1. Introduction

Let $F$ be a totally real number field of degree $n$ over $\mathbb{Q}$ with ring of integers $\mathcal{O}$ and $\sigma_1, \sigma_2, \ldots, \sigma_n$ be all the real embeddings of $F$. Let $\Gamma = SL(2, \mathcal{O})$ be the Hilbert modular group which acts discontinuously on the product of $n$ upper half planes $\mathbb{H}^n$ in the following way:

For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, and $z = (z_1, \cdots, z_n) \in \mathbb{H}^n$, we define $\gamma z = (\gamma_1 z_1, \cdots, \gamma_n z_n)$ where

$$\gamma_i = \begin{pmatrix} \sigma_i(a) & \sigma_i(b) \\ \sigma_i(c) & \sigma_i(d) \end{pmatrix}, \quad \gamma_i z_i = \frac{\sigma_i(a)z_i + \sigma_i(b)}{\sigma_i(c)z_i + \sigma_i(d)} \quad (1 \leq i \leq n).$$

Remark. We may also identify $\Gamma$ with its image in $SL(2, \mathbb{R})^n$ via $\gamma \in \Gamma$, $\gamma = (\gamma_1, \cdots, \gamma_n) \in SL(2, \mathbb{R})^n$.

It is well known that $\Gamma$ has finite co-volume (see [Fr]), i.e.

$$\text{vol}(\Gamma \backslash \mathbb{H}^n) = \int_{\Gamma \backslash \mathbb{H}^n} \frac{dx dy}{(N y)^2} < \infty,$$

where $z = (x_1 + iy_1, \cdots, x_n + iy_n) \in \mathbb{H}^n$, $dx = dx_1 \cdots dx_n$, $dy = dy_1 \cdots dy_n$, and $N y = y_1 \cdots y_n$.

Denote by $S_{2k}(\Gamma)$ ($k \in \mathbb{N}, k \geq 2$) the space of Hilbert modular cusp forms of weight $(2k, \cdots, 2k)$, i.e. the space of holomorphic functions $f(z)$ on $\mathbb{H}^n$ such that

$$(1) \text{ for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \quad f(\gamma z) = N(cz+d)^{2k}f(z),$$

where for $z = (z_1, \cdots, z_n) \in \mathbb{H}^n$.
\[\mathbb{H}^n,\]
\[N(cz + d) = \prod_{i=1}^{n}(\sigma_i(c)z_i + \sigma_i(d)).\]

(2) \(f(z)\) vanishes at all the cusps of \(\Gamma\). (see [Ga] or [Fr])

Let
\[d\mu = \frac{1}{vol(\Gamma\backslash\mathbb{H}^n)} \frac{dxdy}{(Ny)^2}.\]

For \(f\) and \(g\) in \(S_{2k}(\Gamma)\), we define the (normalized) Petersson inner product by
\[\langle f, g \rangle = \int_{\Gamma\backslash\mathbb{H}^n} f(z)\overline{g(z)}(Ny)^{2k}d\mu.\]

It is well known that \(S_{2k}(\Gamma)\) is a finite dimensional Hilbert space. Furthermore, if we let \(J_k = \text{dim}_\mathbb{C}S_{2k}(\Gamma)\), then it was shown by Shimizu [Sh] (using the Selberg trace formula) that
\[J_k = \frac{vol(\Gamma\backslash\mathbb{H}^n)}{(4\pi)^n}(2k - 1)^n + O(1)\]

as \(k \rightarrow \infty\).

One expects the following mass equidistribution conjecture on the Hilbert modular variety \(\Gamma\backslash\mathbb{H}^n\) should be true.

\[\lim_{k \rightarrow \infty} \max_{1 \leq i \leq J_k} \left| \int_{A} (Ny)^{2k}|f_{i,k}(z)|^2d\mu - \int_{A} d\mu \right| = 0\]

where \(A \subset \Gamma\backslash\mathbb{H}^n\) is compact and \(\{f_{i,k}\}_{i=1}^{J_k}\) is the orthonormal Hecke basis of \(S_{2k}(\Gamma)\). For \(n = 1\) (i.e. \(\Gamma = \Gamma(1)\)), this is an analogue of quantum unique ergodicity conjecture, formulated by Rudnick and Sarnak [RS].

This conjecture is still out of reach at the present. However, Luo [Lu] established this conjecture on the average and Lau [La] generalized Luo’s result to the arithmetic surface \(\Gamma_0(N)\backslash\mathbb{H}\). The purpose of this paper is to generalize Luo’s and Lau’s results to the Hilbert modular varieties (Theorem 1 and Corollary 2).
Let \( \{ f_{i,k} \}_{i=1}^{J_k} \) be an orthonormal basis of \( S_{2k}(\Gamma) \). Set
\[
d\mu_k = \frac{1}{J_k} \left( \sum_{i=1}^{J_k} |f_{i,k}(z)|^2 \right) (Ny)^{2k} d\mu.
\]

**Theorem 1.** For any compact subset \( A \subset \Gamma \setminus \mathbb{H}^n \) and any \( 0 < \epsilon < 1 \), we have
\[
\int_A d\mu_k = \int_A d\mu + O_{\epsilon,A} \left( (k^{-1+\epsilon})^n \right)
\]
as \( k \to \infty \).

**Remark 1.** The key ingredients in [Lu] and [La] are the Bergman kernel for the Hecke operator and the Petersson trace formula respectively. Our approach is using the Bergman kernel on \( \Gamma \setminus \mathbb{H}^n \).

**Remark 2.** [Lu] proved a uniform result for all measurable subsets \( A \). In our Theorem 1, the result depends on the compact subset \( A \). But our decay rate is sharper than in [Lu].

**Some properties of \( \Gamma \).** We say that an element \( \gamma (\neq \text{identity}) \) of \( \Gamma \) is **elliptic** (resp: **parabolic** and **hyperbolic**) if all the \( \gamma_i \) are elliptic (resp: parabolic and hyperbolic) in the usual sense (see [Iw]). If \( \gamma (\neq \text{identity}) \) is not of above types, we say that \( \gamma \) is **mixed**. A point \( z \) in \( \mathbb{H}^n \) is called an **elliptic point** if it is fixed by an elliptic element in \( \Gamma \). A point \( \kappa \) in \( \mathbb{R}^n \) (where \( \mathbb{R} = \mathbb{R} \cup \{ \infty \} \)) is called a **cusp** if it is fixed by a parabolic element in \( \Gamma \).

**Proposition 1.** ([Sh] Theorem 6) The number of the \( \Gamma \)-inequivalent elliptic points of \( \Gamma \) is finite.

**Proposition 2.** ([Sh] Lemma 15) Let \( e_1, \cdots, e_s \in \mathbb{H}^n \) be a complete representatives of \( \Gamma \)-inequivalent elliptic points of \( \Gamma \). Then the union of \( \Gamma_{e_i} \setminus \{1\} \) \((1 \leq i \leq s)\) forms a complete representatives of non-conjugate elliptic elements in \( \Gamma \), where \( \Gamma_{e_i} = \{ \gamma \in \Gamma | \gamma e_i = e_i \} \) \((1 \leq i \leq s)\).
Since $\Gamma_{e_i}$ is a discrete subgroup of a compact subgroup, $\Gamma_{e_i}$ is a finite subgroup. Hence we have

**Lemma 1.** There are only finitely many elliptic conjugacy classes in $\Gamma$.

2. Bergman kernel

For $k \in \mathbb{N}$, $k \geq 2$ and $z = (z_1, \cdots, z_n), \ w = (w_1, \cdots, w_n) \in \mathbb{H}^n$, we define the Bergman kernel by

$$B_k(z, w) = \sum_{\gamma \in \Gamma} N(\gamma z - \bar{w})^{-2k} j(\gamma, z)^{-2k}$$

where $N(\gamma z - \bar{w}) = \prod_{i=1}^{n}(\sigma_i(\gamma)z_i - \bar{w}_i)$ and $j(\gamma, z) = N(cz+d), \ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

**Proposition 3.** (1) $B_k(z, w)$ converges absolutely and uniformly for $(z, w)$ in compact subsets of $\mathbb{H}^n \times \mathbb{H}^n$.

(2) For each fixed $w \in \mathbb{H}^n$, $B_k(z, w) \in S_{2k}(\Gamma)$ (as a function of $z$).

*Proof.* The proof can be found in [Ga, 1.14] or [Fr, Chapter II].

**Proposition 4.** If $f \in S_{2k}(\Gamma)$, then

$$f(w) = \left(\frac{2k-1}{4\pi}\right)^n \frac{(2i)^{2kn}}{2} \int_{\Gamma \setminus \mathbb{H}^n} f(z) B_k(z, w)(Ny)^{2k} \frac{dxdy}{(Ny)^2}$$

$$= \left(\frac{2k-1}{4\pi}\right)^n \frac{(2i)^{2kn}}{2} \text{vol}(\Gamma \setminus \mathbb{H}^n) \langle f, B_k(\cdot, w) \rangle$$

where $z = (x_1 + iy_1, \cdots, x_n + iy_n) \in \mathbb{H}^n, \ w \in \mathbb{H}^n$.

*Proof.* See [Ga, 1.14] or [Fr, Chapter II].

For convenience, denote by

$$C_k^{-1} = \left(\frac{2k-1}{4\pi}\right)^n \frac{(2i)^{2kn}}{2} \text{vol}(\Gamma \setminus \mathbb{H}^n)$$

(2.1)
and note that $C_k = \overline{C_k}$ when $k \geq 2$.

For $k \in \mathbb{N}$, $\gamma \in \Gamma$ and $z = (z_1, \ldots, z_n) \in \mathbb{H}^n$, let

$$h(\gamma, z) = N(z - \overline{z})^2 N(\gamma z - \overline{z})^{-2} j(\gamma, z)^{-2}$$

and

$$h_k(\gamma, z) = (h(\gamma, z))^k = N(z - \overline{z})^{2k} N(\gamma z - \overline{z})^{-2k} j(\gamma, z)^{-2k}.$$

**Lemma 2.** $|h_k(\gamma, z)| \leq 1$ for all $z \in \mathbb{H}^n$, and $\gamma \in \Gamma$. Moreover, $|h_k(\gamma, z)| = 1$ if and only if $\gamma = \pm 1$ or $\gamma$ is elliptic and $z$ is its fixed point.

**Proof.** It suffices to prove that when $n = 1$. By definition,

$$|h_k(\gamma, z)| = \left| \frac{z - \overline{z}}{\gamma z - \overline{z}} \cdot \frac{1}{cz + d} \right|^{2k}$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. Let $\gamma z = z' = x' + iy'$ and $z = x + iy$. Then

$$\left| \frac{z - \overline{z}}{\gamma z - \overline{z}} \cdot \frac{1}{cz + d} \right| = \frac{y^{1/2}}{\sqrt{z' + (y+y')^2}} \left( \frac{y}{cz + d} \right)^{1/2} = \frac{y^{1/2} (y')^{1/2}}{\sqrt{y+y'}^2} \leq \frac{y^{1/2} (y')^{1/2}}{y+y'} \leq 1$$

The equality holds if and only if $x = x'$ and $y = y'$, i.e. $\gamma z = z$. Hence the equality holds if and only if $\gamma = \pm 1$ or $\gamma$ is elliptic and $z$ is its fixed point.

**Lemma 3.** For each fixed $k \geq 2$, $\sum_{\gamma \in \Gamma} h_k(\gamma, z)$ converges absolutely and uniformly on any compact subset of $\mathbb{H}^n$.

**Proof.** Note that

$$\sum_{\gamma \in \Gamma} h_k(\gamma, z) = N(z - \overline{z})^{2k} B_k(z, z)$$
and then the result follows from Prop 3.

\begin{align*}
\textbf{Lemma 4.} & \text{ For any } M \in \Gamma, \text{ we have } \\
& h_k(M^{-1}\gamma M, z) = h_k(\gamma, Mz).
\end{align*}

Proof. By a simple computation or see [Fr].

3. Proof of Theorem 1

Before we prove the theorem, we make the following observation. Since $B_k(z, w)$ is a cusp form in $z$ (by Prop 3), we have

\begin{align*}
B_k(z, w) &= \sum_{i=1}^{J_k} \langle B_k(\cdot, w), f_{i,k} \rangle f_{i,k}(z) \\
&= C_k \sum_{i=1}^{J_k} f_{i,k}(w)f_{i,k}(z) \quad \text{(by Prop 4)}.
\end{align*}

Let $w = z$, then we obtain the identity

\begin{equation}
B_k(z, z) = C_k \sum_{i=1}^{J_k} |f_{i,k}(z)|^2,
\end{equation}

where $C_k$ is defined in (2.1).

\textit{Proof of Theorem 1.} Let $\chi_A(z)$ denote the characteristic function of $A$ on $\Gamma \backslash \mathbb{H}^n$. One can extend it (with the same notation) to $\mathbb{H}^n$ as a $\Gamma$-invariant function.
By (3.1) and (2.2),

$$\int_A d\mu_k = \frac{1}{J_k C_k} \int_A B_k(z, z)(Ny)^{2k} d\mu$$

$$= \frac{1}{(2i)^{2kn} J_k C_k} \int_{\Gamma \backslash \mathbb{H}^n} \chi_A(z) \sum_{\gamma \in \Gamma} h_k(\gamma, z) d\mu$$

$$= \frac{1}{(2i)^{2kn} J_k C_k} \left[ \sum_{\gamma \in \Gamma, \gamma \text{ is \ elliptic}} \int_{\Gamma \backslash \mathbb{H}^n} \chi_A(z) h_k(\gamma, z) d\mu + \sum_{\gamma \in \Gamma, \gamma \text{ is \ not \ elliptic}} \int_{\Gamma \backslash \mathbb{H}^n} \chi_A(z) \left( \sum_{\gamma \in \Gamma, \gamma \neq \pm 1} h_k(\gamma, z) \right) d\mu \right] .$$

We estimate the above three summation of integrals in the following cases.

**Case 1.** $\gamma = \pm 1$.

$$\int_{\Gamma \backslash \mathbb{H}^n} \chi_A(z) h_k(\gamma, z) d\mu = \int_{\Gamma \backslash \mathbb{H}^n} \chi_A(z) d\mu = \mu(A).$$

**Case 2.** For $\gamma \in \Gamma$ elliptic, let

$$\Gamma_\gamma = \{ M \in \Gamma : M\gamma = \gamma M \} \quad (the \ centralizer \ of \ \gamma \ in \ \Gamma)$$

and

$$[\gamma] = \{ M^{-1} \gamma M : M \in \Gamma \} .$$

Also let $\Lambda$ be a set of complete representatives of elliptic conjugate classes in $\Gamma$.

**Remark.** $|\Lambda| < \infty$ by Lemma 1.

$$\sum_{\gamma \in \Gamma, \gamma \text{ is \ elliptic}} \int_{\Gamma \backslash \mathbb{H}^n} \chi_A(z) h_k(\gamma, z) d\mu = \sum_{\gamma \in \Lambda} \sum_{\gamma' \in [\gamma]} \int_{\Gamma \backslash \mathbb{H}^n} \chi_A(z) h_k(\gamma', z) d\mu$$

$$= \sum_{\gamma \in \Lambda} \sum_{M \in \Gamma, \gamma' \in \Gamma} \int_{\Gamma \backslash \mathbb{H}^n} \chi_A(z) h_k(M^{-1} \gamma M, z) d\mu$$
Using Lemma 4 and unfolding, we have

\[
\sum_{M \in \Gamma} \int_{\Gamma \setminus \mathbb{H}^n} \chi_A(z)h_k(M^{-1}\gamma M, z) d\mu = \int_{\Gamma \setminus \mathbb{H}^n} \chi_A(z)h_k(\gamma, z) d\mu
\]

\[
= \frac{1}{|\Gamma|} \int_{\mathbb{H}^n} \chi_A(z)h_k(\gamma, z) d\mu
\]

\[
= \frac{1}{|\Gamma|} \int_{\mathbb{H}^n} \chi_A(z) \prod_{i=1}^n h_{k,i}(\gamma_i, z_i) d\mu
\]

\[
\leq \frac{1}{|\Gamma| \text{vol}(\Gamma \setminus \mathbb{H}^n)} \prod_{i=1}^n \int_{\mathbb{H}} h_{k,i}(\gamma_i, z_i) \frac{dx_i dy_i}{y_i^2}
\]

where

\[
h_{k,i}(\gamma_i, z_i) = (z_i - \overline{z}_i)^{2k}(\gamma_i z_i - \overline{z}_i)^{-2k} j(\gamma_i, z_i)^{-2k}.
\]

**Remark.** \(h_{k,i}(M^{-1}\gamma_i M, z_i) = h_{k,i}(\gamma_i, M z_i)\) for any \(M \in SL(2, \mathbb{R})\).

Hence we may assume that each \(\gamma_i\) is of the form

\[
\begin{pmatrix}
\cos \theta_i & \sin \theta_i \\
-\sin \theta_i & \cos \theta_i
\end{pmatrix}
\]

\(\theta_i \neq 0, \pi\).

For convenience, we drop the subscripts \(i\) in \(\gamma_i, z_i, \theta_i\) and etc \(\cdots\).

Now we make change of variables by using the Cayley transform

\[
\mathbb{H} \longrightarrow D (\text{unit disc})
\]

\[
z \longmapsto w = \frac{z-i}{z+i}
\]

and then use the polar coordinates \(w = \rho e^{i\varphi}\) of the unit disc. It yields

\[
\int_{\mathbb{H}} |h_{k,i}(\gamma, z)| dxdy = 4 \int_0^{2\pi} \int_0^1 \frac{(1 - \rho^2)^{2k-2}}{|1 - e^{i\beta} \rho^{2k}|} \rho d\rho d\varphi
\]

\[
= 4\pi \int_0^1 \frac{(1 - t)^{2k-2}}{|1 - e^{i\beta} t^{2k}|} dt
\]

(where \(\beta = 2\theta \neq 0, 2\pi\))
• When $0 \leq t \leq k^{-1+\epsilon}$, $(0 < \epsilon < 1)$. It is easy to see that
\[
\frac{1 - t}{|1 - e^{i\beta t}|} \leq 1.
\]
Hence
\[
\int_0^{k^{-1+\epsilon}} \frac{(1 - t)^{2k-2}}{|1 - e^{i\beta t}|^{2k}} dt = \int_0^{k^{-1+\epsilon}} \left( \frac{1 - t}{|1 - e^{i\beta t}|} \right)^{2k-1} \frac{1}{|1 - e^{i\beta t}|^2} dt \leq \int_0^{k^{-1+\epsilon}} \frac{1}{|1 - e^{i\beta t}|^2} dt \ll k^{-1+\epsilon}.
\]

• When $k^{-1+\epsilon} \leq t \leq 1$. We have $(1 - t)^2 < \frac{1}{4}$ for $k$ sufficient large and then
\[
\frac{2t}{(1 - t)^2} \geq \frac{2k^{-1+\epsilon}}{4} = \frac{1}{2} k^{-1+\epsilon}.
\]
So
\[
\frac{1 - t}{|1 - e^{i\beta t}|} = \frac{1}{|1 + \frac{2t}{(1 - t)^2} (1 - \cos \beta)|^{1/2}} \ll (1 + k^{-1+\epsilon})^{-1/2}.
\]
Hence
\[
\int_{k^{-1+\epsilon}}^1 \frac{(1 - t)^{2k-2}}{|1 - e^{i\beta t}|^{2k}} dt \ll [(1 + k^{-1+\epsilon})^{-1/2}]^{2k-2} = (1 + k^{-1+\epsilon})^{-k+1}
\]
Combining these estimates, we get
\[
\sum_{\gamma \in \Gamma, \gamma \text{ is elliptic}} \int_{\Gamma \backslash \mathbb{H}^n} \chi_A(z) h_k(\gamma, z) d\mu \ll (k^{-1+\epsilon})^n.
\]
Note that here the implicit constant only depends on $\epsilon$.

**Case 3.** Let $\Gamma' = \Gamma \setminus \{\pm 1\} \cup \{ \gamma \in \Gamma : \gamma \text{ is elliptic} \}$.

Since $\sum_{\gamma \in \Gamma'} |h_3(\gamma, z)|$ converges uniformly on $A$ (by Lemma 3) and $|h_3(\gamma, z)| < 1$ for all $z \in A, \gamma \in \Gamma'$ (by Lemma 2), there exists a constant $0 < \lambda < 1$ (depends on $A$) such that $|h_3(\gamma, z)| < \lambda$ for all $z \in A, \gamma \in \Gamma'$. Hence
\[
\int_{\Gamma \backslash \mathbb{H}^n} \chi_A(z) \left( \sum_{\gamma \in \Gamma'} h_k(\gamma, z) \right) d\mu \leq \int_A \sum_{\gamma \in \Gamma'} |h_3(\gamma, z)||h_3(\gamma, z)|^k d\mu \leq \int_A \sum_{\gamma \in \Gamma'} |h_3(\gamma, z)|^2 d\mu \ll (\lambda_1)^k
\]
where $\lambda_1 = (\lambda)^3 < 1$. 

From case 1, 2, 3 and using Shimizu’s asymptotic formula (1.1) for \( J_k \), Theorem 1 follows directly.

4. Some Remarks

Let \( \Gamma \) be a discrete subgroup of \( SL(2, \mathbb{R})^n \) with finite co-volume which satisfies the irreducibility condition below and Assumption(F) on its fundamental domain.

**Irreducibility condition:** The restriction of each of the \( n \) projections

\[
p_j : SL(2, \mathbb{R})^n \longrightarrow SL(2, \mathbb{R}) \quad (1 \leq j \leq n)
\]

to \( \Gamma \) is injective.

**Assumption(F):** Let \( \kappa_v \ (1 \leq v \leq t) \) be a set of complete representatives of \( \Gamma \)-inequivalent cusp of \( \Gamma \). For each \( v \), take a \( g_v \in SL(2, \mathbb{R})^n \) such that \( g_v\kappa_v = \infty \) and put

\[
U_v = \left\{ g_v^{-1}z : \prod_{i=1}^{n} Im(z_i) > d_v, z = (z_1, \cdots, z_n) \right\}
\]

where \( d_v \) is a suitably chosen positive number. Let \( \Gamma_{\kappa_v} = \{ \gamma \in \Gamma : \gamma \kappa_v = \kappa_v \} \) and let \( V_v \) be a fundamental domain of \( \Gamma_{\kappa_v} \) in \( U_v \). Then \( \Gamma \) has a fundamental domain \( F \) of the form

\[
F = F_0 \cup V_1 \cup \cdots \cup V_t
\]

where \( F_0 \) is relatively compact in \( \mathbb{H}^n \).

In this case, Shimizu’s dimension formula (1.1) also holds for \( \Gamma([Sh]) \). Moreover, our propositions, lemmas and theorem in previous sections all remain true for \( \Gamma \). In particular, for a non-zero ideal \( n \) of \( \mathcal{O} \), let

\[
\Gamma_0(n) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathcal{O}) : c \equiv 0 \ modulo \ n \right\}.
\]
Then $\Gamma = \Gamma_0(n)$ satisfies the irreducible condition and Assumption(F). Hence we have the following corollary:

**Corollary 2.** For any compact subset $A \subset \Gamma_0(n) \backslash \mathbb{H}^n$ and any $0 < \epsilon < 1$, we have
\[
\int_A d\mu_k = \int_A d\mu + O_{\epsilon, A} ((k^{-1+\epsilon})^n)
\]
as $k \to \infty$.

**Remark.** Again the decay rate here is sharper than in [La], but the implicit constant depends on the compact subset $A$. In [La], the result is uniform.

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**References**


