponentially nonnegative.

We now turn our attention to necessary conditions for eventual exponential nonnegativity; we first need to quote some results from [43]. Note that in the first theorem below from [43], we have added the assumption that A is not nilpotent; the need for this assumption was first observed in [14].

Theorem 3.9 ([43, Theorem 2.3]) *Let $A \in \mathbb{R}^{n \times n}$ be an eventually nonnegative matrix which is not nilpotent. Then both A and A^T have the Perron-Frobenius property.*

Theorem 3.10 ([43, Theorem 2.4]) *Let both $A \in \mathbb{R}^{n \times n}$ and A^T have the Perron-Frobenius property. If $\rho(A)$ is a simple and the only dominant eigenvalue of A, then*

$$\lim_{k \to \infty} \left(\frac{A}{\rho(A)} \right)^k = xy^T,$$

where x and y are, respectively, right and left nonnegative eigenvectors of A corresponding to $\rho(A)$, satisfying $x^T y = 1$.

Theorem 3.11 *Let $A \in \mathbb{R}^{n\times n}$ be an eventually exponentially nonnegative matrix. Then the following hold:*

(i) *e^A and e^{A^T} have the Perron-Frobenius property.*

(ii) *If $\rho(e^A)$ is a simple eigenvalue of e^A and $\rho(e^A) = e^{\rho(A)}$, then there exists $a_0 \geq 0$ such that for all $a > a_0$,*

$$\lim_{k\to\infty} ((A+aI)/(\rho(A+aI))^k \; = \; xy^T,$$

where x and y are, respectively, right and left nonnegative eigenvectors of A corresponding to $\rho(A)$, satisfying $x^Ty = 1$.

Remark 3.12 Referring to Theorem 3.11, if $(xy^T)_{ij} > 0$, then $\left((A+aI)^k\right)_{ij} > 0$ for all k sufficiently large. In particular, if $xy^T > 0$, then $A + aI$ is eventually nonnegative for all $a > a_0$. If, however, xy^T is nonnegative but not strictly positive, $A + aI$ may fail to be eventually nonnegative for all $a \in \mathbb{R}$.

Questions. To summarize and be specific, some questions arising from the above discussion are the following:

- Under what conditions does eventual nonnegativity imply eventual exponential nonnegativity?
- Under what conditions does eventual exponential nonnegativity imply eventual nonnegativity?
- How are the notions of eventual exponential nonnegativity of A and eventual nonnegativity of e^A related?

4 Points of nonnegative potential

The goals and analysis mentioned here relate to the study of nonnegative dynamics as described in Section 2. The results mentioned and their proofs can be found in [44].

Throughout this section, $A \in \mathbb{R}^{n\times n}$ denotes an eventually exponentially nonnegative matrix with exponential index $t_0 = t_0(A) \geq 0$.

<u>**Goal**</u> Study points of nonnegative potential, namely, study the set

$$X_A(\mathbb{R}^n_+) \; = \; \{x_0 \in \mathbb{R}^n \mid (\exists \hat{t} = \hat{t}(x_0) \geq 0)\,(\forall t \geq \hat{t})\,[e^{tA}x_0 \geq 0]\}. \tag{4.1}$$

In particular, we would like to develop the geometric and analytic properties of $X_A(\mathbb{R}^n_+)$ (and let them lead us to numerical characterizations of its member vectors).

First, let's recall some basic facts and terminology on convex cones in $\mathbb{R}^n$. Our references are [7, Chapter 1] and [47]. A convex set $K \subseteq \mathbb{R}^n$ is called a *convex cone* if $aK \subseteq K$ for all $a \geq 0$. A convex cone is called *polyhedral* if it consists of all finite nonnegative linear combinations of the elements of a finite set. A convex cone K is *pointed* if $K \cap (-K) = \{0\}$

and *solid* if its topological interior is nonempty. A pointed, solid convex cone is called a *proper cone.* The nonnegative orthant $\mathbb{R}^n_+$ is indeed a proper cone; it is also a polyhedral cone, comprising all finite nonnegative combinations of the standard basis vectors. Any subset of $\mathbb{R}^n$ of the form $K = S\mathbb{R}^n_+$, where S is an invertible matrix, is a proper polyhedral cone and referred to as a *simplicial cone.*

Given an eventually exponentially nonnegative matrix $A \in \mathbb{R}^{n\times n}$ with exponential index $t_0 = t_0(A) \geq 0$, define the simplicial cone

$$K = e^{t_0 A}\mathbb{R}^n_+ = \{x_0 \in \mathbb{R}^n \mid (\exists y \geq 0)\,[x_0 = e^{t_0 A}y]\}$$

and consider the sets

$$Y_A(K) \;=\; \{x_0 \in \mathbb{R}^n \mid (\exists \hat{t} = \hat{t}(x_0) \geq 0)\,[e^{\hat{t}A}x_0 \in K]\} \tag{4.2}$$

and

$$X_A(K) \;=\; \{x_0 \in \mathbb{R}^n \mid (\exists \hat{t} = \hat{t}(x_0) \geq 0)\,(\forall t \geq \hat{t})\,[e^{tA}x_0 \in K]\}. \tag{4.3}$$

Lemma 4.1 *Let K, $Y_A(K)$ as defined above. Then $K \subseteq \mathbb{R}^n_+ \subseteq Y_A(K)$.*

Note that the sets $Y_A(K)$, $X_A(K)$ and $X_A(\mathbb{R}^n_+)$ are convex cones. They are not necessarily closed sets, however. For example, when $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, it can be shown that $X_A(\mathbb{R}^2_+)$ consists of the whole upper plane excluding the negative x-axis.

The set $Y_A(K)$ comprises initial points for which the trajectories enter K at some time. The set $X_A(K)$ comprises initial points for which the trajectories enter K at some time and remain in K for all time thereafter. The set of points of nonnegative potential, $X_A(\mathbb{R}^n_+)$, comprises initial points for which the trajectories at some time become nonnegative and remain nonnegative for all time thereafter. It turns out that $Y_A(K)$, $X_A(K)$ and $X_A(\mathbb{R}^n_+)$ coincide:

Proposition 4.2 *Let $A \in \mathbb{R}^{n\times n}$ be an eventually exponentially nonnegative matrix with exponential index $t_0 = t_0(A) \geq 0$ and let $K = e^{t_0 A}\mathbb{R}^n_+$. Then*

$$Y_A(K) \;=\; X_A(\mathbb{R}^n_+) \;=\; X_A(K).$$

Remark 4.3 Referring to Proposition 4.2, observe the following:

(i) If $t_0 = 0$ (i.e., if $A \overset{\mathrm{s}}{\geq} 0$, or equivalently if $e^{tA} \geq 0$ for all $t \geq 0$), then $K = \mathbb{R}^n_+$. In this case, $X_A(\mathbb{R}^n_+)$ coincides with the *reachability cone* of the nonnegative orthant for an essentially nonnegative matrix, which is studied in detail in [41, 42].

(ii) The equality $X_A(\mathbb{R}^n_+) = X_A(K)$, in conjunction with Lemma 4.1, can be interpreted as saying that the simplicial cone $K = e^{t_0 A}\mathbb{R}^n_+$ serves as an *attractor* for trajectories emanating at points of nonnegative potential; in other words, trajectories emanating in $X_A(\mathbb{R}^n_+)$ always reach and remain in $K \subseteq \mathbb{R}^n_+$ after a finite time.

(iii) Our observations so far imply that the trajectory emanating from a point of nonnegative potential will enter cone K; however, it may subsequently exit K while it remains nonnegative, and it will eventually re-enter K and remain in K for all finite time thereafter. This situation is illustrated by the following example.

Example 4.4 Consider the matrix

$$A = \begin{bmatrix} 0.3929 & -0.8393 & 1.1071 & 1.3393 \\ 1.0357 & 0.6964 & -0.5357 & 0.8036 \\ 1.0357 & -0.3036 & 0.4643 & 0.8036 \\ 1.4643 & 1.0536 & -0.9643 & 0.4464 \end{bmatrix}.$$

It can be checked that A and A^T have the strong Perron-Frobenius property and so A is an eventually exponentially positive matrix. Using Matlab and a bisection method, we estimated (within five decimals) the exponential index to be $t_0 = t_0(A) = 2.64378$. The matrices e^A and $e^{t_0 A}$ are

$$e^A = \begin{bmatrix} 3.6277 & -0.7991 & 1.4260 & 3.1345 \\ 3.0341 & 2.2579 & -0.6987 & 2.7958 \\ 3.0341 & -0.4604 & 2.0196 & 2.7958 \\ 3.3050 & 1.4836 & -0.9696 & 3.5701 \end{bmatrix}$$

and

$$e^{t_0 A} = \begin{bmatrix} 91.902 & 3.5982 & 14.0615 & 88.299 \\ 91.499 & 18.162 & 0.3981 & 87.801 \\ 91.499 & 4.0959 & 14.4643 & 87.801 \\ 91.897 & 17.494 & 0 & 88.469 \end{bmatrix}.$$

Hence the cone $K = e^{t_0 A}\mathbb{R}^n_+$ is the cone generated by the columns of the matrix $e^{t_0 A}$ above. Consider now the following trajectory points $x(t) = e^{tA}x(0)$:

$$x_0 = x(0) = \begin{bmatrix} -1.1617 \\ 0.6014 \\ 0.9693 \\ 1.0887 \end{bmatrix}, \quad x(1) = e^A x_0 = \begin{bmatrix} 0.1 \\ 0.2 \\ 1.2 \\ 0 \end{bmatrix}, \quad x(2) = e^{2A} x_0 = \begin{bmatrix} 1.9141 \\ -0.0834 \\ 2.6348 \\ -0.5363 \end{bmatrix},$$

$$e^{(t_0+1)A} x_0 = \begin{bmatrix} 26.7836 \\ 13.2600 \\ 27.3263 \\ 12.6884 \end{bmatrix}, \quad e^{(t_0+2)A} x_0 = \begin{bmatrix} 165.3049 \\ 127.5845 \\ 165.8206 \\ 126.9949 \end{bmatrix}, \quad e^{(2t_0+1)A} x_0 = \begin{bmatrix} 4013.8 \\ 3816.4 \\ 4014.3 \\ 3815.8 \end{bmatrix}.$$

Observe the following: $e^{(t_0+1)A}x_0 \in K$ since $e^A x_0 \in \mathbb{R}^n_+$; $e^{(t_0+2)A}x_0 \notin K$ since $e^{2A}x_0 \notin \mathbb{R}^n_+$; $e^{(2t_0+1)A}x_0 \in K$ since $e^{(t_0+1)A}x_0 \in \mathbb{R}^n_+$; finally, trajectory points $x(t)$ are in K for all $t \geq 2t_0+1$. In other words, the trajectory emanating at x_0 enters K, exits K and eventually re-enters and remains in K for all time thereafter.

In view of the above example, a natural question arises: When is it possible that all trajectories emanating in $X_A(\mathbb{R}^n_+)$ reach and never exit K? This is equivalent to asking whether or not $e^{tA}K \subseteq K$ for all $t \geq 0$. To resolve this question, we invoke the following lemma; see [50, 56].

Lemma 4.5 *Let $A \in \mathbb{R}^{n\times n}$ and $K = S\mathbb{R}^n_+$, where $S \in \mathbb{R}^{n\times n}$ is nonsingular. Then there exists $a \geq 0$ such that $(A + aI)K \subseteq K$ if and only if $e^{tA}K \subseteq K$ for all $t \geq 0$.*

Corollary 4.6 *Let $A \in \mathbb{R}^{n\times n}$ be an eventually exponentially nonnegative matrix with exponential index $t_0 = t_0(A) \geq 0$. Let $K = e^{t_0 A}\mathbb{R}^n_+$. Then $e^{tA}K \subseteq K$ for all $t \geq 0$ if and only if $t_0 = 0$ (or equivalently, if and only if $A \overset{\mathrm{s}}{\geq} 0$).*

Quest for a numerical test for the members of $X_A(K)$

When $A = [a_{ij}] \overset{\mathrm{s}}{\geq} 0$, $X_A(\mathbb{R}^n_+)$ admits a numerical characterization reported in [41] and briefly described in the following. Consider the sequence $\{x_k\}$ generated from x_0 by the Cauchy-Euler finite differences scheme

$$x_k = (I + hA)^k x_0, \quad k = 0, 1, \dots$$

to which we refer as *the discrete trajectory* (*associated with the time-step* h) *emanating from* x_0. Define the quantity

$$h(A) = \sup\{\, h \mid \min_{1\leq i\leq n} (1 + ha_{ii}) > 0\}$$

and notice that $h(A) = \sup\{h \mid (I + hA) \geq 0\} > 0$, as well as that $h(A) = \infty$ when $A \geq 0$. For any $h \in (0, h(A))$, denote by $X_{A,h}(\mathbb{R}^n_+)$ the set of all initial states $x_0 \in \mathbb{R}^n$ that give rise to discrete trajectories $\{x_k\}$ which become and remain (due to nonnegativity of $I + hA$) nonnegative; namely,

$$X_{A,h}(\mathbb{R}^n_+) = \{x_0 \in \mathbb{R}^n \mid (\exists k_0 = k_0(x_0) \geq 0)\, (\forall k \geq k_0)\, [(I + hA)^k x_0 \in \mathbb{R}^n_+]\}.$$

We refer to $X_{A,h}(\mathbb{R}^n)$ as the *discrete reachability cone* (*of* $\mathbb{R}^n_+$ *under* A *with respect to* h). The geometric and algebraic properties of the discrete reachability cone are studied extensively in [40, 41].

Theorem 4.7 ([41]) *Let $A \in \mathbb{R}^{n\times n}$ be an essentially nonnegative matrix and let $h \in (0, h(A))$ such that $(I + hA)$ is invertible. Then $X_A(\mathbb{R}^n_+) = X_{A,h}(\mathbb{R}^n_+)$.*

When $A \overset{\mathrm{s}}{\geq} 0$, Theorem 4.7 suggests a simple test to find out whether a given initial point x_0 belongs to $X_A(\mathbb{R}^n_+)$ or not:

Step 1. Choose a positive $h < h(A)$ such that the iteration matrix $I + hA$ is invertible.

Step 2. Check whether for some nonnegative integer k, $x_k = (I + hA)^k x_0$ is nonnegative (in which case $x_0 \in X_A(\mathbb{R}^n_+)$) or decide that x_k will never be nonnegative (in which case $x_0 \notin X_A(\mathbb{R}^n_+)$).

The numerical analysis of the decision criteria in Step 2 above is provided recently in [45] by examining the sequence $\{x_k\}$. As is the case with Krylov sequences, it generically fails to converge, but its distance from dominant spectral subspaces can serve as an indicator whether or not this sequence will become and remain nonnegative.

As noted in [59], Theorem 4.7 can be generalized from $\mathbb{R}^n_+$ to any simplicial cone K such that $e^{tA}K \subseteq K$ for all $t \geq 0$. Thus, in view of Proposition 4.2, the question arising is whether the above test can be extended to $X_A(\mathbb{R}^n_+) = X_A(K)$, when A is *eventually* exponentially nonnegative with exponential index $t_0 \geq 0$ and $K = e^{t_0 A}\mathbb{R}^n_+$. By Corollary 4.6, however, it follows that the answer is in the negative if $t_0 > 0$ leading to us to another challenge.

Goal Given an eventually exponentially nonnegative matrix A, develop a numerical test for the points of nonnegative potential in terms of discrete trajectories. This effort will likely require a closer examination of the generalized eigenspaces of A as in the proof of Theorem 4.7.

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