HYBRID BOUNDS FOR QUADRATIC WEYL SUMS AND ARITHMETIC APPLICATIONS

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Abstract. Let $D < 0$ be an odd fundamental discriminant and $q$ be a prime number which splits in $\mathbb{Q}(\sqrt{D})$. Given a suitable smooth function $f$ supported on $[X, 2X]$ for $X \geq 1$, we establish a uniform bound in $X$, $D$ and $q$ for

$$\sum_{c \equiv 0 \pmod{q}} W_h(D; c)f(c),$$

where

$$W_h(D; c) := \sum_{\substack{b \pmod{2c} \\ b^2 \equiv D \pmod{4c}}} e\left(\frac{hb}{2c}\right), \quad h \in \mathbb{Z}, \quad e(z) := e^{2\pi iz}$$

is the Weyl sum for roots of the quadratic congruence $x^2 \equiv D \pmod{4c}$. We use this result to study several problems of arithmetic interest, including “level-aspect” versions of equidistribution of quadratic roots, and the asymptotic distribution of traces of CM values of weakly holomorphic modular functions of level $q$. By work of Zagier and Bruinier-Funke, the generating functions for these traces are weight $3/2$ weakly holomorphic modular forms of level $4q$ satisfying Kohnen’s plus space condition.

1. Introduction and statement of results

Let $D < 0$ be a discriminant and define the quadratic Weyl sum

$$W_h(D; c) := \sum_{\substack{b \pmod{2c} \\ b^2 \equiv D \pmod{4c}}} e\left(\frac{hb}{2c}\right), \quad h \in \mathbb{Z}, \quad e(z) := e^{2\pi iz}.$$

In many arithmetic problems one needs a bound for these sums as the modulus $c$ varies in some range, for example

$$W_h(f, D, q) := \sum_{c \equiv 0 \pmod{q}} W_h(D; c)f(c)$$

(1.1)

where $q \geq 1$ is an integer and $f$ is a smooth function supported on $[X, 2X]$ for $X \geq 1$. The main result of this paper is the following hybrid bound for (1.1).

**Theorem 1.1.** Let $D < 0$ be an odd fundamental discriminant and $q$ be a prime number which splits in $\mathbb{Q}(\sqrt{D})$. Let $f : \mathbb{R} \to \mathbb{C}$ be a $C^\infty$ function supported on $[X, 2X]$ for $X \geq 1$ which satisfies

$$f^{(j)} \ll X^{-j}, \quad j = 0, 1, \ldots$$

For $X \gg |D|^{1/2}$, we have

$$W_h(f, D, q) \ll \varepsilon |h|(q|D|X|h|)^{\varepsilon} \min\{A(X, D), B(X, D, q), C(X, D, q)\},$$

where

$$A(X, D) := \frac{(\log(XD)^{1/2})^2}{X^{1/2}}, \quad B(X, D, q) := q^{1/2(XD)^{1/2} \log q},$$

$$C(X, D, q) := q^{1/2(XD)^{1/2} \log q}. $$
where
\[
A(X, D) := X^{3/4}|D|^{1/16}, \quad B(X, D, q) := X^{1/2} \left(1 + \frac{X^{1/4}}{q^{1/4}|D|^{1/8}}\right) \left(1 + \frac{|D|^{1/4}}{q^{1/8}}\right),
\]
\[
C(X, D, q) := X^{1/2} \left(1 + \frac{X^{1/4}}{q^{1/2}|D|^{1/8}}\right) \left(1 + \frac{|D|^{1/4}}{q^{1/8}}\right).
\]

A bound for (1.1) with a power saving in $X$ was first established by Hooley [H] and later improved by Bykovski [B]. In these papers $D$ is fixed and $q = 1$, however, for many arithmetic problems it is crucial to have a wide range of uniformity in at least two of the variables $X, D$ and $q$. Examples of this occur in the groundbreaking work of Duke, Friedlander and Iwaniec [DFI] on the equidistribution of quadratic roots to prime moduli, and the work of Tóth [T] which gives an analog of this result for positive discriminants. Other examples where such uniformity is required occur in Duke, Friedlander and Iwaniec’s recent paper [DFI2], where they established a strong uniform bound for (1.1) (their method works for both positive and negative discriminants). A central role was played by their important work on bilinear forms with Kloosterman fractions [DFI3].

In this paper we study (1.1) from the perspective of period formulas and mean-values of $L$–functions. Our approach is influenced in many ways by [DFI], and our recent joint work with Matt Young [LMY] on hybrid subconvexity and equidistribution of Heegner points in the level aspect. To prove Theorem 1.1, we first express $W_h(f, D, q)$ as the trace of a certain weight zero, smooth Poincaré series for $\Gamma_0(q)$ over the Heegner points of discriminant $D$. After spectrally decomposing the Poincaré series and calculating the spectral coefficients, we are led to estimating, up to a very small error term, an expression of the form
\[
\sum_{|t_g| \leq (qX|D|^{1/2}|h|)^\epsilon} \overline{\rho_g(-h)} \hat{\phi}(t_g) W_{D,g} + \text{continuous spectrum contribution}
\]
where $g$ runs over an orthonormal basis of Maass cusp forms for $\Gamma_0(q)$ with spectral parameter $t_g$. Here $\rho_g(h)$ is the $h$-th Fourier coefficient of $g$, $\hat{\phi}(t)$ is the integral transform
\[
\hat{\phi}(t) := \int_0^\infty \phi(u) K_{it}(u) u^{-3/2} du
\]
with
\[
\phi(u) := f \left( \frac{\pi |h| \sqrt{|D|}}{u} \right),
\]
and $W_{D,g}$ is the trace of $g$ over the Heegner points of discriminant $D$. A formula of Waldspurger [W] and Zhang [Zh] relates $|W_{D,g}|^2$ to $L(g, 1/2)L(g \times \chi_D, 1/2)$ where $\chi_D$ is the Kronecker symbol associated to $\mathbb{Q}(\sqrt{D})$. Various applications of Hölder’s inequality are possible here (see section 7 for further discussion), and we are led naturally to estimating mean-values of different families of $L$–functions. For example, with the choice of exponents 4, 2, 4 in Hölder’s inequality, we are led to estimating the mean-values
\[
M_1 := \sum_{|t_g| \leq (qX|D|^{1/2}|h|)^\epsilon} |\rho_g(h)|^4 |\hat{\phi}(t_g)|^4, \quad M_2 := \sum_{|t_g| \leq (qX|D|^{1/2}|h|)^\epsilon} \frac{L(g \times \chi_D, \frac{1}{2})}{L(\text{sym}^2 g, 1)},
\]
Theorem 1.2.

For $M_3 := \sum_{|t_q| \leq (q^4/|D|^{1/2})^e} L(g, \frac{1}{2})^2$,
where the sums are over Maass newforms $g$ for $\Gamma_0(q)$. We estimate $M_1$ using the
Kuznetsov trace formula, and we estimate $M_3$ using the spectral large sieve inequality (see [DI]).
We estimate $M_2$ in two different ways, first using a hybrid subconvexity bound of
Blomer-Harcos [BH], and second using the following estimate proved in [LMY, Theorem 1.5] for $q$ a prime
with $(q, D) = 1$ and $M \geq 1$,

$$\sum_{|t_q| \leq M} \frac{L(g \times \chi_d, \frac{1}{2})}{L(\text{sym}^2 g, 1)} \ll \epsilon (qM^2 + |D|^{1/2})(|D|Mq)^\epsilon. \quad \text{(1.2)}$$

As mentioned earlier, Duke, Friedlander and Iwaniec [DFI2] established a strong uniform
bound for (1.1) using methods quite different from those in this paper and gave many
interesting arithmetic applications. One application they gave was to the equidistribution
modulo 1 of roots of the quadratic congruence $x^2 \equiv D \pmod{c}$ as $X, |D|, q \to \infty$ (see [DFI2,
Theorem 1.3]). Here we give some examples of results in this direction one can obtain by
employing the bounds in Theorem 1.1. In particular, these bounds yield a wide range of
uniformity in $q$, which has significance for an analog of Linnik’s problem on the least prime
in an arithmetic progression.

Let $I \subset [0, 1]$ be a fixed subinterval of length $\ell(I) > 0$, and define

$$N_I(X, D, q) := |\{b \pmod{2c} : 1 \leq c \leq X, \ c \equiv 0 \pmod{q}, \ b^2 \equiv D \pmod{4c}, \ \frac{b}{2c} \in I\}|.$$

If $X = |D|^{1/2}$, so that the minimum in Theorem 1.1 is $A(X, D)$, we obtain the following

**Theorem 1.2.** For $D$ and $q$ as in Theorem 1.1, we have

$$N_I(|D|^{1/2}, D, q) \sim \frac{12}{\pi^2} \frac{L(\chi_D, 1)}{q + 1} |D|^{1/2} \ell(I)$$
as $q, |D| \to \infty$ subject to the condition $q \leq |D|^{1/16 - \epsilon}$.

In particular, we obtain: given $q$ and $I$, for any $|D|^{1/2} \gg q^{8 + \epsilon}$ we have $N_I(|D|^{1/2}, D, q) > 0$.
This is a natural analog of Linnik’s problem on the least prime in an arithmetic progression:
given $q$ and a residue class $a \pmod{q}$ with $(a, q) = 1$, for any $X \gg q^L$ one has $\pi(X, a, q) > 0$ where

$$\pi(X, a, q) := |\{p \leq X : \ p \ a \ prime, \ p \equiv a \pmod{q}\}|$$
and $L$ is the famous Linnik constant (see [IK, chapter 18]).

On the other hand, if $X = |D|$, so that the minimum in Theorem 1.1 is $B(X, D, q)$, we
obtain the following

**Theorem 1.3.** For $D$ and $q$ as in Theorem 1.1, we have

$$N_I(|D|, D, q) \sim \frac{12}{\pi^2} \frac{L(\chi_D, 1)}{q + 1} |D| \ell(I)$$
as $q, |D| \to \infty$ subject to the condition $|D|^{1/12} \leq q \leq |D|^{1/2 - \epsilon}$.

We can also establish “sparse equidistribution” analogs of these results in which the subin-
terval $I$ is allowed to shrink as a function of $X, D$ and $q$. For example, if $X = |D|^{1/2}$ and $q$
is fixed, we obtain the following
Theorem 1.4. Let $D$ and $q$ be as in Theorem 1.1. If $I_D \subset [0, 1]$ is a subinterval of length $\ell(I_D) = |D|^{-\eta}$ with $\eta > 1/48$, we have

$$N_{I_D}(|D|^{1/2}, D, q) \sim \frac{12 L(\chi_D, 1)}{\pi^2} |D|^{1/2 - \eta}$$

as $|D| \to \infty$.

In particular, we obtain: given $I_D$ with $\ell(I_D) \gg |D|^{-1/48} + \varepsilon$, we have $N_{I_D}(|D|^{1/2}, D, q) > 0$.

In a somewhat different direction, we will use Theorem 1.1 to study the distribution of traces of singular values of weakly holomorphic modular functions of level $q$. To describe this result, let $Q_{D,q}$ be the set of positive definite, integral binary quadratic forms

$$Q(X, Y) = aX^2 + bXY + cY^2$$

of discriminant $b^2 - 4ac = D < 0$ with $a \equiv 0 \pmod{q}$. The group $\Gamma_0^*(q)$ generated by $\Gamma_0(q)$ and the Fricke involution acts on $Q_{D,q}$ with finite quotient. Let $M_0^!(\Gamma_0^*(q))$ be the space of weakly holomorphic modular forms of weight zero for $\Gamma_0^*(q)$. Such a form $f$ is holomorphic on the complex upper half-plane $\mathbb{H}$, meromorphic in the cusps of $\Gamma_0^*(q)$, and has a Fourier expansion in the cusp at $\infty$ of the form

$$f(z) = \sum_{n=1}^{N_f} a_f(-n)e(-nz) + \sum_{n=0}^{\infty} a_f(n)e(nz)$$

for some integer $N_f \geq 1$. Define the trace

$$\text{Tr}_{D,q}^*(f) := \sum_{\tau_Q \in Q_{D,q}/\Gamma_0^*(q)} \frac{f(\tau_Q)}{\#\Gamma_0^*(q)\tau_Q},$$

where

$$\tau_Q = \frac{-b + \sqrt{D}}{2a}$$

is the root of $Q(X, 1)$ in $\mathbb{H}$ and $\Gamma_0^*(q)\tau_Q$ is the stabilizer of $Q$ in $\Gamma_0^*(q)$.

The classical modular $j$-function

$$j(z) := e(-z) + 744 + 196884 \cdot e(z) + \cdots$$

is a weakly holomorphic modular form of weight zero for $\Gamma_0(1)$ whose values $j(\tau_Q)$ are algebraic integers called “singular moduli”. Let $J := j - 744$. Zagier [Z2] proved that the generating function for traces of singular moduli,

$$e(-z) - 2 - \sum_{D < 0} \text{Tr}_{D,1}(J)e(|D|z),$$

is a weight $3/2$ weakly holomorphic modular form for $\Gamma_0(4)$ satisfying Kohnen’s plus space condition. Bruinier and Funke [BF] generalized this result to forms $f \in M_0^!(\Gamma_0^*(q))$. In particular, if $q$ is a prime (or $q = 1$) and $f$ has constant term $a_f(0) = 0$, they proved that the generating function

$$\sum_{D < 0} \text{Tr}_{D,q}^*(f)e(|D|z) + \sum_{n \geq 1} (\sigma_1(n) + q\sigma_1(n/q))a_f(-n) - \sum_{m \geq 1} \sum_{n \geq 1} ma_f(-mn)e(-m^2z)$$

is a weight $3/2$ weakly holomorphic modular form for $\Gamma_0(4)$ satisfying Kohnen’s plus space condition.
is a weight 3/2 weakly holomorphic modular form for $\Gamma_0(4q)$ satisfying Kohnen’s plus space condition. Here $\sigma_1(0) = -1/24$, $\sigma_1(n) = \sum_{t|n} t$ for $n \in \mathbb{Z}_{\geq 0}$ and $\sigma_1(x) = 0$ for $x \notin \mathbb{Z}_{\geq 0}$. In particular, if $q = 1$ and $f = J$, they recovered Zagier’s result.

We are interested in the asymptotic distribution of the traces $\text{Tr}^*_{D,q}(f)$. For traces of singular moduli, Bruinier, Jenkins and Ono [BJO] established the Rademacher type exact formula

$$\text{Tr}_{D,1}(J) = -24H(D) + 2 \sum_{c \in \mathbb{Z}^+} W_1(D; c) \sinh \left( \frac{\pi \sqrt{|D|}}{c} \right),$$

where

$$H(D) := \sum_{Q \in \mathcal{Q}_{D,1}/\Gamma_0(1)} \frac{1}{\#\Gamma_0(1)Q}$$

is the Hurwitz class number. Based on this exact formula, they conjectured that

$$\text{Tr}_{D,1}(J) = -24H(D) + 2 \sum_{1 \leq c < \sqrt{|D|}/3} W_1(D; c) \sinh \left( \frac{\pi \sqrt{|D|}}{c} \right) + o(H(D))$$

as $|D| \to \infty$ through fundamental discriminants. This conjecture was proved by Duke [D] using the equidistribution of CM points.

For traces of weakly holomorphic forms $f \in M_0^!(\Gamma_0^*(q))$, Choi, Jeon, Kang and Kim [CJKK] established the exact formula

$$\text{Tr}^*_{D,q}(f) = -24H_q^*(D) \sum_{h > 0} a_f(-h)c_q(h) + 2 \sum_{h > 0} a_f(-h) \sum_{c \equiv 0 \pmod{q}} W_h(D; c) \sinh \left( \frac{\pi h \sqrt{|D|}}{c} \right),$$

where

$$H_q^*(D) := \sum_{Q \in \mathcal{Q}_{D,q}/\Gamma_0^*(q)} \frac{1}{\#\Gamma_0^*(q)Q}$$

is the “level $q$” Hurwitz class number and

$$c_q(h) := \frac{q^{\alpha+1}}{q+1} \sigma_1(h/q^\alpha) + \sigma_1(h), \quad q^\alpha \parallel h.$$

We will use Theorem 1.1 to establish the following asymptotic formula for $\text{Tr}^*_{D,q}(f)$ with a power saving in $D$.

**Theorem 1.5.** For $D$ and $q$ as in Theorem 1.1 and $f \in M_0^!(\Gamma_0^*(q))$, we have

$$\text{Tr}^*_{D,q}(f) = -24H_q^*(D) \sum_{h > 0} a_f(-h)c_q(h) + 2 \sum_{h > 0} a_f(-h) \sum_{1 \leq c < \sqrt{|D|}/3 \atop c \equiv 0 \pmod{q}} W_h(D; c) \sinh \left( \frac{\pi h \sqrt{|D|}}{c} \right) + O_{f,\varepsilon}(|D|^{7/16+\varepsilon})$$

as $|D| \to \infty$. 
Acknowledgments. We thank Matt Young for several helpful discussions and the referee for a very careful reading of the manuscript leading to an improved exposition. The second author was partially supported by the NSF grant DMS-1162535 during the preparation of this work.

\section{Heegner points}

Let $D < 0$ be an odd fundamental discriminant and $q$ be a prime number which splits in $\mathbb{Q}(\sqrt{D})$. Let $Q_{D,q}$ be the set of positive definite, integral binary quadratic forms

$$Q(X, Y) = [a_Q, b_Q, c_Q](X, Y) = a_Q X^2 + b_Q XY + c_Q Y^2$$

of discriminant $b_Q^2 - 4a_Q c_Q = D$ with $a_Q \equiv 0 \pmod{q}$. The group $\Gamma_0(q)$ acts on $Q_{D,q}$ by

$$Q = [a_Q, b_Q, c_Q] \mapsto Q^\sigma = [a_Q^\sigma, b_Q^\sigma, c_Q^\sigma],$$

where for $\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(q)$ we have

$$a_Q^\sigma = a_Q \alpha^2 + b_Q \alpha \beta + c_Q \beta^2,$$

$$b_Q^\sigma = 2a_Q \alpha \gamma + b_Q (\alpha \delta + \beta \gamma) + 2c_Q \beta \delta,$$

$$c_Q^\sigma = a_Q \gamma^2 + b_Q \gamma \delta + c_Q \delta^2.$$

Given a solution $r \pmod{2q}$ of $r^2 \equiv D \pmod{4q}$ (there are 2 such solutions since $q$ is a prime), we define the subset of forms

$$Q_{D,q,r} := \{Q = [a_Q, b_Q, c_Q] \in Q_{D,q} : b_Q \equiv r \pmod{2q}\}.$$

Then $\Gamma_0(q)$ also acts on $Q_{D,q,r}$, and we have the decomposition (see [GKZ, p. 507])

$$Q_{D,q}/\Gamma_0(q) = \bigcup_{r^2 \equiv D \pmod{4q}} Q_{D,q,r}/\Gamma_0(q).$$

(2.1)

To each form $Q \in Q_{D,q}$ we associate the root

$$\tau_Q = \frac{-b_Q + \sqrt{D}}{2a_Q} \in \mathbb{H}.$$

This is compatible with the group action in the sense that $\sigma \tau_Q = \tau_{Q^\sigma}$ for $\sigma \in \Gamma_0(q)$. Fix a set of representatives for the $\Gamma_0(q)$-equivalence classes of forms in $Q_{D,q}$ and define the set of Heegner points of discriminant $D$,

$$\Lambda_{D,q} := \{\tau_Q : Q \in Q_{D,q}/\Gamma_0(q)\}.$$

Given $r$ and $Q_{D,q,r}$ as above, each set

$$\Lambda_{D,q,r} := \{\tau_Q : Q \in Q_{D,q,r}/\Gamma_0(q)\}$$

is a Gal($H/K$)-orbit of Heegner points of discriminant $D$, where $H$ is the Hilbert class field of $K = \mathbb{Q}(\sqrt{D})$ (see [GZ, pp. 235-236]).
3. Preliminaries on Maass forms

In this section we review some facts we will need concerning Maass forms (see e.g. [I] and [B, section 2]). Let $f_1, f_2 : \mathbb{H} \to \mathbb{C}$ be $\Gamma_0(q)$-invariant functions and $Y_0(q)$ be a fundamental domain for $\Gamma_0(q)$. Define the Petersson inner product
\[ \langle f_1, f_2 \rangle_q := \int_{Y_0(q)} f_1(z) \overline{f_2(z)} \frac{dx dy}{y^2}, \] (3.1)
provided the integral is absolutely convergent.

The hyperbolic Laplacian is defined by
\[ \Delta := -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right). \]

Given a Maass form $g$ with Laplace eigenvalue $\lambda_g$, let $t_g := \sqrt{\lambda_g - 1/4} \in \mathbb{R} \cup (-1/2, 1/2)i$ be the spectral parameter. Let $B_q$ be an orthonormal basis of Hecke-Maass newforms of weight 0 for $\Gamma_0(q)$, and $B_1$ be a basis of Hecke-Maass cusp forms of weight 0 for $\text{SL}_2(\mathbb{Z})$ which is orthonormal with respect to the inner product (3.1). One has the following upper bounds (see [I, section 11.1])
\[ \# \{ g \in B_q : |t_g| \leq T \} \ll qT^2 \quad \text{and} \quad \# \{ g \in B_1 : t_g \leq T \} \ll T^2. \]

Given $g \in B_q \cup B_1$, let $\lambda_g(n)$ be the $n$-th Hecke eigenvalue. Then $\lambda_g(-n) = \pm \lambda_g(n)$ depending on whether $g$ is even or odd, and $\lambda_g(n)$ satisfies
\[ \lambda_g(n) \ll n^{3/2-\delta} \] (3.2)
for some $\delta > 0$. A newform $g \in B_q$ is an eigenfunction for the Fricke involution $z \mapsto -1/qz$, and one has
\[ \lambda_g(q) = \varepsilon(g)q^{-1/2} \] (3.3)
where $\varepsilon(g) = \pm 1$ is the Fricke eigenvalue.

Given $g \in B_1$, define (see [ILS, Proposition 2.6])
\[ g_q(z) := \left( 1 - \frac{q \lambda_g^2(q)}{(q+1)^2} \right)^{-1/2} \left( g(qz) - \frac{q^{1/2} \lambda_g(q)}{q+1} g(z) \right), \] (3.4)
and let $B_1^* := \{ g_q : g \in B_1 \}$. Then an orthonormal basis for the subspace of cusp forms in $L^2(Y_0(q))$ is given by
\[ B := B_q \cup B_1 \cup B_1^*. \]

A form $g \in B_q \cup B_1$ has the Fourier expansion
\[ g(z) = \sqrt{y} \sum_{n \neq 0} \rho_g(n) K_{it_g}(2\pi ny)e(nx), \]
where $\rho_g(n) = \rho_g(1) \lambda_g(n)$ and (see [B, eq. (2.9)])
\[ |\rho_g(1)| = \left( \frac{2 \cosh(\pi t_g)}{L(\text{sym}^2 g, 1)} \right)^{1/2} \times \begin{cases} \left( \frac{q^2}{q+1} \right)^{-1/2}, & g \in B_q \\ (q+1)^{-1/2}, & g \in B_1. \end{cases} \] (3.5)
Similarly, \( g_q \in B_1 \) has the Fourier expansion
\[
g_q(z) = \sqrt{y} \sum_{n \neq 0} \rho_{g_q}(n) K_{i\pi y} (2\pi n y) e(n x),
\]
where \( \rho_{g_q}(n) = \rho_g(1) \lambda_q(n) \) and
\[
\lambda_q(n) := \left( 1 - \frac{q \lambda_q^2(g)}{q + 1} \right)^{-1/2} \left( q^{1/2} \lambda_g(n/q) - \frac{q^{1/2} \lambda_q(q)}{q + 1} \lambda_q(n) \right)
\]
for \( g \in B_1 \), with the convention \( \lambda_g(x) = 0 \) for \( x \in \mathbb{Q} \setminus \mathbb{Z} \).

Using (3.2)–(3.6) and the Hoffstein-Lockhart [HL] bound
\[
L(\text{sym}^2 g, 1) \gg \varepsilon |t_g|^\varepsilon (|t_g| q)^{-\varepsilon},
\]
we obtain
\[
\rho_g(n) \ll \varepsilon |n|^{1/2} |t_g|^{\varepsilon} e^{\pi \varepsilon |t_g|/2}, \quad g \in B.
\]

For the Eisenstein series \( E_a(z, s) \) associated to a cusp \( a \) of \( \Gamma_0(q) \), one has the Fourier expansion (see [I, section 3.4])
\[
E_a(z, \frac{1}{2} + it) = \delta_{a=\infty} y^{1/2+it} + \phi_a(\frac{1}{2} + it) y^{1/2-it} + \sqrt{y} \sum_{n \neq 0} \tau_a(n, t) K_{i\pi y} (2\pi n y) e(n x),
\]
where \( \phi_a(s) \) is a certain meromorphic function and \( \tau_a(n, t) = \rho_a(1, t) \eta_a(n, t) \), where (see [B, eq. (2.13)])
\[
|\rho_a(1, t)| = \left( \frac{4 \cosh(\pi t)}{q |\zeta(q)(1 + 2it)|} \right)^{1/2},
\]
\[
\eta_{\infty}(n, t) := \frac{\eta(n, t)}{q^{1/2+it}} - q^{1/2} \eta(n/q, t), \quad \eta_0(n, t) := \eta(n, t) - q^{-it} \eta(n/q, t),
\]
\[
\eta(n, t) := \sum_{ad=|n|} (a/d)^it,
\]
with the convention \( \eta(x, t) = 0 \) for \( x \in \mathbb{Q} \setminus \mathbb{Z} \).

Using (3.9) and (3.10), we obtain
\[
\tau_a(n, t) \ll \varepsilon |tn|^{\varepsilon} e^{\pi |t|/2}.
\]

Finally, recall that for a function \( f \) in \( L^2(Y_0(q)) \), one has the spectral expansion
\[
f(z) = \frac{\langle f, 1 \rangle_q}{\text{vol}(Y_0(q))} + \sum_{g \in B} \langle f, g \rangle_q g(z) + \sum_a \int_{-\infty}^\infty \langle f, E_a(z, \frac{1}{2} + it) \rangle_q E_a(z, \frac{1}{2} + it) \frac{dt}{4\pi},
\]
which converges in the norm topology (see [I, section 7.3]). If, for example, \( f \) is smooth and compactly supported, then (3.12) converges pointwise absolutely and uniformly on compact sets.
4. Traces of Poincaré series

Let \( \phi : \mathbb{R}_{\geq 0} \to \mathbb{C} \) be a \( C^\infty \) function with compact support and define the Poincaré series
\[
P_{h,\phi}(z) := \sum_{\sigma \in \Gamma_\infty \setminus \Gamma_0(q)} \phi(2\pi |h| \text{Im}(\sigma z)) e(-h \text{Re}(\sigma z)), \quad z \in \mathbb{H},
\]
which is absolutely convergent. We will need the following standard identity (see [By, Lemma 5] and [DFI, section 2]).

Proposition 4.1. Let \( D < -4 \) be an odd fundamental discriminant. Then
\[
\sum_{c \equiv 0 (\text{mod } q)} W_h(D; c) \phi \left( \frac{\pi |h| \sqrt{|D|}}{c} \right) = \sum_{Q \in \mathcal{Q}_{D,q}/\Gamma_0(q)} P_{h,\phi}(\tau_Q).
\]

Proof. For \( c \equiv 0 (\text{mod } q) \) we have
\[
\sum_{Q \in \mathcal{Q}_{D,q}/\Gamma_0(q)} \sum_{\sigma \in \Gamma_\infty \setminus \Gamma_0(q)} e(-h \text{Re}(\sigma \tau_Q)) = \sum_{Q \in \mathcal{Q}_{D,q}/\Gamma_\infty} e(-h \text{Re}(\tau_Q))
\]
\[
= \sum_{b_Q (\text{mod } 2c)} \sum_{b_Q^2 \equiv D (\text{mod } 4c)} e \left( \frac{hb_Q}{2c} \right)
\]
\[
= W_h(D; c),
\]
where we used that the stabilizer of \( Q \) in \( \Gamma_0(q) \) is \( \{ \pm I \} \) for \( D < -4 \). It follows that
\[
\sum_{Q \in \mathcal{Q}_{D,q}/\Gamma_0(q)} P_{h,\phi}(\tau_Q) = \sum_{Q \in \mathcal{Q}_{D,q}/\Gamma_0(q)} \sum_{\sigma \in \Gamma_\infty \setminus \Gamma_0(q)} \phi(2\pi |h| \text{Im}(\sigma \tau_Q)) e(-h \text{Re}(\sigma \tau_Q))
\]
\[
= \sum_{c \equiv 0 (\text{mod } q)} \phi \left( \frac{\pi |h| \sqrt{|D|}}{c} \right) \sum_{Q \in \mathcal{Q}_{D,q}/\Gamma_0(q)} \sum_{\sigma \in \Gamma_\infty \setminus \Gamma_0(q)} e(-h \text{Re}(\sigma \tau_Q))
\]
\[
= \sum_{c \equiv 0 (\text{mod } q)} W_h(D; c) \phi \left( \frac{\pi |h| \sqrt{|D|}}{c} \right).
\]

\( \square \)

5. Spectral expansion of \( W_h(f, D, q) \)

Let \( f : \mathbb{R} \to \mathbb{C} \) be a \( C^\infty \) function supported on \( [X, 2X] \) for \( X \geq 1 \) which satisfies
\[
f^{(j)} \ll X^{-j}, \quad j = 0, 1, \ldots
\]
and define
\[
\phi(u) := f \left( \frac{\pi |h| \sqrt{|D|}}{u} \right). \quad (5.1)
\]
Then \( \phi \) is a \( C^\infty \) function supported on \( [Y^{-1}, 2Y^{-1}] \) for
\[
Y^{-1} = \frac{\pi |h| \sqrt{|D|}}{2X},
\]
and using Faà di Bruno’s formula [F], for example, one obtains the bound
\[ \phi^{(j)} \ll Y^j, \quad j = 0, 1, \ldots \]

By Proposition 4.1 with \( \phi \) defined as in (5.1), we have

\[ W_h(f, D, q) := \sum_{c \equiv 0 \pmod{q}} W_h(D; c)f(c) = \sum_{\tau \in \Lambda_{D,q}} \mathcal{P}_{h,\phi}(\tau). \quad (5.2) \]

Then spectrally expanding \( \mathcal{P}_{h,\phi}(z) \) as in (3.12) and substituting in (5.2) yields

\[ W_h(f, D, q) = h(D) \frac{\langle \mathcal{P}_{h,\phi}, 1 \rangle_q}{\text{vol}(Y_0(q))} + \sum_{g \in B} \langle \mathcal{P}_{h,\phi}, g \rangle_q W_D, g \]
\[ + \sum_a \int_{-\infty}^{\infty} \langle \mathcal{P}_{h,\phi}, E_a(\cdot, \frac{1}{2} + it) \rangle_q W_D, a(t) \frac{dt}{4\pi}, \]

where \( h(D) \) is the class number of \( K \) and the hyperbolic Weyl sums are defined by

\[ W_D, g := \sum_{\tau \in \Lambda_{D,q}} g(\tau) \quad \text{and} \quad W_D, a(t) := \sum_{\tau \in \Lambda_{D,q}} E_a(\tau, \frac{1}{2} + it). \]

Unfolding the Poincaré series gives

\[ \langle \mathcal{P}_{h,\phi}, 1 \rangle_q = 0. \]

Similarly, unfolding gives (see for example [IK, Chapter 16])

\[ \langle \mathcal{P}_{h,\phi}, g \rangle_q = (2\pi|h|^{1/2} \rho_g(-h)\tilde{\phi}(t_g) \]

and

\[ \langle \mathcal{P}_{h,\phi}, E_a(\cdot, \frac{1}{2} + it) \rangle_q = (2\pi|h|^{1/2} \tau_a(-h, t)\tilde{\phi}(t), \]

where \( \tilde{\phi} \) is the integral transform

\[ \tilde{\phi}(t) := \int_0^\infty \phi(u)K_u(u)u^{-3/2}du. \quad (5.3) \]

Finally, combining the preceding calculations yields

\[ W_h(f, D, q) = (2\pi|h|^{1/2} \sum_{g \in B} \rho_g(-h)\tilde{\phi}(t_g)W_D, g \]
\[ + (2\pi|h|^{1/2} \sum_{a} \int_{-\infty}^{\infty} \tau_a(-h, t)\tilde{\phi}(t)W_D, a(t) \frac{dt}{4\pi}. \quad (5.4) \]

6. Period integral formulas

Given \( g \in B \), define the hyperbolic Weyl sum

\[ W_{D,g,r} := \sum_{\tau \in \Lambda_{D,q,r}} g(\tau). \]

Since \( \Lambda_{D,q,r} \) is a \( \text{Gal}(H/K) \)-orbit of Heegner points of discriminant \( D \), it follows from a formula of Waldspurger [W] and Zhang [Zh] that

\[ W_{D, g, r} = \theta_{g, D} \frac{|D|^{1/4} L(g \times \chi_D, \frac{1}{2})^{1/2} L(g, \frac{1}{2})^{1/2}}{q^{1/2} L(\text{sym}^2 g, 1)^{1/2}} \quad \text{if} \quad g \in B_q \cup B_1, \quad (6.1) \]
where \( \theta_{g,D} \) is some complex number satisfying \( |\theta_{g,D}| \leq 10 \). Similarly,

\[
W_{D,g,r} = \theta_{g,D} \frac{|D|^{1/4} L(g \times \chi_D, \frac{1}{2})^{1/2} L(g, \frac{1}{2})^{1/2}}{q^{1/2} L(\text{sym}^2 g, 1)^{1/2}} \quad \text{if} \quad g \in \mathcal{B}^*_r,
\]

where on the right hand side of the identity, \( g \in \mathcal{B}_r \) is the Maass form in the definition of \( g_q \) (see (3.4)). The deduction of these formulas from Waldsburger/Zhang can be found in [LMY, Lemma 5.1], for example.

A similar formula for the Eisenstein series was established by Duke, Friedlander and Iwaniec [DFI4, equation (10.30)],

\[
W_{D,q}(t) = \theta_{D,q,t} \frac{|D|^{1/4} |\zeta(\frac{1}{2} + it)| |L(\chi_D, \frac{1}{2} + it)|}{|\zeta(1 + 2it)|},
\]

(6.2)

where \( \theta_{D,q,t} \) is some complex number satisfying \( |\theta_{D,q,t}| \leq 10 \).

7. CONTRIBUTION OF THE DISCRETE SPECTRUM

In this section we estimate the sum over the Maass forms \( \mathcal{B} \) in (5.4).

We will use the following estimates repeatedly in our analysis (see e.g. [DI] and [DFI]).

**Lemma 7.1.** Let \( \phi : \mathbb{R}_{\geq 0} \to \mathbb{C} \) be a \( C^\infty \) function supported on \([Y^{-1}, 2Y^{-1}]\) which satisfies

\[
\phi^{(j)} \ll Y^j, \quad j = 0, 1, \ldots
\]

The following estimates hold for the integral transform \( \tilde{\phi}(t) \) defined in (5.3).

1. For \( t \in \mathbb{R} \),

\[
\tilde{\phi}(t) \ll Y^{1/2} \log(Y + 6).
\]

2. For \( t \in i\mathbb{R} \) with \( 0 < |t| < 1/2 \),

\[
\tilde{\phi}(t) \ll Y^{1/2} \log(Y + 6)(Y^{it} + Y^{-it}).
\]

3. For \( t \in \mathbb{R} \) with \( |t| \geq 1 \) and \( Y \gg 1 \),

\[
\tilde{\phi}(t) \ll Y^{1/2}(1 + |t|)^{-A} e^{-\frac{\pi}{2} |t|}, \quad A = 0, 1, \ldots
\]

First we estimate the contribution of the oldforms.

**Lemma 7.2.** For \( X \gg |D|^{1/2} \) we have

\[
\sum_{g \in \mathcal{B}_3 \cup \mathcal{B}'_3} \rho_g(-h)\tilde{\phi}(t_g)W_{D,g} \ll \varepsilon |h|^{1/2}(q |D| X |h|)^{\varepsilon} q^{-1} X^{1/2}|D|^{1/6}.
\]

**Proof.** Since \( W_{D,g} \) grows polynomially in \( t_g \), by (3.8) and Lemma 7.1 we may impose the truncation \( t_g \leq (q |D| X |h|)^{\varepsilon} \) with an error term which is \( O((q |D| X |h|)^{-1000}) \) to obtain

\[
\sum_{g \in \mathcal{B}_3 \cup \mathcal{B}'_3} \rho_g(-h)\tilde{\phi}(t_g)W_{D,g} \ll \varepsilon |h|^{1/2}(q |D| Y |h|)^{\varepsilon} Y^{1/2} \log(Y + 6)q^{-\frac{1}{2} + \varepsilon} \sum_{t_g \leq (q |D| Y |h|)^{\varepsilon}} |W_{D,g}|
\]

for \( Y \gg 1 \). By (6.1) and the Conrey-Iwaniec [CI] bound \( L(g \times \chi_D, 1/2) \ll \varepsilon |D|^{1/3} (q |D| Y |h|)^{\varepsilon} \) for \( t_g \leq (q |D| Y |h|)^{\varepsilon} \), we obtain \( W_{D,g} \ll \varepsilon q^{-1/2} |D|^{5/12} (q |D| Y |h|)^{\varepsilon} \). Since there are \( \ll (q |D| Y |h|)^{2\varepsilon} \) oldforms with \( t_g \leq (q |D| Y |h|)^{\varepsilon} \), upon substituting the bound \( Y \asymp X |D|^{-1/2} \) we complete the proof.

Next we estimate the contribution of the newforms.
Lemma 7.3. For $X \gg |D|^{1/2}$ we have
\[
\sum_{g \in B_q} \rho_g(-h) \tilde{\phi}(t_g) W_{D,g} \ll \varepsilon \left| h \right|^{1/2} (q|D|\left| X \right|)^\varepsilon \min\{ A(X,D), B(X,D,q), C(X,D,q) \}, \quad (7.1)
\]
where
\[
A(X,D) := X^{3/4}|D|^{1/16}, \quad B(X,D,q) := X^{1/2} \left( 1 + \frac{X^{1/4}}{q^{1/4}|D|^{1/8}} \right) \left( 1 + \frac{|D|^{1/4}}{q^{1/8}} \right),
\]
\[
C(X,D,q) := X^{1/2} \left( 1 + \frac{X^{1/4}}{q^{1/4}|D|^{1/8}} \right) \left( 1 + \frac{|D|^{1/4}}{q^{1/8}} \right).
\]

Proof. Since $W_{D,g}$ grows polynomially in $t_g$, by (3.8) and Lemma 7.1 we may impose the truncation $|t_g| \leq (q|D|\left| Y \right|)^\varepsilon$ with an error term which is $O((q|D|\left| Y \right|)^{-1000})$ to obtain
\[
\sum_{g \in B_q} \rho_g(-h) \tilde{\phi}(t_g) W_{D,g} \ll \varepsilon \sum_{|t_g| \leq (q|D|\left| Y \right|)^\varepsilon} |\rho_g(h)||\tilde{\phi}(t_g)||W_{D,g}|
\]
for $Y \gg 1$. By (2.1) we have
\[
W_{D,g} \ll |W_{D,g,r}|.
\]

Then from (6.1) we obtain
\[
\sum_{g \in B_q} \rho_g(-h) \tilde{\phi}(t_g) W_{D,g} \ll \varepsilon \frac{|D|^{1/4}}{q^{1/2}} \sum_{|t_g| \leq (q|D|\left| Y \right|)^\varepsilon} |\rho_g(h)||\tilde{\phi}(t_g)| \frac{L(g \times \chi_D, \frac{1}{2})^{1/2}L(g, \frac{1}{2})^{1/2}}{L(\text{sym}^2 g, 1)^{1/2}}.
\]

Various applications of Hölder’s inequality are possible here. For example, applying Hölder’s inequality with exponents 4, 2, 4 yields
\[
\sum_{g \in B_q} \rho_g(-h) \tilde{\phi}(t_g) W_{D,g} \ll \varepsilon \frac{|D|^{1/4}}{q^{1/2}} M_1^{1/4} M_2^{1/2} M_3^{1/4},
\]
where
\[
M_1 := \sum_{|t_g| \leq (q|D|\left| Y \right|)^\varepsilon} |\rho_g(h)|^4 |\tilde{\phi}(t_g)|^4, \quad M_2 := \sum_{|t_g| \leq (q|D|\left| Y \right|)^\varepsilon} \frac{L(g \times \chi_D, \frac{1}{2})^4}{L(\text{sym}^2 g, 1)},
\]
\[
M_3 := \sum_{|t_g| \leq (q|D|\left| Y \right|)^\varepsilon} L(g, \frac{1}{2})^2.
\]

We estimate $M_3$ using a variant of the approximate functional equation [LY, Lemma 2.4] and the following spectral large sieve inequality which was first derived by Deshouillers and Iwaniec [DI] in a slightly weaker form.

Theorem 7.4 ([IK], Theorem 7.24). Let $T \geq 1$ and $N \geq 1$. For any sequence of complex numbers $\{a_n\}_{n=1}^N$, we have
\[
\sum_{g \in \mathcal{B}} \frac{1}{L(\text{sym}^2 g, 1)} \left| \prod_{n=1}^N a_n \lambda_g(n) \right|^2 \ll (qT^2 + N \log(N)) \sum_{n=1}^N |a_n|^2. \quad (7.2)
\]
By [LY, Lemma 2.4], there exists a function $W(x)$, depending on $Q := q(q|D|Y|h|)^\varepsilon$ and $\varepsilon$ only, such that $W(x)$ is supported on $x \leq Q^{1/2+\varepsilon}$ and satisfies
\[ x^j W^{(j)} \ll 1, \]
where the implied constant depends on $j$ and $\varepsilon$ only (not on $Q$), and for which
\[ |L(g, \frac{1}{2})|^2 \ll Q^\varepsilon \int_{-\log(Q)}^{\log(Q)} \left| \sum_{n \geq 1} \frac{\lambda_n(n)}{n^{1+\varepsilon}} W(n) \right|^2 dv + O(Q^{-100}), \quad (7.3) \]
where the implied constant depends on $\varepsilon$, $W$, and the degree of $L(g, s)$ only.

We insert (7.3) into the average over $g$ with $|t_g| \ll (q|D|Y|h|)^\varepsilon$, then apply the spectral large sieve inequality (7.2) and the bound (see [I2])
\[ L(\text{sym}^2 g, 1) \ll \varepsilon (|t_g|^2)^\varepsilon \]
to obtain
\[ M_3 \ll \varepsilon q(q|D|Y|h|)^\varepsilon. \]

We estimate $M_2$ in two different ways. If we apply the Blomer-Harcos [BH] bound $L(g \times \chi_D, 1/2) \ll \varepsilon q^{1/2}|D|^{3/8}(q|D|Y|h|)^\varepsilon$ for $|t_g| \leq (q|D|Y|h|)^\varepsilon$, the bound (3.7), and multiply by the number of newforms which is $\ll q(q|D|Y|h|)^{2\varepsilon}$, we obtain
\[ M_2 \ll \varepsilon (q|D|Y|h|)^\varepsilon q^{3/2}|D|^{3/8}. \]

On the other hand, using the estimate (1.2) we obtain
\[ M_2 \ll \varepsilon (q|D|Y|h|)^\varepsilon (q + |D|^{1/2}). \]

Taken together, these estimates yield
\[ M_2 \ll \varepsilon (q|D|Y|h|)^\varepsilon \min(q^{3/2}|D|^{3/8}, q + |D|^{1/2}). \]

By Proposition 8.2, we have
\[ M_1 \ll \varepsilon |h|^{2+\varepsilon} \left( q^{-1+\varepsilon} Y^2 \log^4(Y + 6) + q^{-2+\varepsilon} Y^2 (Y + Y^{-1} + \log^4(Y + 6)) \log^2(Y + 6) \right). \]

Alternatively, applying Hölder’s inequality with exponents 2, 4, 4 yields
\[ \sum_{g \in B_q} \rho_g(-h) \overline{\phi}(t_g) W_{D,g} \ll \varepsilon \frac{|D|^{1/4}}{q^{1/2}} N_1^{1/2} N_2^{1/4} N_3^{1/4}, \]
where
\[ N_1 := \sum_{|t_g| \leq (q|D|Y|h|)^\varepsilon} |\rho_g(h)|^2 |\overline{\phi}(t_g)|^2, \quad N_2 := \sum_{|t_g| \leq (q|D|Y|h|)^\varepsilon} \frac{L(g \times \chi_D, \frac{1}{2})^2}{L(\text{sym}^2 g, 1)}, \]
\[ N_3 := \sum_{|t_g| \leq (q|D|Y|h|)^\varepsilon} \frac{L(g, \frac{1}{2})^2}{L(\text{sym}^2 g, 1)}. \]

We estimate $N_3$ and $N_2$ using the spectral large sieve inequality as above to obtain
\[ N_3 \ll \varepsilon q(q|D|Y|h|)^\varepsilon \]
and
\[ N_2 \ll \varepsilon (q + q^{1/2}|D|)(q|D|Y|h|)^\varepsilon. \]
Note that to estimate $N_2$, we take $Q := q|D|^2(q|D|Y|h|)^\varepsilon$.

By Proposition 8.1, we have

$$N_1 \ll Y \log^2(Y + 6) + |h|^{1+\varepsilon}q^{-1+\varepsilon}Y \log^2(Y + 6)(Y^{1/2} + Y^{-1/2}).$$

Finally, by combining the estimates for $M_1, M_2$ and $M_3$ (resp. $N_1, N_2$ and $N_3$) and substituting the bounds $Y \asymp X|D|^{-1/2}$ and $q < X$, we obtain (7.1) after a straightforward calculation.

8. Application of the Kuznetsov formula

We begin by showing how to quickly deduce an estimate for $N_1$ from [DFI, eq. (23)].

**Proposition 8.1.** For $Y \gg 1$ we have

$$N_1 \ll Y \log^2(Y + 6) + |h|^{1+\varepsilon}q^{-1+\varepsilon}Y \log^2(Y + 6)(Y^{1/2} + Y^{-1/2}).$$

**Proof.** Let

$$H(t) := \frac{1}{\cosh(\pi t)}h(t)(Y^{2it} + Y^{-2it} + L^2)$$

where $h(t) = 3/(1 + t^2)(4 + t^2)$ and $L = \log(Y + Y^{-1} + 6)$, and define the sum

$$Q := \sum_{g \in B} H(t_g)|\rho_g(h)|^2 + \sum_a \int_{-\infty}^{\infty} H(t)|\tau_a(h, t)|^2 \frac{dt}{4\pi}.$$

Duke, Friedlander and Iwanie [DFI, eq. (23)] used the Kuznetsov trace formula to show

$$Q \ll L^2 + L^2(Y + Y^{-1})^{1/2}h^{1/2}(h, q)^{1/2}\tau(hq).$$

By Lemma 7.1 and positivity, we have

$$N_1 := \sum_{|t_g| \leq (q|D|Y|h|)^\varepsilon} |\rho_g(h)|^2|\tilde{\phi}(t_g)|^2 \ll YQ$$

for $Y \gg 1$. The estimate (8.1) now follows immediately from (8.2).

In the following proposition we adapt the argument in [DFI] to estimate the fourth moment $M_1$.

**Proposition 8.2.** For $Y \gg 1$ we have

$$M_1 \ll Y^{2+\varepsilon}(q^{-1+\varepsilon}Y^2 \log^4(Y + 6) + q^{-2+\varepsilon}Y^2(Y + Y^{-1} + \log^4(Y + 6)) \log^2(Y + 6)).$$

**Proof.** First assume that $(h, q) = 1$. Using the Hecke relations ([I, eq. (8.39)])

$$\lambda_g(m)\lambda_g(n) = \sum_{\ell|(m,n)} \lambda_g(mn/\ell^2),$$

a short calculation yields

$$|\rho_g(h)|^4 = |\rho_g(1)|^2 \sum_{d|h} \sum_{k|h} \rho_g(h^2/d^2)\rho_g(h^2/k^2).$$

Since (see (3.5))

$$|\rho_g(1)|^2 \ll q^{-1+\varepsilon}e^{\pi|t_g|},$$

we have

$$M_1 \ll Y \log^2(Y + 6) + |h|^{1+\varepsilon}q^{-1+\varepsilon}Y \log^2(Y + 6)(Y^{1/2} + Y^{-1/2}).$$

By Proposition 8.1, we have

Finally, by combining the estimates for $M_1, M_2$ and $M_3$ (resp. $N_1, N_2$ and $N_3$) and substituting the bounds $Y \asymp X|D|^{-1/2}$ and $q < X$, we obtain (7.1) after a straightforward calculation.

□
by Lemma 7.1 and positivity we have
\[
M_1 := \sum_{g \in \mathcal{B}} |\rho_g(h)|^4 |\tilde{\phi}(t_g)|^4 \ll e q^{-1+\varepsilon} Y^2 \sum_{d|h} \sum_{k|h} \sum_{g \in \mathcal{B}} \rho_g(h^2/d^2) \rho_g(h^2/k^2) H^*(t_g)
\]
for \(Y \gg 1\), where
\[
H^*(t) := \frac{1}{\cosh(\pi t)} (1 + t^4)^{-1} (Y^{4it} + Y^{-4it} + \log^4(Y + 6)).
\]

The function \(H^*(t)\) clearly satisfies the conditions in [IK, eq. (15.19)]. Therefore by the Kuznetsov trace formula [IK, Theorem 16.3] we have
\[
\sum_{g \in \mathcal{B}} \rho_g(h^2/d^2) \rho_g(h^2/k^2) H^*(t_g) + \sum_a \int_{-\infty}^{\infty} \tau_a(h^2/d^2, t) \tau_a(h^2/k^2, t) H^*(t) \frac{dt}{4\pi} = \delta(h^2/d^2, h^2/k^2) H_0^* + \sum_{c \equiv 0 \mod q} c^{-1} S(h^2/d^2, h^2/k^2; c) \tilde{H}^* \left( \frac{4\pi h^2}{dkc} \right),
\]
where \(S(m, n; c)\) is the Kloosterman sum,
\[
H_0^* := \frac{1}{\pi^2} \int_{-\infty}^{\infty} \sinh(\pi t) H^*(t) t dt
\]
and
\[
\tilde{H}^*(x) := \frac{2i}{\pi} \int_{-\infty}^{\infty} J_{2it}(x) H^*(t) t dt.
\]
Estimating trivially yields
\[
H_0^* \ll \log^4(Y + 6).
\]

We next estimate the sum of Kloosterman sums. Make the change of variables \(s = 1/2 + it\) and shift the line of integration to \(\sigma + it\) with \(1/2 < \sigma < 1\) to obtain
\[
\tilde{H}^*(x) = \frac{2i}{\pi} \int_{(\sigma)} J_{2s-1}(x) H^* \left( (s - 1/2) i \right) (s - 1/2) ds.
\]
By [GR, eq. 8.411.6] and Stirling’s formula,
\[
J_{2s-1}(x) \ll e^{\pi|s|} x^{2s-1}.
\]
We also have
\[
H^* \left( (s - 1/2) i \right) \ll |s|^{-4} e^{-\pi|s|} \left( Y^{2-4\sigma} + Y^{4\sigma-2} + \log^4(Y + 6) \right).
\]
These estimates then yield
\[
\tilde{H}^*(x) \ll x^{2\sigma-1} \left( Y^{2-4\sigma} + Y^{4\sigma-2} + \log^4(Y + 6) \right).
\] (8.4)
By [IK, eq. (16.50)] we have
\[
\sum_{c \equiv 0 \mod q} c^{-1-\omega}|S(m, n; c)| \ll (\omega - 1/2)^{-2} \tau((m, n))(m, n, q)^{1/2} \tau(q) q^{-1/2-\omega} \quad (8.5)
\]
for $1/2 < \omega \leq 1$. Then from (8.4) and (8.5) we obtain
\[
\sum_{c \equiv 0 (\mod q)} c^{-1} S(h^2/d^2, h^2/k^2, c) H^* \left( \frac{4\pi h^2}{dk} \right) \ll \left( \frac{h^2}{dk} \right)^{2\sigma - 1} \left( Y^{4\sigma - 2} + Y^{2-4\sigma} + \log^4(Y + 6) \right) \\
\times (\sigma - 3/4)^{-2}\tau((h^2/d^2, h^2/k^2))(h^2/d^2, h^2/k^2, q)^{1/2}\tau(q)q^{1/2-2\sigma} \ll |h|^{2+\epsilon}q^{-1+\epsilon} \left( Y + Y^{-1} + \log^4(Y + 6) \right) \log^2(Y + 6),
\]
where for the last inequality we substituted
\[
\sigma = \frac{3}{4} + \frac{1}{2\log(Y + 6)}.
\]
By (3.11),
\[
\tau_a(h^2/\ell^2, t) \ll \epsilon |ht|^{\epsilon} e^{\frac{\pi |t|}{2}},
\]
hence we obtain
\[
\sum_a \int_{-\infty}^{\infty} \tau_a(h^2/d^2, t) \tau_a(h^2/k^2, t) H^*(t) \frac{dt}{4\pi} \ll \epsilon |h|^{\epsilon} \log^4(Y + 6).
\]
Combining the preceding estimates yields (8.3) for $(h, q) = 1$.

If $(h, q) > 1$, write $h = q^\alpha n$ for some integer $\alpha \geq 1$ and integer $n$ coprime to $q$. Then by the Hecke relations ([I, (8.39)]) $\lambda_g(q^\alpha n) = \lambda_g(q^\alpha)\lambda_g(n)$ and $\lambda_g(q) = \pm q^{-1/2}$ (see (3.3)), we have
\[
|\rho_g(h)|^4 = q^{-2\alpha}|\rho_g(n)|^4,
\]
and we reduce to the case already considered (with a sharper bound in $q$).

9. Contribution of the continuous spectrum

In this section we estimate the sum over the cusps $a$ in (5.4).

**Lemma 9.1.** For $X \gg |D|^{1/2}$ we have
\[
\sum_a \int_{-\infty}^{\infty} \tau_a(-h, t) \phi(t) W_{D,a}(t) \frac{dt}{4\pi} \ll \epsilon (q|D|X|h|)^{\epsilon} q^{-1/2}X^{1/2}|D|^{1/6}.
\]

**Proof.** Since $W_{D,a}(t)$ grows polynomially in $t$, by (3.11) and Lemma 7.1 we may impose the truncation $|t| \leq (q|D|Y|h|)^{\epsilon}$ with an error term which is $O((q|D|Y|h|)^{-1000})$ to obtain
\[
\int_{-\infty}^{\infty} \tau_a(-h, t) \phi(t) W_{D,a}(t) \frac{dt}{4\pi} \ll \epsilon (q|D|Y|h|)^{\epsilon} Y^{1/2}\log(Y + 6) \int_{|t| \leq (q|D|Y|h|)^{\epsilon}} |W_{D,a}(t)| dt
\]
for $Y \gg 1$. By (6.2), the convexity bound for $\zeta(1/2 + it)$, the Conrey-Iwaniec [CI] bound
\[
L(\chi_D, 1/2 + it) \ll \epsilon |D|^{1/6}(q|D|Y|h|)^{\epsilon} \text{ for } |t| \leq (q|D|Y|h|)^{\epsilon},
\]
and a standard lower bound for $|\zeta(1 + 2it)|$, we obtain $W_{D,a}(t) \ll \epsilon q^{-1/2}|D|^{5/12}(q|D|Y|h|)^{\epsilon}$. Thus
\[
\int_{-\infty}^{\infty} \tau_a(-h, t) \phi(t) W_{D,a}(t) \frac{dt}{4\pi} \ll \epsilon Y^{1/2}\log(Y + 6)q^{-1/2}|D|^{5/12}(q|D|Y|h|)^{\epsilon}.
\]
Upon substituting the bound $Y \ll X|D|^{-1/2}$, we complete the proof. \qed
10. Proof of Theorem 1.1

Theorem 1.1 follows by combining the spectral decomposition (5.4) with the estimates in Lemmas 7.2, 7.3 and 9.1.

11. Proof of Theorems 1.2 and 1.3

We proceed as in [DFI2, section 12] to prove the following proposition, which implies Theorems 1.2 and 1.3.

Proposition 11.1. Let $D$, $q$ and $f$ be as in Theorem 1.1. Let $\phi : \mathbb{R}_{\geq 0} \to \mathbb{C}$ be a $C^\infty$ function supported on a fixed subinterval $I \subset [0, 1]$ of length $\ell(I) > 0$ and

$$N_\phi(f, D, q) := \sum_{c \equiv 0 \pmod{q}} f(c) \sum_{b \equiv 0 \pmod{2c}, b^2 \equiv D \pmod{4c}} \phi \left( \frac{b}{2c} \right).$$

For $X \gg |D|^{1/2}$, we have

$$N_\phi(f, D, q) = \hat{f}(0)\hat{\phi}(0) \frac{12}{\pi^2} \frac{L(\chi_D, 1)}{q + 1} + O_\varepsilon ((|D|X)^\varepsilon \min\{A(X, D), B(X, D, q), C(X, D, q)\}),$$

where

$$\hat{f}(\xi) := \int_{-\infty}^{\infty} f(u)e(\xi u)du$$

(resp. $\hat{\phi}$) is the Fourier transform of $f$ (resp. $\phi$).

Proof. Integrating by parts $A$-times yields

$$\hat{\phi}(h) = \int_{-\infty}^{\infty} \phi(u)e(hu)du \ll |h|^{-A}.$$

Then by Poisson summation and Theorem 1.1, we have

$$N_\phi(f, D, q) = \sum_{h \in \mathbb{Z}} \hat{\phi}(h)W_h(f, D, q)$$

$$= \hat{\phi}(0) \sum_{c \equiv 0 \pmod{q}} f(c)\rho(c) + O_\varepsilon ((|D|X)^\varepsilon \min\{A(X, D), B(X, D, q), C(X, D, q)\}),$$

where

$$\rho(c) := \#\{b \equiv 0 \pmod{2c} : b^2 \equiv D \pmod{4c}\}.$$

Define the $L$-series

$$L(s) := \sum_{c \equiv 0 \pmod{q}} \rho(c)c^{-s}.$$

By Lemma 11.2 we have

$$L(s) = \frac{2}{1 + q^s} \frac{\zeta(s)L(\chi_D, s)}{\zeta(2s)},$$

which has a simple pole at $s = 1$ with residue

$$R := \frac{12}{\pi^2} \frac{L(\chi_D, 1)}{q + 1}.$$
By Mellin inversion,
\[ \sum_{c \equiv 0 \pmod{q}} f(c) \rho(c) = \frac{1}{2\pi i} \int_{(\sigma)} \tilde{f}(s) L(s) ds \quad \text{for} \quad \sigma > 1, \]
where
\[ \tilde{f}(s) := \int_{0}^{\infty} f(u) u^{s-1} du \]
is the Mellin transform of \( f \). Shifting the contour to \((1/2)\), we pick up the residue \( R \) to obtain
\[ \sum_{c \equiv 0 \pmod{q}} f(c) \rho(c) = \hat{f}(0) \frac{12}{\pi^2} \frac{L(\chi_D, 1)}{q + 1} + \frac{1}{2\pi i} \int_{(1/2)} \tilde{f}(s) \frac{\zeta(s) L(\chi_D, s)}{\zeta(s)} \frac{2}{1 + q^s} ds. \]
Integrating by parts \( A \)-times yields
\[ \tilde{f}(s) \ll X^{1/2}|s|^{-A} \quad \text{for} \quad \sigma = 1/2. \]
Then by the Conrey-Iwaniec \([CI]\) bound \( L(\chi_D, 1/2 + it) \ll_{\varepsilon} (1 + |t|)^B |D|^{1/6 + \varepsilon} \) we obtain
\[ \sum_{c \equiv 0 \pmod{q}} f(c) \rho(c) = \hat{f}(0) \frac{12}{\pi^2} \frac{L(\chi_D, 1)}{q + 1} + O_{\varepsilon}(q^{-1/2} X^{1/2} |D|^{1/6 + \varepsilon}). \]

The following lemma is a minor variant of the classical formula
\[ \sum_{c=1}^{\infty} \rho(c) c^{-s} = \frac{\zeta(s) L(\chi_D, s)}{\zeta(2s)} \]
due to Dirichlet (see e.g. \([Z, \text{Proposition 3, (i)}]\)).

**Lemma 11.2.** For \( D \) and \( q \) as in Theorem 1.1, we have
\[ L(s) = \frac{2}{1 + q^s} \frac{\zeta(s) L(\chi_D, s)}{\zeta(2s)}. \]

**Proof.** Observe that
\[ L(s) = \sum_{c \equiv 0 \pmod{q}} \rho(c) c^{-s} = L_q(s) \prod_{p \nmid q} L_p(s) \prod_{p | D} L_p(s), \]
where
\[ L_q(s) := \frac{\rho(q)}{q^s} + \frac{\rho(q^2)}{q^{2s}} + \cdots \]
and
\[ L_p(s) := 1 + \frac{\rho(p)}{p^s} + \frac{\rho(p^2)}{p^{2s}} + \cdots . \]
A calculation yields
\[ L_q(s) = q^{-s}(1 + \chi_D(q))(1 - q^{-s})^{-1} \]
and
\[ L_p(s) = \begin{cases} (1 - p^{-s})^{-1}(1 + \chi_D(p)p^{-s}), & \text{if } p \nmid qD \\ (1 - p^{-s})^{-1}(1 - p^{-2s}), & \text{if } p | D. \end{cases} \]

The result now follows. 

\[ \square \]

12. Proof of Theorem 1.4

We proceed as in [DFI2, section 12] to prove the following proposition, which implies Theorem 1.4.

**Proposition 12.1.** Let \( D, q \) and \( f \) be as in Theorem 1.1. Let \( I_D \subset [0,1] \) be a subinterval of length \( \ell(I_D) = |D|^{-\eta} \) for some \( \eta > 0 \) and \( \phi : \mathbb{R}_{\geq 0} \to \mathbb{C} \) be a \( C^\infty \) function supported on \( I_D \) which satisfies
\[ \phi^{(j)} \ll |D|^{\eta j}, \quad j = 0, 1, \ldots \]

For \( X \gg |D|^{1/2} \), we have
\[ N_\phi(f, D, q) = \hat{f}(0)\hat{\phi}(0)\frac{12 L(\chi_D, 1)}{\pi^2} \frac{1}{q + 1} \times \]
\[ + O_\varepsilon \left( |D|^{2\eta}(q|D|X)^\varepsilon \min \{ A(X, D), B(X, D, q), C(X, D, q) \} \right). \]

**Proof.** Integrating by parts \( A \)-times yields
\[ \hat{\phi}(h) \ll \left( \frac{|D|^{\eta}}{|h|} \right)^A. \]

Then proceeding as in the proof of Proposition 11.1, we have
\[ N_\phi(f, D, q) = \sum_{h \in \mathbb{Z}} \hat{\phi}(h)W_h(f, D, q) \]
\[ = \hat{\phi}(0) \sum_{c \equiv 0 (\text{mod } q)} f(c)\rho(c) + \sum_{|h| \leq |D|^{\eta + \varepsilon}} \hat{\phi}(h)W_h(f, D, q) + O(|D|^{-1000}) \]
\[ = \hat{f}(0)\hat{\phi}(0)\frac{12 L(\chi_D, 1)}{\pi^2} \frac{1}{q + 1} \times \]
\[ + O_\varepsilon \left( |D|^{2\eta}(q|D|X)^\varepsilon \min \{ A(X, D), B(X, D, q), C(X, D, q) \} \right). \]
\[ \square \]

13. Proof of Theorem 1.5

We proceed as in [DFI2, section 14] to prove Theorem 1.5. Let \( \{ \omega_\ell \}_{\ell=0}^\infty \) be a smooth partition of unity such that each constituent \( \omega_\ell \) is supported on \( [Y_\ell, 2Y_\ell] \) with \( Y_\ell := 2^{\ell/2}\sqrt{|D|} \) and satisfies
\[ \omega_\ell^{(j)}(y) \ll y^{-j}, \quad j = 0, 1, \ldots \]

Then the function
\[ f_\ell(y) := \sinh \left( \frac{\pi h\sqrt{|D|}}{y} \right) \omega_\ell(y) \]

satisfies
\[ f^{(j)}_{\ell}(y) \ll \frac{\sqrt{|D|}}{y^{j}}, \quad j = 0, 1, \ldots \]

It follows from Theorem 1.1 with \( f = f_{\ell} \) and \( X = Y_{\ell} \) that
\[
\sum_{\substack{c \geq \sqrt{|D|} \\ c \equiv 0 \pmod{q}}} W_h(D; c) \sinh \left( \frac{\pi h \sqrt{|D|}}{c} \right) \ll \sum_{\ell = 0}^{\infty} |W_h(f_{\ell}, D, q)| \ll \sum_{\ell = 0}^{\infty} \frac{\sqrt{|D|} Y_{\ell}^{3/4} |D|^{1/16} (q|D| Y_{\ell})^\varepsilon}{Y_{\ell}}
\]
\[
\ll \varepsilon |D|^{7/16 + \varepsilon}.
\]

References


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