Math 171: First Semester Calculus

Fall Semester, 2003

First Exam

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Name: ___________________________ Section: ______

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Instructions: Work must be shown to receive credit. No calculators.

Question 1. **(5 points each)** Compute the following limits. If the limit is $\infty$ or $-\infty$, say so. If the limit otherwise does not exist, write DNE for your answer.

A. \[ \lim_{x \to 1} \frac{5x^2 - 7}{6x^2 + 2x} \]

\[ \lim_{x \to 1} \frac{5x^2 - 7}{6x^2 + 2x} = \frac{-2}{8} = -\frac{1}{4} \]

By direct substitution.

B. \[ \lim_{x \to \infty} \frac{5x^2 - 7}{6x^2 + 2x} \]

\[ \lim_{x \to \infty} \frac{5x^2 - 7}{6x^2 + 2x} = \lim_{x \to \infty} \frac{(5x^2 - 7) \frac{1}{x^2}}{6x^2 + 2x} = \lim_{x \to \infty} \frac{5 - \frac{7}{x^2}}{6 + \frac{2}{x}} = \frac{5}{6} \]

C. \[ \lim_{x \to 1^+} \frac{x^2 + 2x - 3}{x - 1} \]

\[ \lim_{x \to 1^+} \frac{x^2 + 2x - 3}{x - 1} = \lim_{x \to 1^+} \frac{(x-1)(x+3)}{x - 1} = \lim_{x \to 1^+} (x+3) = 4 \]

D. \[ \lim_{x \to 1^-} \frac{x^2 + 2x + 3}{x - 1} \]

This limit is $-\infty$ since the numerator approaches 6 and the denominator is small and negative as $x$ approaches 1 from the left.
Question 2.  **(2 points each)** The function \( y = f(x) \) is graphed below. An arrowhead near a dashed line indicates asymptotic behavior. If any requested limit is \( \infty \) or \( -\infty \), say so. If the limit otherwise does not exist, write "DNE" for 'does not exist.'

A. \( \lim_{x \to 4} f(x) = -1 \)

B. \( \lim_{x \to 3^{-}} f(x) = 1 \)

C. \( \lim_{x \to 3^{+}} f(x) = 3 \)

D. \( \lim_{x \to \infty} f(x) = 2 \)

E. \( f(4) = 1 \)
Question 3.  **(5 points for part A; 10 points for part B)**

A. In the box, state the definition of the derivative $f'(x)$:

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{s \to x} \frac{f(s) - f(x)}{s - x}$$

B. Using the definition, prove that if $f(x) = \sqrt{x}$, then $f'(x) = \frac{1}{2\sqrt{x}}$.

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x + h} - \sqrt{x}}{h}$$

$$= \lim_{h \to 0} \frac{(\sqrt{x + h} - \sqrt{x})(\sqrt{x + h} + \sqrt{x})}{h(\sqrt{x + h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{x + h - x}{h(\sqrt{x + h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{h}{h(\sqrt{x + h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x + h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$
Question 4. **(5 points each)** A ball is thrown directly upward from the surface of Planet Stewart. Its height in meters after \( t \) seconds is given by

\[
h(t) = 10t - t^2.
\]

A. Compute the average velocity of the ball in the time interval \( 2 \leq t \leq 4 \) seconds.

\[
\text{Average Velocity} = \frac{\Delta h}{\Delta t} = \frac{h(4) - h(2)}{2} = \frac{40 - 16 - (20 - 4)}{2} = \frac{8}{2} = 4 \text{ m/s}
\]

B. Write down a limit that represents the instantaneous velocity of the ball at \( t = 2 \) seconds.

\[
\text{Inst. Velocity} = \lim_{s \to 0} \frac{h(2 + s) - h(2)}{s}
\]

C. Compute the instantaneous velocity of the ball at \( t = 2 \) seconds either by computing the limit from part B or by any other legitimate method.

\[
\text{Inst. Velocity} = h'(2), \quad h'(t) = 10 - 2t, \quad \text{so Inst. Vel} = 6 \text{ m/s}
\]

Question 5. **(5 points)** State what it means for the function \( f(x) \) to be continuous at the point \( x = a \). (Give the definition.)

The function \( f(x) \) is continuous at \( x = a \) if \( \lim_{x \to a} f(x) = f(a) \) provided both limits exist.
Question 6. **(5 points each)** Find the derivatives of the following functions.

A. \( f(x) = 5x^4 - 7x^2 + 3x + 24 \)

\[ f'(x) = 20x^3 - 14x + 3 \]

B. \( g(t) = e^t \sin t \)

*NOT TESTED (Uses the product rule and exponential functions)*

C. \( h(x) = \frac{2x^2}{x^2 + 1} \)

*NOT TESTED (Uses the quotient rule)*

D. \( F(w) = \sqrt[3]{w^2} \)

\[ F(w) = w^{2/3}; F'(w) = \frac{2}{3} w^{(2/3)-1} = \frac{2}{3} w^{-1/3} = \frac{2}{3\sqrt[3]{w}}. \]
Question 7. **(6 points)** The graph of a differentiable function $y = f(x)$ is shown below. State the intervals on which $f'(x) > 0$ and the intervals on which $f'(x) < 0$.

Intervals where $f'(x) > 0$: The open interval $(3,6)$.

(The interval **must** be open for full credit.)

Intervals where $f'(x) < 0$: The open intervals $(1,3)$ and $(6,7)$.

Question 8. **(9 points)** Show that $\frac{d}{dx} (\tan x) = \sec^2 x$. Hint: $\tan x = \frac{\sin x}{\cos x}$

**NOT TESTED** (Uses the quotient rule)