The Nonnegative Inverse Eigenvalue Problem

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The Nonnegative Inverse Eigenvalue Problem (NIEP):

Find necessary and sufficient conditions on a list of n complex numbers

$$\sigma = (\lambda_1, \lambda_2, \ldots, \lambda_n)$$

for σ to be the spectrum of an $n \times n$ entry-wise nonnegative matrix.

If there exists an $n \times n$ nonnegative matrix A with spectrum σ, we will say that σ is **realizable** and that A **realizes** σ.

The Symmetric Nonnegative Inverse Eigenvalue Problem (SNIEP):

Find necessary and sufficient conditions on a list of n complex numbers

$$\sigma = (\lambda_1, \lambda_2, \ldots, \lambda_n)$$

for σ to be the spectrum of an $n \times n$ symmetric nonnegative matrix.

If there exists an $n \times n$ nonnegative matrix A with spectrum σ, we will say that σ is **symmetrically realizable** and that A is a **symmetric realization** of σ.

Detailed information on the general theory of nonnegative matrices, including NIEP can be found in

- Berman, Plemmons: Nonnegative matrices in the Mathematical Sciences, SIAM, 1994
- Minc: Nonnegative matrices, John Wiley and Sons, 1998

Immediate Necessary Conditions

Realizable list $\sigma = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ satisfies the following conditions:

1. σ is closed under complex conjugation.
2. $s_k = \sum_{i=1}^{n} \lambda_i^k \geq 0$ for $k = 1, 2, \ldots$
3. The Perron eigenvalue

$$\lambda_1 = \max\{|\lambda_i|; \lambda_i \in \sigma\}$$

   lies in σ.

While Kolmogorov, 1937, had asked earlier whether every complex number can arise as an eigenvalue of a nonnegative matrix, the NIEP was first formulated by Suleimanova, 1949.

Suleimanova: If

$$\lambda_1 > 0 \geq \lambda_2 \geq \lambda_3 \geq \ldots \geq \lambda_n,$$

then the list $\sigma = (\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n)$ is the spectrum of a nonnegative matrix if and only if

$$\lambda_1 + \lambda_2 + \lambda_3 + \ldots + \lambda_n \geq 0.$$

Operations that preserve realizability

Brauer

$\sigma_1 = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ realizable $\Rightarrow$

$\sigma_2 = (\lambda_1 + t, \lambda_2, \ldots, \lambda_n)$ realizable ($t \geq 0$)

$\sigma_1 : A,\ Av = \lambda_1 v,\ v \geq 0$

$\sigma_2 : B = A + tvu^T$ for every $u \geq 0$, $u^T v = 1$.

Fiedler, 1974

$\sigma_1 = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\sigma_2 = (\mu_1, \mu_2, \ldots, \mu_m)$ symmetrically realizable, $\lambda_1 \geq \mu_1$, $t \geq 0 \Rightarrow$

$\sigma_3 = (\lambda_1 + t, \lambda_2, \ldots, \lambda_m, \mu_1 - t, \mu_2, \ldots, \mu_n)$ symmetrically realizable

$\sigma_1 : A,\ Au = \lambda_1 u,\ ||u|| = 1$
$\sigma_2 : B,\ Bv = \mu_1 v,\ ||v|| = 1$

$$\sigma_3 : C = \begin{bmatrix} A & \rho u v^T \\ \rho v u^T & B \end{bmatrix}, \rho = \sqrt{t(\lambda_1 - \beta_1 + t)}$$

Idea of combining lists using realizable sublists was developed by several authors:

- Ciarlet
- Salzmann
- Xu
- Kellog and Stephens
- Rojo, Soto, Borobia, Moro.
- Marijuán, Pisonero.

Šmigoc, 2004:

Introduced constructions that extend a nonnegative matrix of the form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

into a nonnegative matrix of the form

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}.$$

$\sigma_1 = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\sigma_2 = (\mu_1, \mu_2, \ldots, \mu_m)$ realizable $\Rightarrow$

$(\lambda_1 + \mu_1, \lambda_2, \ldots, \lambda_n, \mu_2, \ldots, \mu_m)$ realizable.

Guo Wuwen, 1997:

$\sigma_1 = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ realizable, λ_2 real $\Rightarrow$

$\sigma_2 = (\lambda_1 + t, \lambda_2 \pm t, \ldots, \lambda_n)$ realizable, $(t \geq 0)$.

Laffey, 2005:
$\sigma_1 = (\lambda_1, a + \mathrm{i}b, a - \mathrm{i}b, \ldots, \lambda_n)$ realizable $\Rightarrow$

$\sigma_2 = (\lambda_1 + 2t, a - t + \mathrm{i}b, a - t - \mathrm{i}b, \ldots, \lambda_n)$ realizable, $(t \geq 0)$
Small perturbation case → Global case

Guo Wuwen, Guo Siwen, 2007: Different proof.

Guo Wuwen, Guo Siwen, 2007

$\sigma_1 = (\lambda_1, a + \mathrm{i}b, a - \mathrm{i}b, \ldots, \lambda_n)$ realizable $\Rightarrow$

$\sigma_2 = (\lambda_1 + 4t, a + t + \mathrm{i}b, a + t - \mathrm{i}b, \ldots, \lambda_n)$ realizable, $(t \geq 0)$.

Open questions on operations that preserve realizability

Let $\sigma_1 = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, λ_2 real, be the spectrum of a symmetric nonnegative matrix.

Must then

$$\sigma_2 = (\lambda_1 + t, \lambda_2 \pm t, \ldots, \lambda_n),$$

$t > 0$, be the spectrum of a symmetric nonnegative matrix?

$\sigma_1 = (\lambda_1, a + \mathrm{i}b, a - \mathrm{i}b, \ldots, \lambda_n)$ realizable.

- Must $\sigma_2 = (\lambda_1 + 2t, a + t + \mathrm{i}b, a + t - \mathrm{i}b, \ldots, \lambda_n)$, $t > 0$, be realizable?
- Find the "best function" $g(t)$ for which $(\lambda_1 + g(t), a + t + \mathrm{i}b, a + t - \mathrm{i}b, \ldots, \lambda_n)$, $t > 0$, is realizable.

Constructive Methods

Given a class of nonnegative matrices, find large classes of spectra realized by these matrices.

Companion matrix

$$
\begin{aligned}
f(x) &= (x-\lambda_1)(x-\lambda_2)\dots(x-\lambda_n) \\
&= x^n + B_1 x^{n-1} + B_2 x^{n-2} + \dots + B_n.
\end{aligned}
$$

Companion matrix of $f(x)$:

$$
A = \begin{bmatrix}
0 & 1 & 0 & \dots & 0 \\
0 & 0 & 1 & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & 0 \\
0 & 0 & \dots & 0 & 1 \\
-B_n & -B_{n-1} & \dots & -B_2 & -B_1
\end{bmatrix}
$$

A is nonnegative if and only if $B_i \leq 0$ for $i = 1, \dots, n$.

- Friedland, 1978: Suleimanova-type spectra are realizable by a companion matrix.

- Loewy, London, 1978: $(\rho, a+ib, a-ib)$ is realizable if and only if it is realizable by

  $$\alpha I + C,$$

  $\alpha \geq 0$, C a nonnegative companion matrix.

Laffey, Šmigoc, 2006

$\sigma = (\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n)$, $\lambda_1 > 0$,

$$Re(\lambda_i) \leq 0, i = 2, \ldots, n,$$

is realizable if, and only if, it is realizable by a matrix of the form

$$C + \alpha I,$$

where $\alpha \geq 0$ and C is a companion matrix with trace zero.

Rojo, Soto, 2001

Found sufficient conditions for the realizability of spectra by nonnegative circulant matrices.

Kim, Ormes, Roush, 2000

Realizability by nonnegative integer matrices.

The authors obatined realizations of $\prod_{j=1}^{n}(1-\lambda_j t)$ as $\det(I - tA(t))$ where $A(t) \in \mathbb{Z}_+[t]$, and these lead to realizations by block matrices, with the companion matrices as blocks on the diagonal.

Laffey, 1998

Considered the realization of spectra by matrices of the form:

$$\left[\begin{array}{cccc|ccc} & C(f) & & & & & \\ & & & & 1 & & \\ \hline & & & & & & \\ & & & & & C(g) & \\ u & v & w & s & & & \end{array}\right]$$

Laffey, Šmigoc

Matrices of the form

$$\left(\begin{array}{ccc|cc|cccc}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\alpha_3 & \alpha_2 & \alpha_1 & 1 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \beta_2 & \beta_1 & 1 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
c & b & a & f & e & \gamma_4 & \gamma_3 & \gamma_2 & \gamma_1
\end{array}\right)$$

yield realizations of real spectra with two positive entries.

$$f(x) = f_1(x)f_2(x)f_3(x) - (ex + f)f_1(x) - (ax^2 + bx + c)$$

Laffey, Šmigoc, 2008:

$$\sigma_N(t) = (3+t, 3-t, -2, -2, -2, \underbrace{0, \ldots, 0}_{N}).$$

If $t \geq 3^{-N/4}\sqrt{2}$, $\sigma_N(t)$ can be realized by a matrix

$$A_N(t) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \ldots & 0 \\ \mathbf{9} & \mathbf{6} & \mathbf{0} & 0 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \mathbf{3} & \mathbf{0} & 1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 0 & 0 & & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \mathbf{C}(\mathbf{v_N}) & & \\ 0 & 0 & 0 & 0 & 0 & & & & \\ -\mathbf{c_N} & -\mathbf{b_N} & -\mathbf{a_N} & -\mathbf{f_N} & -\mathbf{e_N} & & & & \end{bmatrix}$$

Butcher, Chartier, 1999: Doubly companion matrix.

$f(x) = x^n + p_1 x^{n-1} + \ldots + p_n$ and $g(x) = x^n + q_1 x^{n-1} + \ldots + q_n$

$$C(f,g) = \begin{bmatrix} -q_1 & 1 & 0 & \ldots & 0 \\ -q_2 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -q_{n-1} & 0 & \ldots & 0 & 1 \\ -p_n - q_n & -p_{n-1} & \ldots & \ldots & -p_1 \end{bmatrix}$$

has characteristic polynomial

$$\mathrm{Trunc}_n(f(x)g(x))/x^n,$$

where $\mathrm{Trunc}_n(h(x))$ means the polynomial obtained from $h(x)$ by deleting terms of degree smaller than n.

Let $C = C(f)$ be the companion matrix of a given monic polynomial $f(x)$. Reams (1994) observed that one can write the Newton identities in the form

$$CS_1 = S_2$$

where S_1 and S_2 are nonnegative and S_1 is unitriangular. Then the matrix $R = S_1^{-1} S_2$ is similar to C and under certain conditions R is nonnegative.

Laffey, Meehan, 1998

Solution of the NIEP for $n = 4$ uses matrices of the form

$$\alpha I + \begin{bmatrix} a & 1 & 0 & 0 \\ b & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ u & v & w & q \end{bmatrix}$$

Torre-Mayo, Abril-Raymundo, Alarcia-Estevez, Marijuan,Pisonero, 2007: Another constructive solution to NIEP for $n = 4$.

Leal-Duarte, Johnson, 2004

Solved the NIEP for the case where the realizing matrix is an arbitrary nonnegative diagonal matrix added to a nonnegative matrix whose graph is a tree.

Holtz, 2005

Solved the NIEP for symmetric nonnegative matrices of the form

$$\begin{bmatrix} 0 & \dots & \dots & 0 & a_1 \\ \vdots & & \iddots & a_2 & b_1 \\ \vdots & \iddots & \iddots & \iddots & 0 \\ 0 & a_2 & \iddots & \iddots & \vdots \\ a_1 & b_1 & 0 & \dots & 0 \end{bmatrix}.$$

Soules, 1983:

Constructed real orthogonal matrices U with the property that if $D = \mathrm{diag}(d_1, d_2, \ldots, d_m)$ with $d_1 \geq d_2 \geq \ldots \geq d_n$, then $U^T DU$ has all its off-diagonal entries nonnegative.

$$\frac{1}{2\sqrt{6}} \begin{bmatrix} \sqrt{6} & \sqrt{2} & 2 & 2\sqrt{3} \\ \sqrt{6} & \sqrt{2} & 2 & -2\sqrt{3} \\ \sqrt{6} & \sqrt{2} & -4 & 0 \\ \sqrt{6} & -3\sqrt{2} & 0 & 0 \end{bmatrix}$$

Further development and application of Soules matrices:

- Elsner, Nabben, Neumann, 1998
- McDonald, Neumann, 2000
- Shaked-Monderer, 2004
- Loewy, McDonald, 2004
- Nabben, 2007
- Chen, Neumann, Shaked-Monderer, 2008

Open Problem

Find other good classes of matrices.

Existence Results

Boyle, Handelman, 1991

A list of complex numbers $\sigma = (\lambda_1, \ldots, \lambda_n)$ is the nonzero spectrum of some nonnegative matrix if:

1. $\lambda_1 > |\lambda_j|$ for $j = 2, 3, \ldots, n$.
2. σ is closed under complex conjugation.
3. For all positive integers k and m:

$$s_k = \lambda_1^k + \ldots + \lambda_n^k \geq 0,$$

and $s_k > 0$ implies $s_{mk} > 0$.

Square nonnegative matrices A and B are equivalent if there exists N and nonnegative matrices P_i and Q_i with sizes so that P_iQ_i, Q_iP_i exists and

$$A = P_1Q_1,\ Q_1P_1 = P_2Q_2\ldots, Q_{N-1}P_{N-1} = P_NQ_N,\ Q_NP_N = B.$$

Open Problem

Finding a constructive proof of Boyle- Handelman result with a "good" bound on the number of zeros required for the realization.

Johnson, Laffey, Loewy, 1996

Suppose that $\sigma = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ with N zeros added is the spectrum of a symmetric nonnegative matrix.

Then σ with $\binom{n+1}{2}$ zeros added is the spectrum of a symmetric nonnegative matrix.

Laffey, Meehan: Adding zeros the the spectrum can help to realize it by a symmetric nonnegative matrix.

Example:
Loewy and Hartwig, McDonald and Neumann:
$(3+t, 3-t, -2, -2, -2)$ is realizable by a symmetric nonnegative matrix only for $t \geq 1$.

Laffey, Šmigoc: $(3+t, 3-t, -2, -2, -2, 0)$ is realizable by a symmetric nonnegative matrix for $t \geq \frac{1}{3}$.

Necessary Conditions

Immediate Necessary Conditions

Realizable list $\sigma = (\lambda_1, \lambda_2, \dots, \lambda_n)$ satisfies the following conditions:

1. σ is closed under complex conjugation.
2. $s_k = \sum_{i=1}^{n} \lambda_i^k \geq 0$ for $k = 1, 2, \dots$
3. The Perron eigenvalue

$$\lambda_1 = \max\{|\lambda_i|; \lambda_i \in \sigma\}$$

lies in σ.

Dmitriev, Dynkin, 1946

If λ_1 is the Perron eigenvalue of a nonnegative matrix with spectrum $(\lambda_1, \ldots, \lambda_n)$, then for $j = 2, \ldots, n$:

$$\mathrm{Re}(\lambda_j) + |\mathrm{Im}(\lambda_j)| \leq \lambda_1.$$

Johnson (1980), Loewy and London (1978)

$$n^{k-1} s_{km} \geq s_m^k$$

for all positive integers k, m.

Goldberger, Neumann, 2008

Proved the following necessary condition conjectured by Boyle and Handelman.

If $\sigma = (\lambda_1, \ldots, \lambda_k, 0, \ldots, 0)$ is realizable, then:

$$\prod_{j=1}^{k}(x - \lambda_j) \leq x^k - \lambda_1^k,$$

for all $x \geq \lambda_1$, where λ_1 is the Perron root.

Let B be an $M-$matrix or inverse $M-$matrix with characteristic polynomial

$$f(x) = x^n + p_1 x^{n-1} + \ldots + p_n.$$

Holtz (2005) proved that if

$$q_0 = 1, q_i = \frac{(-1)^i p_i}{\binom{n}{i}}, i \geq 1$$

then Newton's inequalities

$$q_i^2 \geq q_{i-1} q_{i+1}, \quad 1 = 1, 2, 3, \quad \text{...}$$

hold.

If A is a nonnegative matrix with Perron root ρ, then taking $B = \rho I - A$ yields a set of inequalities on the coefficients of the polynomial

$$\Pi_{j=1}^{n}(x - \lambda_j)$$

which are necessary for the realizability of

$$\sigma = (\lambda_1, \ldots, \lambda_n).$$

Holtz shows that these inequalities are not in general consequences of the other known necessary conditions.

Guo, 1997

If $(\lambda_2, \dots, \lambda_3)$ is a list of complex numbers closed under complex conjugation, then there is a least real number $\lambda_1 \geq 0$ such that

$$(h, \lambda_2, \dots, \lambda_n)$$

is realizable for all $h \geq \lambda_1$.

Laffey, 1998

Let $\sigma = (\lambda_1, \ldots, \lambda_n)$ be realizable.

- σ is extreme if for all $\epsilon > 0$

  $$(\lambda_1 - \epsilon, \lambda_2 - \epsilon \ldots, \lambda_n - \epsilon)$$

  is not realizable.
- σ is Perron extreme if for all $\epsilon > 0$

  $$(\lambda_1 - \epsilon, \lambda_2, \ldots, \lambda_n)$$

  is not realizable.

Laffey, 1998

Suppose σ is an extreme spectrum and A is a realizing matrix for A. Then there exists a nonzero nonnegative matrix Y with

$$AY = YA \text{ and } \mathrm{trace}(AY) = 0.$$

Result also hold for symmetric realizations and for Perron extreme spectra.

Used by Laffey, Meehan to solve NIEP for $n = 4$ and by Loewy, McDonald to study SNIEP for $n = 5$.

Open Problem

In the case that $s_1 = 0$, $Y = I$ satisfies the above conditions, so finding a more restrictive concept of extremality with some similar commuting result in the trace 0 case would be welcome.

Open Questions

Guo for symmetric matrices. Suppose that

$$\sigma = (\lambda_1, \lambda_2, \quad ... \quad , \lambda_n)$$

is the spectrum of a nonnegative symmetric matrix with Perron root λ_1, and let $t > 0$. Must

$$\sigma = (\lambda_1 + t, \lambda_2 - t, \quad ... \quad , \lambda_n)$$

be the spectrum of a nonnegative symmetric matrix.

Suppose that

$$(\lambda_1, \lambda_2, \overline{\lambda}_2, \ldots, \lambda_n)$$

is realizable with Perron root λ_1 and $t > 0$. Must

$$(\lambda_1 + 2t, \lambda_2 + t, \overline{\lambda}_2, +t, \ldots, \lambda_n)$$

be realizable.

Find a good definition of extremality with an associated commutation result in the trace 0 case.

Find a constructive proof of the Boyle- Handelman theorem with a ’good’ bound on the number N of zeros that need to be appended to a given list for realizability.

Get better understanding of the influence of adding zeros to the spectrum in the case of the SNIEP.

Make further progress on complete solutions of the NIEP and SNIEP for $n = 5, 6, ...$

The NIEP for real spectra.

Find matrix patterns which accommodate every realizable spectrum and have ’nice’ properties (For example, their characteristic polynomials are easy to compute, as for matrices close to companion matrices, they are sparse, ...).

Obtain better bounds on spectral gap

Distinguish between realizable spectra and those realizable by positive matrices. Borobia-Moro, 1998