SIMULTANEOUS NONVANISHING OF AUTOMORPHIC $L$-FUNCTIONS

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Abstract. Let $\pi$ be a unitary cuspidal automorphic representation of $GL_2(\mathbb{A}_\mathbb{Q})$. We show that there are infinitely many primitive Dirichlet characters $\chi \pmod{q}$ such that

$$L\left(\frac{1}{2}, \pi \times \chi\right)L\left(\frac{1}{2}, \chi\right) \neq 0.$$ 

1. Introduction and main results

The problem of determining whether an automorphic $L$-function is vanishing at the central point has been intensively studied. In many cases, the problem is related to a deep arithmetic problem such as Birch and Swinnerton-Dyer conjecture, nonvanishing of theta lifting, etc. Moreover, the question of simultaneous nonvanishing of twists of automorphic $L$-functions is also interesting. This has been studied by many people (see [Ak], [Kh], [Li], [Liu], [MV], [RR], and [Xu], for example). In particular, Ramakrishnan and Rogawski [RR] proved a simultaneous nonvanishing result for $GL_2 \times GL_2$ and $GL_2$ $L$-functions. More precisely, they showed that there are infinitely many primes $N$ such that $L(1/2, f \times \chi)L(1/2, f) \neq 0$, where $f$ is a newform for the congruence subgroup $\Gamma_0(N)$ and $\chi$ is a Dirichlet character. In this paper, we will prove an analogous result for $GL_2 \times GL_1$ and $GL_1$ $L$-functions.

Let $D = \{q : q = pr \text{ where } Q^{3/4} < p \leq 2Q^{3/4}, \ Q^{1/4} < r \leq 2Q^{1/4}, \ p, r \text{ are primes}\}$. Let $\pi$ be a unitary cuspidal automorphic representation of $\mathbb{A}_\mathbb{Q}$ and let $\chi$ be an even primitive character modulo $q$. We will prove the following asymptotic formula for the first moment of product of $L$-functions.

**Theorem 1.1.** For any $\varepsilon > 0$, we have

$$\sum_{q \in D} \sum_{\chi \pmod{q}}^+ L\left(\frac{1}{2}, \pi \times \chi\right)L\left(\frac{1}{2}, \chi\right) = A(Q) + O(Q^{15/8 + \varepsilon}),$$

where $\sum^+$ denotes summation over even primitive characters, and

$$A(Q) := \frac{1}{2} \sum_{q \in D} q \asymp \frac{Q^2}{\log^2 Q}.$$ 

The implied constant depends on $\pi$ and $\varepsilon$.

Theorem 1.1 is similar to the third moment of $L(1/2, \chi)$. For prime modulus $q$, Young [Y] proved an asymptotic formula with power saving for the fourth moment of $L(1/2, \chi)$ without averaging over $q$. A problem related to the fourth moment of $L(1/2, \chi)$ is to consider the

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second moment of $L(1/2, f \times \chi)$ where $f$ is a fixed Hecke cusp form. This was studied by Hoffstein and Lee [HL] and they proved a simultaneous nonvanishing result for $GL(2)$ $L$-functions twists by $\chi$.

As a corollary of Theorem 1.1, we have the following analogous result of [RR].

**Corollary 1.2.** Let $\pi$ be a unitary cuspidal automorphic representation of $GL_2(\mathbb{A}_\mathbb{Q})$. For each $Q$ large enough, there exists a primitive Dirichlet character $\chi \pmod{q}$ with $Q < q \leq 4Q$ such that

$$L(\frac{1}{2}, \pi \times \chi)L(\frac{1}{2}, \chi) \neq 0.$$  

In fact we will prove a stronger result than Corollary 1.2. Note that (see (3.1))

$$\# \left\{ \chi \pmod{q} : q \in \mathcal{D}, \chi \text{ is even} \right\} \approx \frac{Q^2}{\log^2 Q}.$$  

Using Theorem 1.1, the upper bound for the second moment (Theorem 4.2) and Cauchy’s inequality, we prove the following nonvanishing theorem.

**Theorem 1.3.** For any $\varepsilon > 0$,

(1.2) \quad \# \left\{ \chi \pmod{q} : q \in \mathcal{D}, \chi \text{ is even}, L(\frac{1}{2}, \pi \times \chi)L(\frac{1}{2}, \chi) \neq 0 \right\} \gg Q^{2-\varepsilon}.

## 2. Review of $L$-functions

We review the $L$-functions that will be used in this paper. Let $\Gamma(s) = \pi^{-s/2} \Gamma(s/2)$. For an even primitive character $\chi$ modulo $q$, let $L_\infty(s, \chi) = \Gamma(s)$ and let $L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$. The completed $L$-function $\Lambda(s, \chi) := L_\infty(s, \chi)L(s, \chi)$ is holomorphic and satisfies the functional equation

$$\Lambda(s, \chi) = \epsilon(s, \chi)\Lambda(1 - s, \overline{\chi})$$  

with $\epsilon(s, \chi) = q^{1/2-s}W(\chi)$. Here $W(\chi) = \tau(\chi)/q^{1/2}$ is the root number of $\chi$ and $\tau(\chi)$ is the Gauss sum of $\chi$. Let $\pi$ be a unitary cuspidal automorphic representation of $GL_2(\mathbb{A}_\mathbb{Q})$ with conductor $f_\pi$. The completed standard $L$-function $\Lambda(s, \pi) := L_\infty(s, \pi)L(s, \pi)$ is holomorphic and satisfies the functional equation

$$\Lambda(s, \pi) = \epsilon(s, \pi)\Lambda(1 - s, \overline{\pi})$$  

with $\epsilon(s, \pi) = f_\pi^{1/2-s}W(\pi)$ and $W(\pi)$ is the root number of $\pi$. Here $L(s, \pi) = \sum_{n=1}^{\infty} \frac{\lambda_\pi(n)}{n^s}$ with $\lambda_\pi(1) = 1$ and

$$L_\infty(s, \pi) = \Gamma_\mathbb{R}(s - \alpha_1)\Gamma_\mathbb{R}(s - \alpha_2).$$  

Let $\beta = \max\{\Re(\alpha_1), \Re(\alpha_2)\}$. Then $\beta \leq \frac{7}{64}$ by Kim and Sarnak ([KS]). From now on, we will assume $f_\pi$ and $q$ are coprime. The twisted $L$-function $L(s, \pi \times \chi)$ is defined by

$$L(s, \pi \times \chi) = \sum_{n=1}^{\infty} \frac{\lambda_\pi(n)\chi(n)}{n^s}.$$
The completed \( L \)-function \( \Lambda(s, \pi \times \chi) := L_{\infty}(s, \pi)L(s, \pi \times \chi) \) is also holomorphic and satisfies the functional equation

\[
\Lambda(s, \pi \times \chi) = \epsilon(s, \pi \times \chi)\Lambda(s, \bar{\pi} \times \bar{\chi})
\]

where \( \epsilon(s, \pi \times \chi) = (f_q q^2)^{\frac{1}{2} - s}W(\pi \times \chi) \) and \( W(\pi \times \chi) = W(\pi)\omega_{\pi}(q)\chi(f_q)W(\chi)^2 \) is the root number. Here \( \omega_{\pi} \) is the central character of \( \pi \).

3. Proof of Theorem 1.1

The proof of Theorem 1.1 is based on Iwaniec’s method in [Iw] which was also adopted by Luo [Lu]. We start with an approximate functional equation (see [LRS]). For \( \psi \in C_c^\infty(0, \infty) \) with \( \int_0^\infty \psi(x) \frac{dx}{x} = 1 \), set

\[
k(s) = \int_0^\infty \psi(y)y^s \frac{dy}{y}.
\]

Then \( k(s) \) is entire, rapidly decreasing in vertical strips, and \( k(0) = 1 \). For \( x > 0 \), define

\[
V_i(y) = \frac{1}{2\pi i} \int_{(2)} k(s)y^{-s} \frac{ds}{s}, \quad V_2(y) = \frac{1}{2\pi i} \int_{(2)} k(-s)L_\infty(\frac{1}{2} + s, \pi)L_\infty(\frac{1}{2} + s, \bar{\chi})y^{-s} \frac{ds}{s}.
\]

We have the estimate for \( V_i(y) \) (see [Lemma 3.1, LRS]).

**Lemma 3.1.** \( V_1(y) \) and \( V_2(y) \) satisfy the following:

1. \( V_i(y) \ll y^{-A} \) for any \( A > 0 \). (\( i = 1, 2 \).)
2. \( V_1(y) = 1 + O(y^A) \) for any \( A > 0 \).
3. \( V_2(y) \ll 1 + y^{1/2 - \beta - \varepsilon} \) for any \( \varepsilon > 0 \).

By considering the integral below with \( \sigma > 0 \),

\[
\frac{1}{2\pi i} \int_{(\sigma)} k(s)L(s + \frac{1}{2}, \pi \times \chi)L(s + \frac{1}{2}, \chi) \frac{Q^{5/4}}{\tau_q^3} \frac{ds}{s},
\]

we derive the approximate functional equation:

**Lemma 3.2.**

\[
L(\frac{1}{2}, \pi \times \chi)L(\frac{1}{2}, \chi) = \sum_{m=1}^\infty \sum_{n=1}^\infty \frac{\lambda_\pi(m)\chi(m)\chi(n)}{(mn)^{1/2}} V_1 \left( \frac{mnQ^{5/4}}{\tau_q^3} \right) + W(\pi \times \chi)W(\chi) \sum_{m=1}^\infty \sum_{n=1}^\infty \frac{\lambda_{\bar{\pi}}(m)\overline{\chi}(m)\overline{\chi}(n)}{(mn)^{1/2}} V_2 \left( \frac{mnQ^{5/4}}{Q^{5/4}} \right).
\]

By Lemma 3.2, we have

\[
\sum_{\chi \mod q}^+ \Lambda(\frac{1}{2}, \pi \times \chi)L(\frac{1}{2}, \chi) = \sum_{m=1}^\infty \sum_{n=1}^\infty \frac{\lambda_\pi(m)}{(mn)^{1/2}} V_1 \left( \frac{mnQ^{5/4}}{\tau_q^3} \right) \sum_{\chi \mod q}^+ \chi(m)\chi(n) + W(\pi)\omega_{\pi}(q)q^{-3/2} \sum_{m=1}^\infty \sum_{n=1}^\infty \frac{\lambda_{\bar{\pi}}(m)}{(mn)^{1/2}} V_2 \left( \frac{mnQ^{5/4}}{Q^{5/4}} \right) \sum_{\chi \mod q}^+ \chi(f_q)\overline{\chi}(m)\overline{\chi}(n)\tau(\chi)^3.
\]
Let
\[ A_q(m, n) := \sum_{\chi (\text{mod } q)}^+ \chi(m)\chi(n) \]
and
\[ B_q(m, n) := \sum_{\chi (\text{mod } q)}^+ \chi(\chi(m))\chi(n)\tau(\chi)^3. \]

We obtain
\[ \sum_{q \in \mathcal{D}} \sum_{\chi (\text{mod } q)}^+ L\left(\frac{1}{2}, \pi \times \chi\right)L\left(\frac{1}{2}, \chi\right) \]
\[ = \sum_{q \in \mathcal{D}} A_q(1, 1)V_1 \left(\frac{Q^{5/4}}{f \pi q^3}\right) + \sum_{m \neq 1} \lambda_\pi(m) A_q(m, n)V_1 \left(\frac{mnQ^{5/4}}{f \pi q^3}\right) \]
\[ + W(\pi) \sum_{m=1}^\infty \sum_{n=1}^\infty \frac{\lambda_\pi(m)}{(mn)^{1/2}} V_2 \left(\frac{mn}{Q^{5/4}}\right) \sum_{q \in \mathcal{D}} \omega_\pi(q)q^{-3/2}B_q(m, n) \]
\[ := M + E_1 + E_2. \]

We will show \( M = A(Q) + O \left(\frac{Q^{7/4}}{\log^2 Q}\right) \) and \( E_1, E_2 \ll Q^{15/8+\epsilon} \). Theorem 1.1 follows from these estimates. \( \square \)

3.1. **The main term** \( M \). First we note that the orthogonality of even primitive characters is given by the Lemma below.

**Lemma 3.3.** If \((mn, q) = 1\), then
\[ A_q(m, n) = \frac{1}{2} \left( \sum_{d|q} \phi(d)\mu\left(\frac{q}{d}\right) + \sum_{d|q} \phi(d)\mu\left(\frac{q}{d}\right) \right). \]

Otherwise \( A_q(m, n) = 0 \).

The proof of this is standard and can be easily derived from [IK, (3.8)]. Now we will show \( M \) contributes the main term.

**Lemma 3.4.**
\[ M = A(Q) + O \left(\frac{Q^{7/4}}{\log^2 Q}\right) \]
and
\[ A(Q) = \frac{1}{2} \sum_{q \in \mathcal{D}} q \asymp \frac{Q^2}{\log^2 Q}. \]

**Proof.** For \( q = pr \in \mathcal{D} \) with \( Q^{3/4} < p \leq 2Q^{3/4} \) and \( Q^{1/4} < r \leq 2Q^{1/4} \), by Lemma 3.3 one has
\[ A_q(1, 1) = \frac{1}{2} (\phi(q) - \phi(p) - \phi(r) + 2). \]
By the prime number theorem, we deduce that
\begin{equation}
\sum_{q \in \mathcal{D}} A_q(1, 1) = A(Q) + O\left( \frac{Q^{7/4}}{\log^2 Q} \right),
\end{equation}
with $A(Q) = \frac{1}{2} \sum_{q \in \mathcal{D}} q \asymp \frac{Q^2}{\log^2 Q}$. By Lemma 3.1,
\begin{equation*}
M = \sum_{q \in \mathcal{D}} A_q(1, 1) + O(Q^{-100}) = A(Q) + O\left( \frac{Q^{7/4}}{\log^2 Q} \right).
\end{equation*}
\[ \square \]

3.2. Estimate of $E_1$. We will show that $E_1$ only contributes the error term.

**Lemma 3.5.** For any $\varepsilon > 0$, we have
\[ E_1 \ll Q^{15/8 + \varepsilon}. \]

**Proof.** Let
\begin{equation*}
A(m, n) = \sum_{q \in \mathcal{D}} A_q(m, n) V_1\left( \frac{mnQ^{5/4}}{\ell_q q^3} \right).
\end{equation*}
For $mn \neq 1$, by Lemma 3.1 and 3.3 we have
\begin{equation}
A(m, n) \ll \sum_{q \in \mathcal{D}} \sum_{d \mid (q, mn \pm 1)} \phi(d) \ll \sum_{d \mid (q, mn \pm 1)} \frac{Q}{d} \phi(d) \ll Q(mn)^\varepsilon.
\end{equation}
By Lemmas 3.1 and (3.2), we have
\begin{equation*}
E_1 \ll \sum_{mn \leq Q^{7/4 + \varepsilon}} \sum_{d \mid (q, mn \pm 1)} \frac{\lambda_\pi(m)}{\lambda_\pi(m)} Q(mn)^\varepsilon + O(Q^{-100})
\ll Q \sum_{n \leq Q^{7/4 + \varepsilon}} n^{-1/2 + \varepsilon} \sum_{m \leq Q^{7/4 + \varepsilon}} \frac{\lambda_\pi(m)}{m^{1/2 - \varepsilon}} + O(Q^{-100})
\ll Q^{15/8 + \varepsilon}.
\end{equation*}
For the last inequality, we use the Cauchy’s inequality and the following estimate (from Rankin-Selberg theory)
\begin{equation}
\sum_{m \leq X} |\lambda_\pi(m)|^2 \ll X
\end{equation}
for any $X \geq 1$.
\[ \square \]

3.3. Estimate of $E_2$. We will show that $E_2$ only contributes the error term. We need the following Poisson summation formula.

**Lemma 3.6.** Let $X \geq 1$. Let $F(y) = \frac{1}{x} (1 + y^2)^{-1}$ and let $\tilde{F}(y) = e^{-2\pi |y|}$ which is the Fourier transform of $F$. For a primitive character $\chi$ modulo $p$, we have
\begin{equation*}
\sum_{k=-\infty}^{\infty} \chi(k) F\left( \frac{k}{X} \right) = \frac{X}{\tau(\chi)} \sum_{h=-\infty}^{\infty} \chi(h) \tilde{F}\left( \frac{Xh}{p} \right).
\end{equation*}
The proof of this Lemma is standard. For the convenience of the reader, we include the proof here.

Proof.

\[
\sum_{k=-\infty}^{\infty} \chi(k) F \left( \frac{k}{X} \right) = \sum_{a \pmod{p}} \chi(a) \sum_{n=-\infty}^{\infty} F \left( \frac{pn+a}{X} \right) = \sum_{a \pmod{p}} \chi(a) \sum_{h=-\infty}^{\infty} \frac{X}{p} e^{2\pi i \frac{ah}{p}} \tilde{F} \left( \frac{Xh}{p} \right) = \frac{X}{p} \tau(\chi) \sum_{h=-\infty}^{\infty} \tilde{\chi}(h) \tilde{F} \left( \frac{Xh}{p} \right),
\]

where we use \( p = |\tau(\chi)|^2 \), \( \tau(\chi) = \chi(-1) \tau(\chi) \) and \( \tilde{F}(y) \) is even in the last identity. □

Lemma 3.7. For any \( \varepsilon > 0 \), we have

\[ E_2 \ll Q^{15/8+\varepsilon}. \]

Proof. Let

\[ B(m, n) = \sum_{q \in \mathcal{D}} \omega_\pi(q) q^{-3/2} B_q(m, n), \]

then

\[ E_2 = W(\pi) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\lambda_\pi(m)}{(mn)^{1/2}} V_2 \left( \frac{mn}{Q^{5/4}} \right) B(m, n). \]

By Cauchy’s inequality, Lemma 3.1, and (3.3)

\[
E_2 \ll \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\lambda_\pi(m)^2}{mn} \left| V_2 \left( \frac{mn}{Q^{5/4}} \right) \right|^2 \right)^{1/2} \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |B(m, n)|^2 \right)^{1/2} \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{mn}{Q^{5/4}} |B(m, n)|^2 \right)^{1/2} \lesssim Q^\varepsilon \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} F \left( \frac{mn}{Q^{5/4}} \right) |B(m, n)|^2 \right)^{1/2},
\]

where \( F(y) = \frac{1}{\pi} (1 + y^2)^{-1} \).

For \( q = pr \in \mathcal{D} \) with \( Q^{3/4} < p \leq 2Q^{3/4} \) and \( Q^{1/4} < r \leq 2Q^{1/4} \), each primitive character \( \chi \pmod{q} \) can be factorized as \( \chi = \chi_1 \chi_2 \) where \( \chi_1 \pmod{p} \) and \( \chi_2 \pmod{r} \) are primitive characters. Moreover

\[ \tau(\chi_1 \chi_2) = \chi_1(r) \chi_2(p) \tau(\chi_1) \tau(\chi_2). \]
Hence (for convenience, we omit the conditions of \( p \) and \( r \) below)

\[
B(m, n) = \frac{1}{2} \sum_{q \in \mathcal{D}_k} \sum_{\chi \pmod{q}} \chi(1 + \chi(-1))\chi(\tau)m\tau(\chi)^3
\]

\[
= \frac{1}{2} \sum_{p} \sum_{r} \omega_\pi(p)\omega_\pi(r)p^{-3/2}r^{-3/2} \sum_{\chi_1 \pmod{p}} \sum_{\chi_2 \pmod{r}} \chi_1^3(r)\chi_2^3(p)\tau(\chi_1)^3\tau(\chi_2)^3
\]

\[\times \left( \chi_1^3(mn\tau)\chi_2^3(mn\tau) + \chi_1(-mn\tau)\chi_2(-mn\tau) \right) \ll \sum_{r} \sum_{\chi_2 \pmod{r}} \sum_{\chi_1 \pmod{p}} \chi_1^3(r)\tau^3(\chi_1)(\pm mn\tau)^3(\chi_1),\]

where \( \sum^* \) denotes summation over primitive characters. Let

\[
K := \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} F \left( \frac{mn}{Q^{5/4}} \right) |B(m, n)|^2.
\]

By (3.5) and the Cauchy's inequality, we have

\[
K \ll \sum_{k=1}^{\infty} d(k) F \left( \frac{k}{Q^{5/4}} \right) \left( \sum_{r} \sum_{\chi_2 \pmod{r}} \sum_{\chi_1 \pmod{p}} \chi_1^3(r)\tau^3(\chi_1) \right)^2
\]

\[
\ll Q^\varepsilon \sum_{k=1}^{\infty} F \left( \frac{k}{Q^{5/4}} \right) \left( \sum_{r} \sum_{\chi_2 \pmod{r}} \sum_{\chi_1 \pmod{p}} \chi_1^3(r)\tau^3(\chi_1) \right)^2
\]

\[
\ll Q^{1/2+\varepsilon} \sum_{k=1}^{\infty} F \left( \frac{k}{Q^{5/4}} \right) \sum_{r} \sum_{\chi_2 \pmod{r}} \sum_{\chi_1 \pmod{p}} \chi_1^3(r)\tau^3(\chi_1) |B(m, n)|^2
\]

Opening the square and using the orthogonality relation in summing over \( \chi_2 \), we have

\[
(3.6) \quad K \ll Q^{1/2+\varepsilon} \sum_{r} \phi(r) \sum_{p_1 \equiv p_2 \pmod{r}} (p_1p_2)^{-3/2}
\]

\[\times \left| \sum_{\chi_1 \pmod{p_1}} \sum_{\chi_3 \pmod{p_2}} \chi_1^3(\tau)\chi_3^3(\tau) \sum_{k=-\infty}^{\infty} F \left( \frac{k}{Q^{5/4}} \right) \chi_3(\pm \tau^3) \right|.
\]

Now we estimate (3.6) by considering the following cases.

**Case 1.** If \( p_1 = p_2 = p \) and \( \chi_1 = \chi_3 \), the contribution is

\[
Q^{1/2+\varepsilon} \sum_{r} \phi(r) \sum_{p} p^{-3} \left| \sum_{\chi \pmod{p}} \tau(\chi) \right|^6 \sum_{k=-\infty}^{\infty} F \left( \frac{k}{Q^{5/4}} \right) \ll Q^{15/4+\varepsilon}.
\]
Case 2. If \( p_1 = p_2 = p \) but \( \chi_1 \neq \chi_3 \), then \( \overline{\chi_1} \chi_3 \) is a primitive character modulo \( p \). By Lemma 3.6, the contribution from this is

\[
Q^{1/2+\varepsilon} \sum_r \phi(r) \sum_{p \equiv p_2 (\text{mod } r)} (p_1 p_2)^{-3/2} \sum^*_{\chi_1 \overline{\chi_3} \equiv \chi_1 \not\equiv \chi_3} \frac{\tau^3(\chi_1) \overline{\tau^3(\chi_3)}}{\tau(\chi_1 \overline{\chi_3})} \sum_{h=-\infty}^\infty \chi_1 \overline{\chi_3}(\pm hr^3 f_\pi) \widetilde{F}\left(\frac{Q^{5/4} h}{p_1 p_2}\right) \quad \leq Q^{-A}
\]

for any \( A > 0 \).

Case 3. If \( p_1 \neq p_2 \), then \( \overline{\chi_1} \chi_3 \) is a primitive character modulo \( p_1 p_2 \). By Lemma 3.6, the contribution from these terms is

\[
Q^{1/2+\varepsilon} \sum_r \phi(r) \sum_{\substack{p_1 \equiv p_2 (\text{mod } r) \\ p_1 \neq p_2}} (p_1 p_2)^{-3/2} \sum^*_{\chi_1 \overline{\chi_3} (\text{mod } p_2)} \frac{\tau^3(\chi_1) \overline{\tau^3(\chi_3)}}{\tau(\chi_1 \overline{\chi_3})} \sum_{h=-\infty}^\infty \chi_1 \overline{\chi_3}(\pm hr^3 f_\pi) \widetilde{F}\left(\frac{Q^{5/4} h}{p_1 p_2}\right) \quad \leq Q^{15/4+\varepsilon}
\]

By (3.4), this equals

\[
Q^{1/2+\varepsilon} \sum_r \phi(r) \sum_{\substack{p_1 \equiv p_2 (\text{mod } r) \\ p_1 \neq p_2}} (p_1 p_2)^{-3/2} \sum^*_{\chi_1 \overline{\chi_3} (\text{mod } p_2)} \frac{\tau^2(\chi_1) \chi_1(\pm hr^3 f_\pi p_2) \chi_3(\pm hr^3 f_\pi p_2)}{\chi_1 (\text{mod } p_1) \chi_3 (\text{mod } p_2)} \quad \leq Q^{15/4+\varepsilon}
\]

For the previous inequality, we use if \( (a, p) = 1 \),

\[
\sum^*_{\chi (\text{mod } p)} \tau^2(\chi) \overline{\chi}(a) = (p - 2) \sum_{uv \equiv a (\text{mod } p)} e\left(\frac{u + v}{p}\right) \quad \leq p^{3/2}
\]

by Weil’s bound.

By cases 1-3, we have \( K \leq Q^{15/4+\varepsilon} \) and hence \( E_2 \leq Q^{15/8+\varepsilon} \). \( \square \)

4. UPPER BOUND FOR THE SECOND MOMENT

In this section we will prove an upper bound for the second moment of the product of \( L \)-functions. The proof will make use of the following large sieve inequality.

**Lemma 4.1.** Let \( N \geq 1 \). For any complex number \( a_n \), we have

\[
\sum_{q \leq Q} \frac{1}{\phi(q)} \sum^*_{\chi (\text{mod } q)} \left| \sum_{n \leq N} a_n \chi(n) \right|^2 \leq (Q^2 + N - 1) \sum_{n \leq N} |a_n|^2.
\]

**Proof.** See [IK, Theorem 7.11]. \( \square \)
**Theorem 4.2.** For any $\varepsilon > 0$, we have

\begin{equation}
\sum_{q \in \mathcal{D}} \sum_{\chi(\text{mod } q)}^+ \left| L\left(\frac{1}{2}, \pi \times \chi \right) L\left(\frac{1}{2}, \chi \right) \right|^2 \ll Q^{2+\varepsilon}.
\end{equation}

**Proof.** By Lemma 3.1 and 3.2, we have

\begin{equation}
\ll \sum_{q \in \mathcal{D}} \sum_{\chi(\text{mod } q)}^+ \left| \sum_{mn \leq Q^{7/4+\varepsilon}} \frac{\lambda_\pi(m) \chi(m) \chi(n)}{(mn)^{1/2}} V_1 \left( \frac{mnQ^{5/4}}{f_\pi q^3} \right) \right|^2 \\
\ll \sum_{q \leq Q} \frac{q}{\phi(q)} \sum_{\chi(\text{mod } q)}^+ \left| \sum_{k \leq Q^{7/4+\varepsilon}} \left( \frac{1}{k^{1/2}} V_1 \left( \frac{kQ^{5/4}}{f_\pi q^3} \right) \sum_{d | k} \lambda_\pi(d) \right) \chi(k) \right| \\
\ll Q^{2+\varepsilon}.
\end{equation}

The other term in (4.2) is estimated in the same way, and this completes the proof of the theorem. \[\square\]

5. **Proof of Theorem 1.3**

By Theorem 1.1, we have

\begin{equation}
\sum_{q \in \mathcal{D}} \sum_{\chi(\text{mod } q)}^+ \left| L\left(\frac{1}{2}, \pi \times \chi \right) L\left(\frac{1}{2}, \chi \right) \right| \gg \frac{Q^2}{\log^2 Q}.
\end{equation}
On the other hand, by Cauchy’s inequality and Theorem 4.2, we have
\[ \sum_{q \in \mathcal{D}} \sum_{\chi \pmod{q}}^+ \left| L\left(\frac{1}{2}, \pi \times \chi \right) L\left(\frac{1}{2}, \chi \right) \right| \]
\[ \leq \left( \sum_{q \in \mathcal{D}} \sum_{\chi \pmod{q}}^+ 1 \right)^{1/2} \left( \sum_{q \in \mathcal{D}} \sum_{\chi \pmod{q}}^+ \left| L\left(\frac{1}{2}, \pi \times \chi \right) L\left(\frac{1}{2}, \chi \right) \right|^2 \right)^{1/2} \]
\[ \leq \left( \sum_{q \in \mathcal{D}} \sum_{\chi \pmod{q}}^+ 1 \right)^{1/2} Q^{1+\varepsilon}. \] (5.2)

The Theorem follows from (5.1) and (5.2). \(\square\)

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**References**


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