§Appendix C  Sigma Notation

Let \( a_1, a_2, \ldots, a_n, \ldots \) be a sequence of numbers.

- \( a_2 \) is the second number in the sequence
- \( a_i \) is the \( i \)th number in the sequence

define

\[
\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n
\]

where \( i \) is the index of summation

1 is the starting value of \( i \) in this example

\( n \) is the final value of \( i \) in this example

Example. Let \( a_i = i \)

\[
\sum_{i=1}^{4} i = 1 + 2 + 3 + 4 = 10
\]

Example. Let \( a_i = i^2 \)

\[
\sum_{i=1}^{4} i^2 = 1^2 + 2^2 + 3^2 + 4^2
\]

\[= 1 + 4 + 9 + 16 = 30\]

Four Basic Properties of Sums

Let \( a_1, a_2, \ldots \) and \( b_1, b_2, \ldots \) be sequences of numbers and let \( c \) be a constant.

1. \( \sum_{i=1}^{n} c = nc \)
2. \( \sum_{i=1}^{n} c a_i = c \sum_{i=1}^{n} a_i \)
3. \( \sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i \)
4. \( \sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i \)

Why?

1. \( \sum_{i=1}^{3} c = c + c + c = 3c \)
2. \( \sum_{i=1}^{3} c a_i = c a_1 + c a_2 + c a_3 = c (a_1 + a_2 + a_3) = c \sum_{i=1}^{n} a_i \)
3. \( \sum_{i=1}^{n} (a_i + b_i) = (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) = (a_1 + a_2 + a_3) + (b_1 + b_2 + b_3) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i \)
4. similar to 3.
Two Sums of Powers

A. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

B. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

Proof of (A)

\[
\begin{align*}
1 & + 2 + \ldots + (n-1) + n \\
n & + (n-1) + \ldots + 2 + 1 \\
(n+1) & + (n+1) + \cdots + (n+1) + (n+1) \\
& = n(n+1)
\end{align*}
\]

This is equal to $2 \sum_{i=1}^{n} i$, so we must divide by 2 to get the final result.

Proof of (B)

recall $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

Consider the following telescoping sum\[
\begin{align*}
\sum_{i=1}^{n} [(1 + i)^3 - i^3] & = (2^3 - 1^3) + (3^3 - 2^3) + \ldots + (n^3 - (n-1)^3) + ((n+1)^3 - n^3) \\
& = (n+1)^3 - 1 \\
& = n^3 + 3n^2 + 3n \quad [1]
\end{align*}
\]
on the other hand\[
\begin{align*}
\sum_{i=1}^{n} [(1 + i)^3 - i^3] & = \sum_{i=1}^{n} (1 + 3i + 3i^2) \\
& = \sum_{i=1}^{n} 1 + \sum_{i=1}^{n} 3i + \sum_{i=1}^{n} 3i^2 \\
& = \sum_{i=1}^{n} 1 + 3 \sum_{i=1}^{n} i + 3 \sum_{i=1}^{n} i^2 \\
& = n + 3 \frac{n(n+1)}{2} + 3S \quad [2]
\end{align*}
\]
where $S$ is the sum we seek. Equating [1] with [2]

\[
\begin{align*}
n + \frac{3}{2} (n^2 + n) + 3S & = n^3 + 3n^2 + 3n \\
3S & = n^3 + \frac{3}{2} n^2 + \frac{1}{2} n \\
S & = \frac{2n^3 + 3n^2 + n}{6} = \frac{n(2n^2 + 3n + 1)}{6} = \frac{n(n+1)(2n+1)}{6}
\end{align*}
\]

\[\blacksquare\]
§5.1 Areas and Distances

The Area Problem

Find the area of the region $S$ lying under the graph of $y = f(x)$ on the interval from $x = a$ to $x = b$.

$$\langle \ldots a \ldots b \ldots x^- \ldots, \ldots, \text{curve, } f, S \rangle$$

Approx. the area under $f$ by a sum of rectangles.

$$\langle a \ldots b \ldots, \ldots, f, \text{four rectangles, } x_1^*, x_2^*, x_3^*, x_4^* \rangle$$

It's easy to compute the area of a rectangle

$$\langle \text{rectangle, width } \Delta x \text{ of subinterval, height } f(x_i^*), x_i^* \rangle$$

sample point for $i^{TH}$ rectangle

Take the limit as the number of rectangles increases to $\infty$. 

Example. Estimate the area $A$ under the graph of $f(x) = x$ from $x = 0$ to $x = 1$.

The width of each rectangle is $\Delta x = 1/4$. Area of rectangles

\[ R_4 = \Delta x \cdot \frac{1}{4} + \Delta x \cdot \frac{1}{2} + \Delta x \cdot \frac{3}{4} + \Delta x \cdot 1 \]

\[ = \Delta x \left( \frac{1}{4} + \frac{1}{2} + \frac{3}{4} + 1 \right) \]

\[ = \Delta x \cdot \frac{1+2+3+4}{4} \]

\[ = \frac{1}{4} \cdot \frac{10}{4} = \frac{10}{16} = \frac{5}{8} \]

Clearly $A < R_4$
Alternatively, let $x^*_i$ be the left hand endpoint of each subinterval.

$\langle 0 \ldots 1, \ 0 \ldots 1, \ f(x) = x, \text{rectangles} \rangle$

Clearly

$L_4 < A < R_4$

$\frac{3}{8} < A < \frac{5}{8}$

Improve estimate by dividing area into 8 strips

$L_8 < A < R_8$

$\frac{7}{16} < A < \frac{9}{16}$

Area of rectangles

$L_4 = \Delta x \cdot \frac{0}{4} + \Delta x \cdot \frac{1}{4} + \Delta x \cdot \frac{1}{2} + \Delta x \cdot \frac{3}{4}$

$= \Delta x \cdot \frac{0+1+2+3}{4}$

$= \frac{1}{4} \cdot \frac{6}{4} = \frac{6}{16} = \frac{3}{8}$
Example. Obtain the exact area under $f(x) = x$ from $x = 0$ to $x = 1$ by taking the limit as $n$ increases.

Let the sample points be right hand endpoints of each subinterval. 

\[ \langle 0 \ldots 1, \ldots, f(x) = x, \text{rectangles of heights } \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots \rangle \]

\[ R_n = \Delta x \cdot \frac{1}{n} + \Delta x \cdot \frac{2}{n} + \cdots + \Delta x \cdot \frac{n}{n} \]

\[ = \Delta x \cdot \frac{1}{n} (1 + 2 + \cdots + n) \]

\[ = \frac{1}{n^2} (1 + 2 + \cdots + n) \]

\[ = \frac{1}{n^2} \frac{n(n+1)}{2} = \frac{n+1}{2n} \]

Exact area

\[ A = \lim_{n \to \infty} R_n \]

\[ = \lim_{n \to \infty} \frac{n+1}{2n} \]

\[ = \frac{1}{2} \]

Alternatively, let the sample points be left hand endpoints of each subinterval.

heights of rectangles = \[ \frac{0}{n}, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n} \]

\[ L_n = \Delta x \cdot \frac{0}{n} + \Delta x \cdot \frac{1}{n} + \cdots + \Delta x \cdot \frac{n-1}{n} \]

\[ = \Delta x \cdot \frac{1}{n} (0 + 1 + \cdots + (n-1)) \]

\[ = \frac{1}{n^2} (0 + 1 + \cdots + (n-1)) \]

\[ = \frac{1}{n^2} \frac{(n-1)n}{2} = \frac{n-1}{2n} \]

Exact Area

\[ A = \lim_{n \to \infty} L_n \]

\[ = \lim_{n \to \infty} \frac{n-1}{2n} = \frac{1}{2} \]

\[ \blacksquare \]
Fact: We can take the height of the \(i^{th}\) rectangle as \(f(x_i^*)\) for any number \(x_i^*\) in the \(i^{th}\) subinterval.

\[
A = \lim_{n \to \infty} \left[ f(x_1^*) \Delta x + f(x_2^*) \Delta x + \cdots + f(x_n^*) \Delta x \right]
\]

\[
A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x
\]

This formula works for any continuous function \(f\)

\[\langle \ldots a \ldots b \ldots, 0 \ldots, i^{th} \text{ rect.}, \text{width } \Delta x, x_i^*, \text{height } f(x_i^*) \rangle\]

The Distance Problem

Find the distance traveled by an object if its velocity is known

If the velocity is constant

\[
\text{distance} = \text{velocity} \times \text{time}
\]

Example

120 miles = 60 \(\text{miles/hour}\) \(\times\) 2 hours

What if the velocity varies?

\[\langle \ldots a \ldots b \ldots, \text{time } t^*, \ldots, \text{velocity } v^*, \text{curve} \rangle\]

define time subintervals of length \(\Delta t\)

pick sample time \(t_i^*\) in each subinterval

approximate the velocity in each subinterval as \(v(t_i^*)\)
\[ d = \text{distance traveled} \]
\[ \approx v(t_1^*)\Delta t + v(t_2^*)\Delta t + v(t_3^*)\Delta t + v(t_4^*)\Delta t \]
\[ = \sum_{i=1}^{4} v(t_i^*)\Delta t \]

The exact distance is obtained in the limit as we use more and more subintervals.

\[ d = \lim_{n \to \infty} \sum_{i=1}^{n} v(t_i^*)\Delta t \]

§5.2 The Definite Integral

\[ \langle \ldots \ a \ \ldots \ b \ \ldots \ x, \ \ldots, \ \text{curve} \ f \rangle \]

define the definite integral

Steps

Divide \([a, b]\) into \(n\) subintervals of equal width

\[ \Delta x = (b - a)/n \quad \langle \text{add subinterval } i \rangle \]
\[ x_i = a + i\Delta x \quad \langle \text{add } x_0, x_{i-1}, x_i, x_n \rangle \]

Choose sample points in these subintervals

\[ x_1^*, x_2^*, \ldots x_n^* \quad \langle \text{add } x_i^* \rangle \]

Construct the Riemann Sum

\[ \sum_{i=1}^{n} f(x_i^*)\Delta x \]

Take the limit \(n \to \infty\) to obtain the definite integral

\[ \int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x \]

Example. The distance problem

\[ d = \int_a^b v(t) \, dt \]

\[ \square \]
Notation (integral sign, lower limit of integration $a$, upper limit $b$, integrand, ghost of $\Delta x$)

Note that integral does not depend on “$x$”

$$\int_{a}^{b} f(x)dx = \int_{a}^{b} f(t)dt$$

$x$ and $t$ are dummy variables.

Example. Express

$$I = \lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{1 + \left(\frac{i}{n}\right)^2} \cdot \frac{1}{n}$$

as an integral on $[0,1]$.

Let $\Delta x = \frac{1}{n}$ and $f(x_i) = \frac{1}{1+(x_i^*)^2}$ where $x_i^* = \frac{i}{n}$.

Then $I = \int_{0}^{1} \frac{1}{1+x^2} dx$
Interpretation of the Definite Integral

\[ \int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{n=1}^{\infty} f(x_i^*) \Delta x \]

For \( f \) positive and \( b > a \), the Riemann sum approximates the area under the curve.

\[ \langle \ldots a \ldots b \ldots x_-, \ldots, \text{graph of } f, \text{area } A, 4 \text{ rectangles} \rangle \]

Suppose \( f \) can be positive or negative

\[ \langle \ldots a \ldots b \ldots x_-, \ldots, f, A_1 \text{ under } f, A_2 \text{ above } f, \text{ rectangles} \rangle \]

\[ \langle \text{here } f(x_i^*) > 0 \ldots \sum f(x_i^*) \Delta x \approx A_1, \text{ here } f(x_i^*) < 0 \ldots \sum f(x_i^*) \Delta x \text{ is a negative quantity whose magnitude } \approx A_2 \rangle \]

\[ \int_a^b f(x)\,dx = A_1 - A_2 = \text{net area under or above } f \text{ from } a \text{ to } b \]
Evaluating Integrals

Recall

1. \( \sum_{i=1}^{n} c = nc \)

2. \( \sum_{i=1}^{n} c a_i = c \sum_{i=1}^{n} a_i \)

3. \( \sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i \)

4. \( \sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i \)

where \( a_1, a_2, \ldots \) and \( b_1, b_2, \ldots \) are sequences of numbers and \( c \) is a constant (an expression that does not depend on the index of summation \( i \)).

Recall two sums of powers

1. \( \sum_{i=1}^{n} i = \frac{1}{2} n(n + 1) \)

2. \( \sum_{i=1}^{n} i^2 = \frac{1}{6} n(n + 1)(2n + 1) \)

Example. Prove that \( \int_{a}^{b} x \, dx = \frac{1}{2} (b^2 - a^2) \)

\(<...a \ldots b\ldots x\ldots\text{-}, \text{partition points, } \Delta x, \Delta x = \frac{b-a}{n}>\)

choose sample points in each subinterval

an easy choice is the right hand endpoints

\( x_1^* = a + \Delta x \)

\( x_2^* = a + 2\Delta x \)

\( x_i^* = a + i\Delta x \)

Begin with the definition

\[ \int_{a}^{b} x \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x \]

\[ = \lim_{n \to \infty} \sum_{i=1}^{n} x_i^* \Delta x \]
\[ \lim_{n \to \infty} \sum_{i=1}^{n} (a + i\Delta x)\Delta x = \lim_{n \to \infty} \sum_{i=1}^{n} a\Delta x + i\Delta x^2 \]

\[ \lim_{n \to \infty} \left( \sum_{i=1}^{n} a\Delta x + \sum_{i=1}^{n} i\Delta x^2 \right) \]

\[ \int_a^b x \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} a\Delta x + \sum_{i=1}^{n} i\Delta x^2 \]

\[ \lim_{n \to \infty} \left( a\Delta x \sum_{i=1}^{n} 1 + \Delta x^2 \sum_{i=1}^{n} i \right) \]

\[ \lim_{n \to \infty} \left( a \frac{b-a}{n} n + \left( \frac{b-a}{n} \right)^2 \frac{1}{2} n(n+1) \right) \]

\[ \lim_{n \to \infty} \left( a(b-a) + (b-a)^2 \frac{1}{2} \frac{n + 1}{n} \right) \]

\[ = a(b-a) + \frac{1}{2} (b-a)^2 \lim_{n \to \infty} \frac{n+1}{n} \]

\[ = a(b-a) + \frac{1}{2} (b-a)^2 \]

\[ = (b-a) \left( a + \frac{1}{2} (b-a) \right) \]

\[ = \frac{1}{2} (b-a)(b+a) \]

\[ = \frac{1}{2} (b^2 - a^2) \quad \text{we are done!} \]

Example. Prove that \( \int_0^1 x^2 \, dx = \frac{1}{3} \)

\[ \left< 0 \ldots 1 \ldots x-, 0 \ldots 1 \ldots y-, y = x^2 \right> \]
width of subintervals \( \Delta x = \frac{1}{n} \)

sample points \( x_i^* = \frac{i}{n} \)

\[
\int_0^1 x^2 \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} (x_i^*)^2 \Delta x
\]

\[
= \lim_{n \to \infty} \sum_{i=1}^{n} \left( \frac{i}{n} \right)^2 \frac{1}{n}
\]

\[
= \lim_{n \to \infty} \frac{1}{n^3} \sum_{i=1}^{n} i^2
\]

\[
= \lim_{n \to \infty} \frac{1}{n^3} \frac{1}{6} n(n + 1)(2n + 1)
\]

\[
= \lim_{n \to \infty} \frac{2n^3 + \ldots}{6n^3}
\]

\[
= \frac{1}{3}
\]

Properties of the definite integral

Reversing limits of integration

\[
\int_a^b f(x) \, dx = - \int_b^a f(x) \, dx
\]
Four basic properties of integrals

1. \[ \int_a^b c \, dx = c(b - a) \]

2. \[ \int_a^b c f(x) \, dx = c \int_a^b f(x) \, dx \]

3. \[ \int_a^b f(x) + g(x) \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx \]

4. \[ \int_a^b f(x) - g(x) \, dx = \int_a^b f(x) \, dx - \int_a^b g(x) \, dx \]

Notes

2. constant multiple rule
3. integral of sum is sum of integrals
4. integral of difference is difference of integrals.

Property 1 has a simple interpretation

\[ \langle \ldots a \ldots b \ldots, \ldots c \ldots, \text{function } c, \text{ area } = c(b - a) \rangle \]

Example. Evaluate \[ I = \int_0^3 (5 - 2x) \, dx \]

use the difference rule

\[ I = \int_0^3 5 \, dx - \int_0^3 2x \, dx \]

property \#1 and constant multiple rule

\[ = 5(3 - 0) - 2 \int_0^3 x \, dx \]

rule that \( \int_a^b x \, dx = \frac{1}{2}(b^2 - a^2) \)

\[ = 15 - 2 \left( \frac{1}{2} \right)(9 - 0) \]

\[ = 6 \]

Addition Property wrt Interval of Integration

For any real nos. \( a, b \) and \( c \)

5. \[ \int_a^c f(x) \, dx + \int_c^b f(x) \, dx = \int_a^b f(x) \, dx \]

Picture for \( a < c < b \) and \( f \geq 0 \)

\[ \langle \ldots a \ldots c \ldots b \ldots, x^-, \ldots, f, A, A_1, A_2 \rangle \]
Example: Evaluate

\[ I = \int_3^4 f(x)dx + \int_1^3 f(x)dx + \int_1^4 f(x)dx \]

by addition property reverse limits

so \( I = 0 \)

Comparison Property of Integrals

6. If \( f(x) \geq 0 \) and \( b > a \) then

\[ \int_a^b f(x)dx \geq 0 \]

7. If \( f(x) \geq g(x) \) and \( b > a \) then

\[ \int_a^b f(x)dx \geq \int_a^b g(x)dx \]

8. If \( m \leq f(x) \leq M \) then

\[ m(b - a) \leq \int_a^b f(x)dx \leq M(b - a) \]

Example. Show that

\[ \int_2^5 \sqrt{x^2 - 1} \, dx \leq 10.5 \]

without evaluating the integral.

Solution. For \( x > 1 \):

\[ \sqrt{x^2 - 1} < \sqrt{x^2} = x \]

Then by property 7:

\[ \int_2^5 \sqrt{1 - x^2} \, dx \leq \int_2^5 x \, dx \]

\[ = \frac{1}{2}(5^2 - 2^2) = \frac{1}{2}(25 - 4) = \frac{21}{2} \]

\[ = 10.5 \]
Example. Show that
\[ \frac{\pi}{6} < \int_{0}^{\pi/3} \cos(x) \, dx < \frac{\pi}{3} \]
without evaluating the integral.

\[ \langle 0 \cdots \frac{\pi}{3} \cdots \frac{\pi}{2} \cdots x, 0 \cdots 1 \cdots y, \cos(x) \rangle \]

on the interval \([0, \pi/3]\)
\[ \frac{1}{2} \leq \cos(x) \leq 1 \]
by property 8
\[ \frac{1}{2} \left( \frac{\pi}{3} - 0 \right) < \int_{0}^{\pi/3} \cos(x) \, dx < 1 \left( \frac{\pi}{3} - 0 \right) \]
which gives the result we are trying to show ■

§5.3 The Evaluation Theorem
If \( f \) is continuous on the closed interval \([a, b]\), then
\[ \int_{a}^{b} f(x) \, dx = F(b) - F(a) \]
where \( F \) is any antiderivative of \( f \), that is \( F' = f \).

Example. \( \int_{a}^{b} x \, dx = F(b) - F(a) \)
where \( F(x) = \frac{1}{2} x^2 + C \).
\[ F(b) = \frac{1}{2} b^2 + C, \quad F(a) = \frac{1}{2} a^2 + C \]
\[ \int_{a}^{b} x \, dx = \frac{1}{2} (b^2 - a^2) \]
Notice that the constant of integration \( C \) cancels. We may as well set \( C = 0 \). ■

Example. \( \int_{0}^{1} x^2 \, dx = F(1) - F(0) \)
where \( F(x) = \frac{1}{3} x^3 \)
\[ F(1) = \frac{1}{3}, \quad F(0) = 0 \]
\[ \int_{0}^{1} x^2 \, dx = \frac{1}{3} \]
■
Example. The distance problem.

$v$ velocity
$s$ position
distance traveled
$a$ starting time
$b$ ending time

we have seen previously

\[ d = \int_a^b v(t) \, dt \]

recall \( s'(t) = v(t) \)

By the Evaluation Theorem

\[ \int_a^b v(t) \, dt = s(b) - s(a) \] = distance traveled! \[\blacksquare\]

Proof of the Evaluation Theorem.

Partition \([a, b]\) into \(n\) subintervals of equal width.

\[ \langle ... a \ldots b \ldots x_-, x_0, x_1, x_2, \ldots, x_{n-1}, x_n \rangle \]

width of each subinterval \( \Delta x = (b - a)/n \)

Let \( F \) be any antiderivative of \( f \).

Write \( F(b) - F(a) \) as a telescoping sum

\[ F(b) - F(a) = F(x_n) - F(x_0) \]

\[ = (F(x_n) - F(x_{n-1})) + (F(x_{n-1}) - F(x_{n-2})) + \]

\[ ... + (F(x_2) - F(x_1)) + (F(x_1) - F(x_0)) \]

Apply the Mean Value Theorem to \( F \) on an arbitrary
subinterval \( i: [x_{i-1}, x_i] \)

\[ F(x_i) - F(x_{i-1}) = F'(x^*_i)(x_i - x_{i-1}) = f(x^*_i)\Delta x, \]
where $x_i^*$ is some no. on the interval $(x_{i-1}, x_i)$.

Then

$$F(b) - F(a) = f(x_n^*)\Delta x + f(x_{n-1}^*)\Delta x + \cdots + f(x_1^*)\Delta x$$

$$= \sum_{i=1}^{n} f(x_i^*)\Delta x$$

the last expression is a Riemann sum!

Let $n \to \infty$

$$\lim_{n \to \infty} F(b) - F(a) = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

$F(b) - F(a)$ does not depend on $n$ and the limit on the right is the integral. Therefore we get

$$F(b) - F(a) = \int_a^b f(x)dx$$

\[\blacksquare\]

Note $F(b) - F(a) = F(x)|_a^b$

Then the Evaluation Theorem may be written

$$\int_a^b f(x)dx = F(x)|_a^b$$

where $F$ is any antiderivative of $F$.

Example. Evaluate $\int_0^1 x^{3/7} dx$

For $f(x) = x^n$, an antiderivative of $f$ is

$$F(x) = \frac{1}{n+1}x^{n+1}$$

For $n = 3/7$

$$F(x) = \frac{x^{10}}{10}$$

Thus

$$\int_0^1 x^{3/7} dx = \frac{7}{10} x^{10/7} \bigg|_0^1$$

$$= \frac{7}{10} (1)^{10/7} - \frac{7}{10} (0)^{10/7}$$

$$= \frac{7}{10} - 0$$

$$= \frac{7}{10} \blacksquare$$
Example. Evaluate $I = \int_{0}^{\pi/4} \sec^2(\theta) d\theta$

Recall $\frac{d}{d\theta} \tan(\theta) = \sec^2(\theta)$

Then $I = \tan(\theta) \big|_{0}^{\pi/4} = \tan\left(\frac{\pi}{4}\right) - \tan(0) = 1$

The Evaluation Theorem in Applications

Interpret derivatives as rates of change

$s$ position $v$ velocity

$Q$ electric charge $I = \frac{dQ}{dt}$ electric current

$y = h(x)$ height of trail $\frac{dy}{dx} = h'(x)$ slope of trail

$x$ miles from start

rewrite the evaluation theorem in terms of rate of change

Net Change Theorem
The integral of a rate of change is a net change

\[ \int_{a}^{b} F'(x)dx = F(b) - F(a) \]

where $F'(x)$ is the rate of change of $F$ wrt $x$ and $F(b) - F(a)$ is the net change in $F$.

Note that the word “net” connotes that there can be positive and negative contributions to the rate of change

Examples.

The integral of velocity gives the net change in position

\[ \int_{a}^{b} v(t)dt = s(b) - s(a) \]

The integral of current gives the net charge passing through a wire

\[ \int_{a}^{b} I(t)dt = Q(b) - Q(a) \]

The integral of the slope of trail gives the net change in height of trail

\[ \int_{a}^{b} h'(x)dx = h(b) - h(a) \]
Particle Motion (back and forth) along a straight line

\[ s(t) \]  \quad \text{position at time } t \\
\[ v(t) \]  \quad \text{velocity at time } t \\
\[ |v(t)| \]  \quad \text{speed at time } t

Consider the following particle path

\[ \langle 0 \ldots 4 \ldots s-, \ t_1 \text{ at } 0, \ t_2 \text{ at } 4, \ t_3 \text{ at } 3 \rangle \]

the particle’s displacement (distance traveled)

\[ s(t_3) - s(t_1) = 3 = \int_{t_1}^{t_3} v(t) \, dt \]

the total distance the particle travels

\[ 5 = 4 + 1 = \int_{t_1}^{t_3} |v(t)| \, dt = \int_{t_1}^{t_2} v(t) \, dt + \int_{t_2}^{t_3} (-v(t)) \, dt \]

Example. The velocity function for a particle moving along a line is

\[ v(t) = 4 - 2t \]  \quad \text{meters/sec} 

Find (a) the displacement and (b) the total distance traveled over the time interval

\[ 0 \leq t \leq 3 \text{ seconds} \]

(a) \[ \int_{0}^{3} v(t) \, dt = \int_{0}^{3} (4 - 2t) \, dt \]
\[ = 4t - t^2 \bigg|_{0}^{3} \]
\[ = (12 - 9) - (0 - 0) \]
\[ = 3 \text{ meters} \]

(b) \[ \int_{0}^{3} |v(t)| \, dt = \int_{0}^{2} v(t) \, dt + \int_{2}^{3} (-v(t)) \, dt \]
\[ = \int_{0}^{2} (4 - 2t) \, dt + \int_{2}^{3} (2t - 4) \, dt \]
\[ = (4t - t^2) \bigg|_{0}^{2} + (t^2 - 4t) \bigg|_{2}^{3} \]
\[ = ((8 - 4) - (0 - 0)) + ((9 - 12) - (4 - 8)) \]
\[ = (4 - 0) + ((-3) - (-4)) \]
\[ = 4 + 1 = 5 \text{ meters} \]
Indefinite Integrals

The symbol $$\int f(x)\,dx$$

is called the **indefinite integral** and means the general antiderivative of $$f(x)$$.

If $$F$$ is any antiderivative of $$f$$ then

$$\int f(x)\,dx = F(x) + C$$

$$C$$ is the **constant of integration**.

**Antiderivative Formulas using indefinite integral notation**

Let $$k$$ be a constant and let $$f$$ and $$g$$ be functions

$$\int k\,dx = kx + C$$

$$\int k\ f(x)\,dx = k\int f(x)\,dx + C$$

$$\int f(x) + g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int f(x) - g(x)\,dx = \int f(x)\,dx - \int g(x)\,dx$$

we also have formulas for specific functions

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$$ \quad valid for $$n \neq -1$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C$$

$$\int a^x\,dx = \frac{a^x}{\ln(a)} + C$$

$$\int \sin(x)\,dx = -\cos(x) + C$$

$$\int \cos(x)\,dx = \sin^{-1}(x) + C$$

$$\int \sec^2(x)\,dx = \tan(x) + C$$

$$\int \sec(x)\tan(x)\,dx = \sec(x) + C$$

$$\int \frac{1}{1 + x^2}\,dx = \tan^{-1}(x) + C$$

$$\int \frac{1}{\sqrt{1 - x^2}}\,dx = \sin^{-1}(x) + C$$
Example. Find the indefinite integral
\[ I = \int (\cos(x) - 2\sin(x)) \, dx \]
solution
\[ I = \int \cos(x) \, dx - \int 2 \sin(x) \, dx \]
\[ = \int \cos(x) \, dx - 2 \int \sin(x) \, dx \]
\[ = \sin(x) + 2 \cos(x) + C \quad \blacksquare \]

Example. Find the indefinite integral
\[ I = \int \left( \frac{1}{5x} + 3 \sec(x) \tan(x) \right) \, dx \]
solution
\[ I = \int \frac{1}{5x} \, dx + \int 3 \sec(x) \tan(x) \, dx \]
\[ = \frac{1}{5} \int \frac{1}{x} \, dx + 3 \int \sec(x) \tan(x) \, dx \]
\[ = \frac{1}{5} \ln |x| + 3 \sec(x) + C \quad \blacksquare \]

§5.4 The Fundamental Theorem of Calculus

The “Area so far” function
\[ \langle \ldots a \ldots x \ldots b \ldots t \ldots, 0 \ldots, f(t), \text{area } g(x) \text{ from } a \text{ to } x \rangle \]

\[ g(x) = \int_{a}^{x} f(t) \, dt \]
If \( f > 0 \), \( g \) is the “area so far” under \( f \)

Example. Let \( a = 1 \) and \( f(t) = \frac{1}{t} \).
\[ g(x) = \int_{1}^{x} \frac{1}{t} \, dt \]
\[ = \ln |t| \big|_{1}^{x} \]
\[ = \ln(x) - \ln(1) \]
\[ = \ln(x) \quad \text{where } x > 0 \]

Notice that
\[ g'(x) = \frac{1}{x} = f(x) \quad \text{This is not a coincidence!} \quad \blacksquare \]
The Fundamental Theorem of Calculus, part 1 (FTC 1)

If $f$ is continuous on $[a, b]$, then the function $g$ defined by

$$g(x) = \int_a^x f(t) \, dt$$

is an antiderivative of $f$, that is

$$g'(x) = f(x) \quad \text{for} \quad a \leq x \leq b.$$

Plausibility Argument. Why is $g'(x) = f(x)$?

From the graph

$$g(x + h) - g(x) \approx f(x)h$$

this becomes more accurate as $h \to 0$

$$g'(x) = \lim_{h \to 0} \frac{f(x)h + \text{small correction}}{h} = f(x)$$

Our text gives a rigorous proof.

Using Leibniz notation, the FTC 1 becomes

$$\frac{d}{dx} \int_a^x f(t) \, dt = f(x)$$

Example. Find the derivative of

$$g(x) = \int_{-1}^x \sqrt{t^3 + 1} \, dt$$

Solution $f(t) = \sqrt{t^3 + 1}$ is continuous on $[-1, \infty)$. 

$$g'(x) = \sqrt{x^3 + 1}$$
Example. Find the derivative of

$$y = \int_{x^2}^{\pi} \frac{\sin(t)}{t} \, dt$$

Solution. $f(t) = \frac{\sin(t)}{t}$ is continuous for $t > 0$.

Reverse the limits of integration.

$$y = -\int_{\pi}^{x^2} \frac{\sin(t)}{t} \, dt$$

$$= -\int_{\pi}^{u} \frac{\sin(t)}{t} \, dt$$

where $u = x^2$. By the FTC 1

$$\frac{dy}{du} = -\frac{\sin(u)}{u}$$

By the chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= -\frac{\sin(u)}{u} \cdot \frac{du}{dx}$$

$$= -\frac{\sin(x^2)}{x^2} \cdot 2x$$

$$= -\frac{2}{x} \sin(x^2) \quad \blacksquare$$

The Fundamental Theorem of Calculus (FTC)

Suppose $f$ is continuous on $[a, b]$

1. $\frac{d}{dx} \int_{a}^{x} f(t) \, dt = f(x)$

2. $\int_{a}^{b} f(x) \, dx = F(b) - F(a)$

where $F$ is any antiderivative of $f$. This is the Evaluation Theorem!

Part 2 can be rewritten

$$\int_{a}^{b} F'(x) \, dx = F(b) - F(a)$$

Roughly speaking, the FTC says that differentiation and integration are inverse processes!

Recall the Mean Value Theorem:

If $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$, then there is a no. $c$ in $(a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$
Mean Value Theorem for Integrals

If $f$ is continuous on $[a, b]$, then there is a no. $c$ in $(a, b)$ such that

$$f(c) = \frac{\int_a^b f(x)dx}{b - a}$$

Proof. Let $F$ be the “Area so far” function

$$F(x) = \int_a^x f(t)dt$$

$F'(x) = f(x)$ by FTC 1.

$F$ is continuous on $[a, b]$ and differentiable on $(a, b)$.

By the Mean Value Theorem, there is a no. $c$ in $[a, b]$ such that

$$F'(c) = \frac{F(b) - F(a)}{b - a}$$

By FTC 2 (the Evaluation Theorem), this becomes

$$f(c) = \frac{\int_a^b f(x)dx}{b - a}$$

which is what we want to prove! ■

Geometrical Interpretation

$\langle ... \ a \ ... \ b \ ... \ x, ..., f \rangle$  \quad  \langle \text{add } c, f(c), \text{ rectangle} \rangle$

Note:

$$\int_a^b f(x)dx = f(c)(b - a)$$

area under curve = area of rectangle

is the average value of $f$ on $[a, b]$
Example

a) Find the average value of $f(x) = x^3$ on $[0,1]$

$$f_{\text{ave}} = \frac{1}{1-0} \int_0^1 x^3 \, dx = \frac{1}{4} x^4 \bigg|_0^1 = \frac{1}{4}$$

b) Find $c$ such that $f_{\text{ave}} = f(c)$

$$f(c) = c^3 = \frac{1}{4}$$

$$c = \left(\frac{1}{4}\right)^{\frac{1}{3}} \approx 0.63$$

$\blacksquare$

STOP §5.5 The Substitution Rule

Differential Notation revisited

Consider $y = f(x)$. By the definition of derivative:

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

this is not exactly a ratio...

Let $dx$ be any real no. and define

$$dy = f'(x) \, dx$$

another real no. This allows us to interpret the left hand side of

$$\frac{dy}{dx} = f'(x)$$

as a true ratio.

Practice. Let's write $u = g(x)$. Then $du = g'(x) \, dx$

Let $u = x^2$. Then $du = 2x \, dx$

?? Let $u = \sqrt{x}$. Then $du =$

?? Let $u = \ln |x|$. Then $du =$

?? Let $u = \tan \theta$. Then $du =$
Integration by Substitution

Recall the indefinite integral
\[ \int F'(x) \, dx = F(x) + C \]

Recall the chain rule
\[
F'(x) = f'(g(x)) \cdot g'(x)
\]
Thus
\[ \int f'(g(x))g'(x) \, dx = f(g(x)) + C \]

Example. Let \( F(x) = \sin(x^2) \)
\[
F'(x) = \cos(x^2) \cdot 2x
\]
so \( \int \cos(x^2)2x \, dx = \sin(x^2) + C \)

This is taking the chain rule backwards!

How to recognize \( f'(g(x)) \cdot g'(x) \) in an integrand?

Trick – “Integration by Substitution”

Suppose you have
\[ I = \int f'(g(x))g'(x) \, dx \]
but you might not recognize the form.

Guess the inner function of the composition
\( u = g(x) \)
form the differential
\[ du = g'(x) \, dx \]
substitute into \( I \)
\[ I = \int f'(u) \, du \]
\[ = f(u) + C \]
\[ = f(g(x)) + C \]
Example. Evaluate

\[ I = \int 2x \cos(x^2) \, dx \]

guess the inner function

\[ u = x^2 \]

form the differential

\[ du = 2x \, dx \]

substitute into \( I \)

\[ I = \int \cos(u) \, du \]

= \sin(u) + C

= \sin(x^2) + C  \quad \blacksquare

Example. Evaluate

\[ I = \int \sqrt{3x + 4} \, dx \]

guess the inner function

\[ u = 3x + 4 \]

form the differential

\[ du = 3 \, dx \]

solve for \( dx \)

\[ dx = \frac{1}{3} \, du \]

substitute into \( I \)

\[ I = \int u^{1/2} \frac{1}{3} \, du \]

= \frac{1}{3} \cdot \frac{2}{3} u^{3/2} + C

= \frac{2}{9} (3x + 4)^{3/2} + C

can check by differentiation

\[ \frac{d}{dx} \left( \frac{2}{9} (3x + 4)^{3/2} + C \right) = \frac{2}{9} \cdot \frac{3}{2} (3x + 4)^{1/2}(3) + 0 = (3x + 4)^{1/2} \]

\[ \blacksquare \]
Example. Evaluate

\[ I = \int \tan^2(x) \sec^2(x) \, dx \]

guess the inner function

\[ u = \tan(x) \]

form the differential

\[ du = \sec^2(x) \, dx \]

substitute into \( I \)

\[ I = \int u^2 \, du = \frac{1}{3} u^3 + C = \frac{1}{3} \tan^3(x) + C \]

Example. Evaluate

\[ I = \int \frac{x}{1 + x^4} \, dx \]

guess the inner function

\[ u = x^2 \]

form the differential

\[ du = 2x \, dx \]

solve for \( x \, dx \)

\[ x \, dx = \frac{1}{2} \, du \]

substitute into \( I \)

\[ I = \int \frac{1}{1 + u^2} \frac{1}{2} \, du = \frac{1}{2} \tan^{-1}(u) + C = \frac{1}{2} \tan^{-1}(x^2) + C \]
Evaluating Definite Integrals By Substitution

Recall the Fundamental Theorem of Calculus, part 2

\[ \int_{a}^{b} F'(x) \, dx = F(b) - F(a) \]

**Method I** Find the indefinite integral by substitution and then apply the FTC, part 2.

**Example.** Evaluate

\[ \int_{-1}^{0} \sqrt{3x + 4} \, dx \]

let

\[ u = 3x + 4, \quad du = 3 \, dx, \quad \frac{1}{3} \, du = dx \]

then

\[
\int \sqrt{3x + 4} \, dx = \int u^{1/2} \cdot \frac{1}{3} \, du = \frac{1}{3} \cdot \frac{2}{3} u^{3/2} + C
\]

\[ = \frac{2}{9} (3x + 4)^{3/2} + C \]

Now let \( F(x) = \frac{2}{9} (3x + 4) \). Then by FTC 1

\[
I = F(0) - F(-1)
\]

\[ = \frac{2}{9} \cdot 4^{3/2} - \frac{2}{9} \cdot 1^{3/2} \]

\[ = \frac{2}{9} (8 - 1) \]

\[ = \frac{14}{9} \quad \blacksquare \]
Method II. Transform limits of integration while substituting and apply FTC 2.

This method is better once you get used to it because it involves less writing.

Example. Evaluate

\[ I = \int_{-1}^{0} \sqrt{3x + 4} \, dx \]

Let

\[ u = 3x + 4, \quad du = 3dx, \quad \frac{1}{3} du = dx \]

Also note that if \( x = -1 \) then \( u = 1 \)

if \( x = 0 \) then \( u = 4 \)

then

\[ I = \int_{1}^{4} \frac{1}{3} \, du \]

\[ = \frac{1}{3} \cdot \frac{2}{3} u^\frac{3}{2} \bigg|_{1}^{4} \]

\[ = \frac{2}{9} \left( \frac{3}{2} - 1^\frac{3}{2} \right) \]

\[ = \frac{2}{9} (8 - 1) = \frac{14}{9} \] ■

Example. Evaluate

\[ I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^2(x) \sec^2(x) \, dx \]

Let

\[ u = \tan(x) \]

\[ du = \sec^2(x) \, dx \]

Also note that if \( x = -\frac{\pi}{4} \) then \( u = -1 \)

if \( x = \frac{\pi}{4} \) then \( u = 1 \)

then

\[ I = \int_{-1}^{1} u^2 \, du \]

\[ = \frac{1}{3} u^3 \bigg|_{-1}^{1} \]

\[ = \frac{1}{3} (1^3 - (-1)^3) \]

\[ = \frac{2}{3} \] ■
Example. Evaluate

\[ I = \int_{1}^{2} \frac{\cos \left( \frac{\pi}{x} \right)}{x^2} \, dx \]

Let \( u = \frac{\pi}{x} = \pi x^{-1}. \)

Then \( du = \pi (-x^{-2}) \, dx \) and \(-\frac{1}{\pi} \, du = x^{-2} \, dx.\)

If \( x = 1 \) then \( u = \pi \)
\[ x = 2 \quad u = \frac{\pi}{2} \]
Then
\[ I = \int_{\frac{\pi}{2}}^{\pi} \cos(u) \left( -\frac{1}{\pi} \, du \right) \]
\[ = \frac{1}{\pi} \int_{\frac{\pi}{2}}^{\pi} \cos(u) \, du \]
\[ = \frac{1}{\pi} \sin(u) \bigg|_{\frac{\pi}{2}}^{\pi} \]
\[ = \frac{1}{\pi} \left( \sin(\pi) - \sin \left( \frac{\pi}{2} \right) \right) \]
\[ = \frac{1}{\pi} \left( 0 - 1 \right) \]
\[ = -1/\pi \]

[\[\Box\]\]

Example. Evaluate

\[ I = \int_{2}^{4} \frac{1}{x \ln(x)} \, dx \]

Let \( u = \ln(x). \)

Then \( du = \frac{1}{x} \, dx \)

If \( x = 2 \) then \( u = \ln 2 \)
\[ x = 4 \quad u = \ln 4 = \ln 2^2 = 2 \ln 2 \]
Then
\[ I = \int_{\ln 2}^{\ln 4} \frac{1}{u} \, du \]
\[ = \ln u \bigg|_{\ln 2}^{\ln 4} \]
\[ = \ln(\ln 4) - \ln(\ln 2) \]
\[ = \ln(2 \ln 2) - \ln(\ln 2) \]
\[ = \ln 2 + \ln(\ln 2) - \ln(\ln 2) \]
\[ = \ln 2 \]

[\[\Box\]\]