The IVP \( y' = f(t, y) \), \( y(t_0) = y_0 \) can be integrated and rewritten as follows:

\[
\int_{t_0}^{t} y'(\tau) \, d\tau = \int_{t_0}^{t} f(\tau, y(\tau)) \, d\tau \Rightarrow y(t) = y_0 + \int_{t_0}^{t} f(\tau, y(\tau)) \, d\tau \quad (1)
\]

A function \( y = \phi(t) \) will solve the IVP iff it solves (1). The proofs of the Existence Theorem that I've seen all rely on constructing a solution as the limit of the following sequence:

\[
\begin{align*}
\phi_0(t) &= y_0 \\
\phi_1(t) &= y_0 + \int_{t_0}^{t} f(\tau, \phi_0(\tau)) \, d\tau \\
\phi_2(t) &= y_0 + \int_{t_0}^{t} f(\tau, \phi_1(\tau)) \, d\tau \\
&\vdots \\
\phi_n(t) &= y_0 + \int_{t_0}^{t} f(\tau, \phi_{n-1}(\tau)) \, d\tau \\
&\vdots
\end{align*}
\]

It can be shown that the function \( y(t) = \lim_{n \to \infty} \phi_n(t) \) solves the original IVP on some interval \( t_0 - \delta \leq t \leq t + \delta \) (under the assumptions of the Existence Theorem).

Here's a proof of Thm 7.15:

Under the assumptions of Thm 7.15, we have

\[
|\phi(t) - y(t)| = \left| \int_{t_0}^{t} [f(\tau, x(\tau)) - f(\tau, y(\tau))] \, d\tau \right|
\]

\[
\leq |x(t) - y(t)| + \int_{t_0}^{t} |f(\tau, x(\tau)) - f(\tau, y(\tau))| \, d\tau \quad \text{(here, assumed } t > t_0) \]

\[
\leq |x_0 - y_0| + \int_{t_0}^{t} M |x(\tau) - y(\tau)| \, d\tau.
\]

Call this \( U(\delta) \).

Then \( U(\delta) = M |x(t) - y(t)| \leq MU(t) \).
\[ \Rightarrow U'(t) - M U(t) = 0 \Rightarrow e^{-M(t-t_0)} U(t) - e^{-M(t-t_0)} U(t_0) = 0 \]

\[ \Rightarrow \int_0^t (e^{-M(t-t_0)} U(t')) \, dt' \leq \int_0^t 0 \, dt' \Rightarrow e^{-M(t-t_0)} U(t) - U(t_0) \leq 0 \]

\[ \Rightarrow U(t) \leq U(t_0) e^{M(t-t_0)} = |x_0 - y_0| e^{M(t-t_0)} \]

So \[ |x(t) - y(t)| \leq U(t) \leq |x_0 - y_0| e^{M(t-t_0)} \], assuming \( t \geq t_0 \).

If \( t < t_0 \), get \[ |x(t) - y(t)| \leq |x_0 - y_0| e^{M(t_0-t)} \].

Overall, \[ |x(t) - y(t)| \leq |x_0 - y_0| e^{M|t-t_0|} \].