Table of Trig Functions

<table>
<thead>
<tr>
<th>Function</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sin(x) )</td>
<td>( \cos(x) )</td>
</tr>
<tr>
<td>( \tan(x) = \frac{\sin(x)}{\cos(x)} )</td>
<td>( \cot(x) = \frac{\cos(x)}{\sin(x)} )</td>
</tr>
<tr>
<td>( \sec(x) = \frac{1}{\cos(x)} )</td>
<td>( \csc(x) = \frac{1}{\sin(x)} )</td>
</tr>
</tbody>
</table>

Obtain co-functions in right column from left column by replacing \( \sin(x) \rightarrow \cos(\pi - x) \) and \( \cos(x) \rightarrow \sin(\pi - x) \).

Two Famous Triangles

(\text{sketch } 45^\circ-45^\circ-90^\circ \text{ triangle, label angles and sides})

Pythagorean theorem

\( \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = 1 \quad \text{or} \quad \frac{3}{4} + \frac{1}{2} = 1 \)

\( \sin \left(\frac{\pi}{6}\right) = \frac{1}{2}, \cos \left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}, \tan \left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}} \)

\( \sin \left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}, \cos \left(\frac{\pi}{3}\right) = \frac{1}{2}, \tan \left(\frac{\pi}{3}\right) = \sqrt{3} \)

Pythagorean theorem

\( \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = 1 \quad \text{or} \quad \frac{1}{2} + \frac{1}{2} = 1 \)

\( \sin \left(\frac{\pi}{4}\right) = \cos \left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}, \tan \left(\frac{\pi}{4}\right) = 1 \)
Derivatives of Trig Functions

\[
\frac{d}{dx} \tan(x) = \frac{d}{dx} \frac{\sin(x)}{\cos(x)}
= \frac{(\sin(x))' \cos(x) - (\cos(x))' \sin(x)}{\cos^2(x)}
= \frac{\cos(x) \cos(x) - \sin(x) (-\sin(x))}{\cos^2(x)}
= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)}
= \frac{1}{\cos^2(x)} = \sec^2(x)
\]

where \(\sec(x) = 1/\cos(x)\).

\[
\frac{d}{dx} \sec(x) = \frac{d}{dx} \frac{1}{\cos(x)}
= \frac{(1)' \cos(x) - 1 \cdot (\cos(x))'}{\cos^2(x)}
= \frac{-\sin(x)}{\cos^2(x)}
= \frac{1}{\cos(x)} \cdot \frac{\sin(x)}{\cos(x)}
= \sec(x) \cdot \tan(x)
\]

similarly, obtain the following table (to memorize)

<table>
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<tr>
<th>Function</th>
<th>Derivative</th>
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</tr>
</thead>
<tbody>
<tr>
<td>(\sin(x))</td>
<td>(\cos(x))</td>
<td>(\cos(x))</td>
<td>(-\sin(x))</td>
</tr>
<tr>
<td>(\tan(x))</td>
<td>(\sec^2(x))</td>
<td>(\cot(x))</td>
<td>(-\csc^2(x))</td>
</tr>
<tr>
<td>(\sec(x))</td>
<td>(\sec(x) \cdot \tan(x))</td>
<td>(\csc(x))</td>
<td>(-\csc(x) \cdot \cot(x))</td>
</tr>
</tbody>
</table>

to obtain the right hand column from the left hand column, replace functions by co-functions and multiply by \(-1\).

\[
\frac{d}{dx} x \sec(x)
\]

\[
\frac{d}{d\theta} \tan^{-1}(\theta)
\]

\[
\frac{-1}{\theta^2 + 1}
\]
Example. Find an equation of the line tangent to the curve

\[ y = x \tan(x) \]

at the point \( \left( \frac{\pi}{4}, \frac{\pi}{4} \right) \).

Solution.

Point-slope form of tangent line

\[ y - y_0 = m(x - x_0) \]

Need slope

\[ \frac{dy}{dx} = \tan(x) + x \sec^2(x) \]

\[ \frac{dy}{dx} \bigg|_{\frac{\pi}{4}} = \tan \left( \frac{\pi}{4} \right) + \frac{\pi}{4} \sec^2 \left( \frac{\pi}{4} \right) \]

where \( \sec \left( \frac{\pi}{2} \right) = \frac{1}{\cos \left( \frac{\pi}{4} \right)} = \sqrt{2} \)

\[ \frac{dy}{dx} \bigg|_{\frac{\pi}{4}} = 1 + \frac{\pi}{4} \cdot 2 = 1 + \frac{\pi}{2} \]

equation for tangent line

\[ y - \frac{\pi}{4} = \left( 1 + \frac{\pi}{2} \right) \left( x - \frac{\pi}{4} \right) \]

§2.5 The Chain Rule

Composition

Example. Consider \( F(x) = \sqrt{x^2 + 1} \)

Let \( f(u) = \sqrt{u} \), \( g(x) = x^2 + 1 \)

\( F \) is the composition of \( f \) and \( g \)

Function machines:

\[ \langle x \rightarrow [g] \rightarrow x^2 + 1 = u \rightarrow [f] \rightarrow \sqrt{u} = \sqrt{x^2 + 1} \rangle \]

Notation for composition

\[ F(x) = f\left(g(x)\right) \]

\( f \) is the outer function,

\( g \) is the inner function
Derivatives of Compositions

Let \( y = F(x) = f(g(x)) \)
or \( y = f(u), \ u = g(x). \)

The Chain Rule

If \( g \) is differentiable at \( x \) and \( f \) is differentiable at \( u \),
then
\[
F'(x) = f'(u)g'(x)
\]
The derivative of a composition of two functions is
the product of their derivatives

Write another way
\[
F'(x) = f'(g(x))g'(x)
\]

Leibniz notation

Let \( y = f(u) \) and \( u = g(x) \) and
\[
y = F(x) = f(g(x))
\]

then
\[
F'(x) = f'(u)g'(x)
\]

\[
\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}
\]

Easy to remember: looks as though a term “du”
cancels on the right hand side!

Example. Let \( F(x) = \sqrt{x^2 + 1} \). Find \( F'(x) \).

Write as a composition
\[
u = g(x) = x^2 + 1
\]
\[
y = f(u) = \sqrt{u} \quad \text{where} \quad \sqrt{u} = u^{\frac{1}{2}}
\]
apply the chain rule
\[
\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}
\]
\[
= \frac{1}{2} u^{-\frac{1}{2}} \cdot 2x
\]
\[
= (x^2 + 1)^{-\frac{1}{2}} \cdot x = \frac{x}{\sqrt{x^2 + 1}} \]

\( \blacksquare \)
Example. Let $y = \cos(3x)$. Find $\frac{dy}{dx}$.

Write as a composition

$y = \cos(u)$ where $u = 3x$

apply the chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$

$$= -\sin(u) \cdot 3$$

$$= -3\sin(3x)$$

Let $y = \sqrt{\sin(x)}$. Find $\frac{dy}{dx}$.

?? Class Practice

Let $y = (x^2 + 1)^{10}$. Find $\frac{dy}{dx}$.
Plausibility Argument for the Chain Rule

Recall the limit definition of derivative
\[
\frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}
\]

Change notation

Let \(\Delta x = h\)

\(\Delta y = f(x + h) - f(x)\)

then

\[
\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}
\]

Statement of Chain Rule

Let \(y = F(x) = f(g(x))\) or \(y = f(u)\) with \(u = g(x)\)

If \(g\) is differentiable at \(x\), and \(f\) is differentiable at \(u\), then

\[F'(x) = f'(u)g'(x)\]

or \[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}
\]

Plausibility Argument

Let \(\Delta u\) be the change in \(u\) corresponding to \(\Delta x\).

\[
\Delta u = g(x + \Delta x) - g(x)
\]

The corresponding change in \(y\) is

\[
\Delta y = f(u + \Delta u) - f(u)
\]

Then

\[
\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}
\]

\[
= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}
\]

as long as \(\Delta u \neq 0\)

\[
= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}
\]

product rule for limits

but \(\Delta u \to 0\) as \(\Delta x \to 0\), so

\[
\frac{dy}{dx} = \lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}
\]

\[
= \frac{dy}{du} \cdot \frac{du}{dx}
\]

The only problem is that we may have \(\Delta u = 0\). ■
Generalized Power Rule

Combine the power rule and the chain rule

Consider

\[ F(x) = g(x)^n \]

write as a composition

\[ u = g(x) \]

\[ y = f(u) = u^n \]

apply the chain rule

\[ \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} \]

\[ = n \ u^{n-1} \frac{du}{dx} \]

this is frequently written

\[ \frac{d}{dx} g(x)^n = n \ g(x)^{n-1} \ g'(x) \]

Example. Consider

\[ F(x) = (x^3 + 4x)^{10} \]

this has the form given above, with

\[ g(x) = x^3 + 4x \]

\[ n = 10 \]

then

\[ F'(x) = 10 (x^3 + 10x)^9 \frac{d}{dx} (x^3 + 4x) \]

\[ = 10(x^3 + 10x)^9 (3x^2 + 4) \]

Example.

\[ F(x) = \frac{1}{\sqrt{\sin(x)}} = (\sin(x))^{-\frac{1}{2}} \]

\[ F'(x) = -\frac{1}{2} (\sin(x))^{-\frac{3}{2}} \cos(x) \]

WARNING It is a common mistake to forget the factor \( g'(x) \)!
**Compositions of Three Functions**

Consider

\[ F(t) = f \left( g(h(t)) \right) \]

write as a composition

\[ y = f(u), \quad u = g(x), \quad x = h(t) \]

chain rule

\[
\frac{dy}{dt} = \frac{dy}{du} \frac{du}{dx} \frac{dx}{dt}
\]

it looks as though ‘\(du\)’ and ‘\(dx\)’ factors cancel

Example

\[ F(t) = \sin^2(4t) \]

write as a composition

\[ y = u^2, \quad u = \sin(x), \quad x = 4t \]

chain rule

\[
\frac{dy}{dt} = \frac{dy}{du} \frac{du}{dx} \frac{dx}{dt}
\]

\[
= 2u \cdot \cos(x) \cdot 4
\]

\[
= 8u \cos(x)
\]

\[
= 8 \sin(x) \cos(x)
\]

\[
= 8 \sin(4t) \cos(4t)
\]

◼
Example

Find the line tangent to the graph of \( y = \sin^2(4t) \) at 
\( t = \frac{\pi}{16} \).

Solution. Point slope form of a tangent line

\[ y - y_0 = m(t - t_0) \]

\( t_0 = \pi/16, \quad y_0 = \sin^2 \left( \frac{\pi}{4} \right) = \left( \frac{1}{\sqrt{2}} \right)^2 = \frac{1}{2} \)

slope? \( m = \frac{dy}{dx} \bigg|_{t=\frac{\pi}{16}} \)

\[ = 8 \sin(4t) \cos(4t) \bigg|_{t=\frac{\pi}{16}} \]

\[ = 8 \sin \left( \frac{\pi}{4} \right) \cos \left( \frac{\pi}{4} \right) \]

\[ = 8 \left( \frac{1}{\sqrt{2}} \right) \left( \frac{1}{\sqrt{2}} \right) \]

\[ = 4 \]

gives the tangent line

\[ y - \frac{1}{2} = 4 \left( t - \frac{\pi}{16} \right) \]

?? Class practice differentiating!

Find \( \frac{dy}{dx} \)

1. \( y = (x^4 - 3x^2 + 6)^3 \)

2. \( y = 2x\sqrt{x^2 + 1} \)
3. $y = \left(x + \frac{1}{x^2}\right)^{\sqrt{7}}$

4. $y = \frac{t}{1-t^2}$

5. $y = \sin(\sec(x))$

6. $y = \cos(\tan(\sqrt{1 + t^2}))$
§2.6 Implicit Differentiation

circle of radius 1
\langle ... 0 ... x-, ... 0 ... y-, circle \rangle

\[ x^2 + y^2 = 1 \]
defines \( y \) implicitly as a function of \( x \)
solve for \( y \): \( y^2 = 1 - x^2 \)
two possibilities
\[ y = (1 - x^2)^{\frac{1}{2}} = \sqrt{1 - x^2} \quad [1a] \]
\[ y = -(1 - x^2)^{\frac{1}{2}} = -\sqrt{1 - x^2} \quad [1b] \]

derivatives
\[ \frac{dy}{dx} = \frac{1}{2} (1 - x^2)^{-\frac{1}{2}} (-2x) = \frac{-x}{\sqrt{1-x^2}} \quad [2a] \]
\[ \frac{dy}{dx} = -\frac{1}{2} (1 - x^2)^{-\frac{1}{2}} (-2x) = \frac{x}{\sqrt{1-x^2}} \quad [2b] \]

Using Implicit Differentiation is easier in many cases!
\[ x^2 + y^2 = 1 \quad [3] \]
regard \( y \) as a function of \( x \)
\[ y = f(x) \quad \text{or} \quad y^2 = f(x)^2 \]
recall the generalized power rule
\[ \frac{d}{dx} f(x)^2 = 2 f(x)f'(x) \]
write this as
\[ \frac{d}{dx} y^2 = 2 y y' \]
must remember \( y' = \frac{dy}{dx} \)!
differentiate equation [3] implicitly with respect to $x$

$$2x + 2y \frac{dy}{dx} = 0$$

solve for $dy/dx$

$$y \frac{dy}{dx} = -x$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

from [1], two possibilities

$$\frac{dy}{dx} = \frac{-x}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx} = \frac{x}{\sqrt{1-x^2}}$$

these match [2]!

Example. Find $\frac{dy}{dx}$ if

$$x^{-1} + y^{-1} = 1.$$ \[1\]

Solution. Regard $y$ as a function of $x$

$$y = f(x)$$

Use the generalized power rule

$$\frac{d}{dx} f(x)^{-1} = -f(x)^{-2} f'(x)$$

$$\frac{d}{dx} y^{-1} = -y^{-2} \frac{dy}{dx}$$

differentiate [1] implicitly with respect to $x$

$$-x^{-2} - y^{-2} \frac{dy}{dx} = 0$$

solve for $\frac{dy}{dx}$

$$-y^{-2} \frac{dy}{dx} = \frac{1}{x^2}$$

$$\frac{dy}{dx} = -\frac{y^2}{x^2} \quad \blacksquare$$
Example. Find the equation of the line tangent to the curve 
\[2 \cos(x) \sin(y) = 1\] 
at \((x_0, y_0) = \left(\frac{\pi}{4}, \frac{\pi}{4}\right)\).

Solution. Recall \(\sin \left(\frac{\pi}{4}\right) = \cos \left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}\).

Think of \(y\) as a function of \(x\)

\[\frac{d}{dx} \sin(y) = \cos(y) \frac{dy}{dx}\]

differentiate [1] implicitly

\[-2 \sin(x) \sin(y) + 2 \cos(x) \cos(y) \frac{dy}{dx} = 0\]

solve for \(\frac{dy}{dx}\)

\[2 \cos(x) \cos(y) \frac{dy}{dx} = 2 \sin(x) \sin(y)\]

\[\frac{dy}{dx} = \tan(x) \tan(y)\]

Slope of curve at \(x = y = \pi/4\)

\[\frac{dy}{dx} \bigg|_{x=y=\frac{\pi}{4}} = \tan \left(\frac{\pi}{4}\right) \tan \left(\frac{\pi}{4}\right) = 1 \cdot 1 = 1\]
?? Class practice with implicit differentiation

1. Let \( x^3 + x^2 y + 4 y^2 = c \), where \( c \) is a constant.
   Find \( \frac{dy}{dx} \).

   Solution: \( \frac{dy}{dx} = \frac{-3x^2 + 2xy}{x^2 + 8y} \)

2. Let \( \sin(x) + \cos(y) = \sin(x) \cos(y) \). Find \( \frac{dy}{dx} \).

   Solution: \( \frac{dy}{dx} = \frac{\cos(x)}{\sin(x) - 1} \cdot \frac{\cos(y) - 1}{\sin(y)} \)
Find an equation of the line tangent to the following curve at the given point.

$$2(x^2 + y^2)^2 = 25(x^2 - y^2)$$ at the point (3,1)

**Solution.** Differentiate implicitly with respect to $x$

$$4(x^2 + y^2)(2x + 2yy') = 25(2x - 2yy')$$

Solve for $y'$

$$4(x^2 + y^2) 2yy' + 50yy' = -8(x^2 + y^2)x + 50x$$

$$yy'(8y(x^2 + y^2) + 50y) = -8x^3 - 8xy^2 + 50x$$

$$y' = \frac{-8x^3 - 8xy^2 + 50x}{8y(x^2 + y^2) + 50y}$$

$$= \frac{2x(-4x^2 - 4y^2 + 25)}{2y(4x^2 + 4y^2 + 25)}$$

evaluate at $x = 3$ and $y = 1$

$$y' = \frac{6(-36 - 4 + 25)}{2(36 + 4 + 25)} = \frac{6(-15)}{2(65)} = \frac{-9}{13}$$

point slope form of tangent line

$$y - y_0 = m(x - x_0)$$

$$y - 1 = \frac{-9}{13}(x - 3)$$

in slope intercept form

$$y = \frac{-9}{13}x + \frac{27}{65} + 1$$

$$= \frac{-9}{13}x + \frac{40}{13}$$

(show Maple image of curve and tangent line) ■
§2.7 Related Rates

Example. A descending balloonist lets helium escape at a rate of 10 cubic feet / minute. Assume the balloon is spherical. How fast is the radius decreasing when the radius is 10 feet?

Given

\[ V = \text{volume of balloon} \]
\[ \frac{dV}{dt} = -10 \quad \text{(feet)}^3 \text{minute} \]
\[ r = \text{radius of balloon} = 10 \text{ feet} \]

Find \( \frac{dr}{dt} \) when \( r = 10 \)

Equation relating these quantities

\[ V = \frac{4}{3} \pi r^3 \quad [1] \]

differentiate with respect to time using the chain rule

generalized power rule: \( \frac{d}{dt} g(t)^n = n g(t)^{n-1} \)

consider radius \( r \) a function of time \( t \)

\[ \frac{d}{dt} r^3 = 3r^2 \frac{dr}{dt} \]

Differentiate [1] with respect to time to get

\[ \frac{dV}{dt} = \frac{4}{3} \pi \left( 3 r^2 \frac{dr}{dt} \right) = 4\pi r^2 \frac{dr}{dt} \]

Solve for \( \frac{dr}{dt} \)

\[ \frac{dr}{dt} = \frac{1}{4\pi r^2} \frac{dV}{dt} \]

\[ \frac{dr}{dt} \bigg|_{r=10} = \frac{1}{4 \pi} \frac{1}{100} (-10) \frac{(\text{feet})^3}{\text{min}} \]

\[ = \frac{-1}{40 \pi \text{ min}} \approx -0.008 \frac{\text{feet}}{\text{min}} \approx -0.1 \frac{\text{inches}}{\text{min}} \]
Problem Solving Strategy

1. Read problem carefully
2. Draw picture if possible
3. Assign notation to given information and unknown rate
4. Write down an equation relating the given information and the function whose rate is unknown
5. Differentiate with respect to time
6. Solve for the unknown rate

Example. The length of a rectangle is increasing at a rate of 7 cm/s and its width is increasing at a rate of 7 cm/s. When the length is 6 cm and the width is 4 cm, how fast is the area of the rectangle increasing?

2. \(<\text{picture}>\)

3. Given information
\[L(t) = 6\text{cm}, \quad W(t) = 4\text{cm}\]
\[L'(t) = 7\text{ cm/s}, \quad W'(t) = 7\text{ cm/s}\]

Unknown rate is \(A'(t)\), where \(A = LW\).

4. \(A(t) = L(t)W(t)\)
5. \(A'(t) = L'(t)W(t) + L(t)W'(t)\)
6. \[= 7\frac{\text{cm}}{s} \cdot 4\text{ cm} + 6\text{ cm} \cdot 7\frac{\text{cm}}{s}\]
\[= 28\frac{\text{cm}^2}{s} + 42\frac{\text{cm}^2}{s} = 70\frac{\text{cm}^2}{s} \quad \blacksquare\]
Example. One end of a 13 foot ladder is on the ground and the other end rests on a vertical wall. If the bottom end is drawn from the wall at 3 feet/second, how fast is the top of the ladder sliding down the wall when the bottom is 5 feet from the wall?

\[ ... x \ldots x', ... y \ldots y', \text{ ladder betw. } x \& y, \ l = 13 \text{ ft} \]

Pythagorean theorem
\[ x^2 + y^2 = l^2 \]

differentiate with respect to time
\[ 2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \]
solve for \( y \)
\[ \frac{dy}{dt} = -\frac{x}{y} \frac{dx}{dt} \]
to evaluate \( dy/dt \) we need \( y \)
\[ y^2 = l^2 - x^2 = 169 - 25 = 144 \]
\[ y = \sqrt{144} = 12 \]

then
\[ \left. \frac{dy}{dt} \right|_{x=5 \text{ ft}} = -5 \cdot 3 \text{ feet/sec} = -\frac{5 \text{ feet}}{4 \text{ sec}} \]

given information
\[ l = 13 \text{ ft}, \quad x = 5 \text{ ft}, \quad \frac{dx}{dt} = 3 \text{ ft./sec.} \]

unknown rate
\[ \frac{dy}{dt} = ? \]
Example. Pat walks 5 feet/second towards a street light whose lamp is 20 feet above the ground. If Pat is 6 feet tall, find how rapidly Pat’s shadow changes in length.

Given information

\[ y = 20 \text{ feet}, \quad h = 6 \text{ feet}, \quad \frac{dx}{dt} = -5 \text{ feet/sec} \]

Unknown rate

\[ \frac{ds}{dt} = ? \]

by similar triangles

\[
\frac{y}{x+s} = \frac{h}{s} \\
y s = h (x + s) \\
(y - h)s = hx \\
s = \frac{h}{y-h} x
\]

differentiate wrt time

\[
\frac{ds}{dt} = \frac{h}{y-h} \frac{dx}{dt} \\
= \frac{6}{20-6} \left(-5 \text{ feet/sec}\right) \\
= -\frac{15 \text{ feet}}{7 \text{ sec}} \]

\[ \Box \]
Example. The beacon of a lighthouse 1 mile from the shore makes 5 rotations per minute. Assuming that the shoreline is straight, calculate the speed at which the spotlight sweeps along the shoreline as it lights up sand 2 miles from the lighthouse.

\[0 \ldots x \ldots x \ldots , \ldots 0 \ldots y \ldots y \ldots ,\] lighthouse at \((0, y)\), beam to \((x, 0)\), length \(L\), angle \(\theta\)

from the definition of tangent

\[
\tan(\theta) = \frac{x}{y}
\]

differentiate wrt time

\[
\sec^2(\theta) \frac{d\theta}{dt} = \frac{1}{y} \frac{dx}{dt}
\]

using that \(y\) is a constant

\[
\frac{dx}{dt} = y \sec^2(\theta) \frac{d\theta}{dt}
\]

we need \(\sec^2(\theta)\)

\[
\cos(\theta) = \frac{y}{L} \quad \sec(\theta) = \frac{L}{y} = 2
\]

then

\[
\frac{dx}{dt} = \frac{(1 \text{ mile}) \cdot 4 \cdot 10\pi \text{ radians}}{\text{minute}}
\]

\[
= 40\pi \frac{\text{miles}}{\text{minute}}
\]

\[
= 40\pi \frac{\text{miles}}{\text{minute}} \cdot 60 \frac{\text{minutes}}{\text{hour}}
\]

\[
= (40)(60)\pi \frac{\text{miles}}{\text{hour}}
\]

\[
\approx 7539 \frac{\text{miles}}{\text{hour}}
\]

\[\text{STOP}\]
§2.8 Linear Approximations and Differentials

Linear or Tangent Line Approximation

\[ y = y_0 = m(x - x_0) \]

Point-slope form of the tangent line

\[ (x_0, y_0) = (a, f(a)) \]

Point

\[ m = f'(a) \]

Slope

Then

\[ y - f(a) = f'(a)(x - a) \]

\[ y = L(x) = f(a) + f'(a)(x - a) \]

Example. Find the linear approximation to

\[ f(x) = \sqrt{1 + x} \text{ near } x = 0. \]

\[ f(x) = (1 + x)^{1/2} \]

\[ f'(x) = \frac{1}{2}(1 + x)^{-1/2} \]

\[ f'(0) = \frac{1}{2} \]

\[ L(x) = f(0) + f'(0)(x - 0) = 1 + \frac{1}{2}x \]

this straight line approximates \(\sqrt{1 + x}\) near \(x = 0.\)

\[ \langle -1 \ldots 0 \ldots 1 \ldots x, 0 \ldots 1 \ldots y, f(x), L(x) \rangle \]
Example (continued). Use linear approximation to estimate \( \sqrt{1.1} \).

\[
\sqrt{1.1} = \sqrt{1 + 0.1} = f(0.1)
\]

\[
L(0.1) = 1 + \frac{1}{2}(0.1) = 1.05
\]

Exact value \( \sqrt{1.1} = 10.488 \ldots \)

Example (continued). For what values of \( x \) is the linear approximation

\[
\sqrt{1+x} \approx 1 + \frac{1}{2}x
\]

accurate to within 0.05? This means

\[
|L(x) - f(x)| < 0.05
\]

or

\[
L(x) - 0.05 < f(x) < L(x) + 0.05
\]

(show Maple output)

From the plot, the linear approximation is accurate to within 0.05 for

\(-0.54 < x < 0.74\)

Realistic Example. Estimate the amount of paint required to paint a spherical tank of radius 20 feet with a coat 0.01 inches thick.

Exact calculation: volume of sphere

\[
V(r) = \frac{4}{3} \pi r^3
\]

volume between two concentric spheres of radii \( r \) and \( a \)

\[
\Delta V = V(r) - V(a) = \frac{4}{3} \pi r^3 - \frac{4}{3} \pi a^3
\]

\( a = 20 \text{ feet} = 240 \text{ inches} \)

\( r = 240.01 \text{ inches} \)

\( \Delta V \approx 7238.53 \text{ cubic inches} \approx 31 \text{ gallons} \)

1 gallon = 231 cubic inches
Approximate calculation:

Linear approximation, usual notation

\[ f(x) \approx f(a) + f'(a)(x - a) \]

Use \( V(r) \) instead of \( f(x) \)

\[ V(r) \approx V(a) + V'(a)(x - a) \]

\[ \Delta V = V(r) - V(a) \approx V'(a)(r - a) \]

\[ V'(r) = \frac{d}{dr} \frac{4}{3} \pi r^3 = \frac{4}{3} \pi (3r^2) = 4 \pi r^2 \]

and

\[ V'(a) = 4\pi a^2 \]

then

\[ \Delta V = 4 \pi a^2 (r - a) \]

\[ = \text{surface area of sphere } \times \text{ thickness of paint} \]

\[ = 4 \pi (240 \text{ inches})^2 (0.01 \text{ inches}) \]

\[ \approx 7237.23 \text{ cubic inches} \]

the error is only 0.3 cubic inches!

STOP Differentials

\( \langle 0 \ldots x-, 0 \ldots y-, f, P, L, x, x + \Delta x \rangle \)

tangent line approximation

\[ L(x + \Delta x) = f(x) + f'(x)\Delta x \]

write

\[ dx = \Delta x \]

\[ dy = L(x + \Delta x) - f(x) \]

\[ = f'(x)dx \]

\( \langle \text{add } dx = \Delta x, \Delta y, \ dy = f'(x)dx \rangle \)

\[ \Delta y = f(x + \Delta x) - f(x) \]

is approximated by

\[
\begin{array}{c}
\frac{dy}{dx} = f'(x) \\
\end{array}
\]

Note that \( \frac{\text{rise}}{\text{run}} = \frac{dy}{dx} = f'(x) \). This is clever notation.
Example. Estimate the amount of paint required to
paint a spherical tank of radius 20 feet with a coat of
paint 0.01 inches thick.

Volume of sphere

\[ V(r) = \frac{4}{3} \pi r^3 \]

approximate amount of paint

\[ dV = V'(r) dr = 4 \pi r^2 dr \]

if \( dr = 0.01 \) inches, then

\[ dV = 4 \pi (240 \text{ inches})^2 (0.01 \text{ inches}) \]

the same result as above. This is a shorthand for
linear approximation. ■

Newton’s Method – An Application of Linear
Approximation

\( \langle 0 \ldots r \ldots x-, 0 \ldots y-, f \rangle \)

A root of \( f \) is a number \( r \) such that \( f(r) = 0 \).

How to find roots geometrically?

guess \( x_1 \langle \text{add} \rangle \)

let \( L_1 \) be the line tangent to \( f \) at \( x_1 \langle \text{add} \rangle \)

follow \( L_1 \) to the \( x \) axis, get an intersection at \( x_2 \langle \text{add} \rangle \)

repeat this process

if the initial guess was good, the sequence \( x_1, x_2, \ldots \)
rapidly converges to \( r \)

How to find roots algebraically?

Make an initial guess \( x_1 \).

Find the line tangent to \( f \) at \( x = x_1 \).

\[ L_1(x) = f(x_1) + f'(x_1)(x - x_1) \]

by the linear or
tangent line approximation!

\( L_1 \) intersects the \( x \) axis at \( x_2 \). Find \( x_2 \).

\[ L_1(x_2) = 0 = f(x_1) + f'(x_1)(x_2 - x_1) \]

\[ -f(x_1) = f'(x_1)(x_2 - x_1) \]
\[- \frac{f(x_1)}{f'(x_1)} = x_2 - x_1\]

\[x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}\]

Find the line tangent to \(f\) at \(x = x_2\).

\[L_2(x) = f(x_2) + f'(x_2)(x - x_2)\]

\(L_2\) intersects the \(x\) axis at \(x_3\). Find \(x_3\).

\[L_2(x_3) = 0 = f(x_2) + f'(x_2)(x_3 - x_2)\]

\[-f(x_2) = f'(x_2)(x_3 - x_2)\]

\[-f(x_2) \over f'(x_2) = x_3 - x_2\]

\[x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}\]

repeat this procedure to obtain a general formula for \(n = 1, 2, \ldots\)

\[x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\]

If the initial guess \(x_1\) was good, the iterates \(x_2, x_3, \ldots\) will quickly approach \(r\).

Example. \(f(x) = x^2 - 2\)

\[\langle -2 \ldots -1 \ldots 0 \ldots 1 \ldots 2 \ldots x-, -2 \ldots 0 \ldots 2 \ldots y-\rangle\]

\(f', r = \sqrt{2}\)

\[f'(x) = 2x\]

from the boxed formula above

\[x_{n+1} = x_n - \frac{x_n^2 - 2}{2 x_n}\]

guess \(x_1 = 2\)

\[x_2 = 2 - \frac{4 - 2}{4} = 2 - \frac{1}{2} = 3 \over 2\]

\[x_3 = \frac{3}{2} - \frac{9 - 2}{3} = \frac{3}{2} - \frac{1}{3} = \frac{9}{6} - \frac{2}{6} = \frac{7}{6} = 1.4167\ldots\]

Exact value of positive root
\[ r = \sqrt{2} = 1.4142 ... \]

Transparency – How Newton’s Method can go wrong!