WSU Math 315 Test I (Fall, 2015)

Name: ____________________________ ID: ____________________________ Section No.: ______

Show your work in detail. There are 8 problems.

[10] 1. Find the general solution for the following differential equation:

\[ \frac{dy}{dt} = -2y - 5t. \]

First order linear, sos

\[ y' + 2y = -5t , \quad m = e^{2t}, \quad y_0 = (e^{2t})' = -5te^{2t}. \]

Integrate, get

\[ e^{2t}y = -5\left(\frac{1}{2}te^{2t} - \frac{1}{4}e^{2t}\right) + C \rightarrow y = -\frac{5}{2}t + \frac{5}{4} + Ce^{-2t} \]

[10] 2. Solve the following first-order differential equation with the initial value:

\[ \frac{dy}{dt} = (ty)^2 + 4t^2, \quad y(0) = 0. \]

\[ \frac{dy}{dt} = t^2y^2 + 4t^2 = t^2(y^2 + 1) \rightarrow \frac{1}{y^2+1} \, dy = t^2 \, dt \]

\[ \Rightarrow \frac{1}{2} \arctan(y) = \frac{1}{3} t^3 + C \Rightarrow \arctan\left(\frac{y}{2}\right) = \frac{2}{3} t^3 + D \]

\[ \Rightarrow y = \tan\left(\frac{2}{3} t^3 + D\right). \quad y(0) = 0 \Rightarrow D = 0. \]

(Looking back to the earlier line)

\[ \arctan\left(\frac{y}{2}\right) = \ldots, \quad D \text{ can only be } 0 \]

So

\[ y = \tan\left(\frac{2}{3} t^3\right) \]
For the differential equation \( y' = y(4 - y^2) \),

(a) Find all equilibrium solutions. \( y = 0, \ y = \pm 2 \).

(b) Determine whether or not each equilibrium solution is asymptotically stable or unstable as \( t \) goes to \( \infty \).

\[
\begin{aligned}
&y = 2 \quad \text{asympt. stable} \\
y = 0 \quad \text{unstable} \\
y = -2 \quad \text{asympt. stable}
\end{aligned}
\]

For the differential equation
\[
\frac{M}{\frac{dx}{dx} - \frac{dy}{dy}} + \frac{N}{\frac{dx}{dx} - \frac{dy}{dy}} dy = 0
\]

(a) Verify that the equation is exact. \( M_y = 2xy, \ N_x = 2xy, \) so exact.

(b) Find the general solution.

\[
F(x,y) = \frac{\frac{1}{2}x^2y^2}{x^2y - \frac{1}{4}x^4 + 5x + \cos y}
\]

So \( \frac{1}{2}x^2y^2 - \frac{1}{4}x^4 + 5x + \cos y = C \)
5. For the differential equation:
\[ y'' + 4y' + 3y = 0. \]

Find a fundamental set of solutions.

Find the Wronskina determinant for the fundamental set of solutions.

\[ r^2 + 4r + 3 = 0 \implies r = -3, -1, \text{ so } \{e^{-3t}, e^{-t}\}. \]

\[ W(e^{-3t}, e^{-t}) = \begin{vmatrix} e^{-3t} & e^{-t} \\ -3e^{-3t} & -e^{-t} \end{vmatrix} = 2e^{-4t}. \]

6. This problem has two parts.

Find the general solution for the following differential equation.

\[ y'' + y' + y = 0. \]

What is the limit of the general solution as \( t \to \infty \)?

\[ r^2 + r + 1 = 0 \implies r = -\frac{1 \pm \sqrt{-3}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i. \]

So

\[ y = Ce^{-\frac{1}{2}t} \cos\left(\frac{\sqrt{3}}{2}t\right) + De^{-\frac{1}{2}t} \sin\left(\frac{\sqrt{3}}{2}t\right). \]

\[ \lim_{t \to \infty} (y) = 0, \text{ due to the } e^{-\frac{1}{2}t} \text{ envelope.} \]
7. Find the solution for the following initial-value problem

\[ y'' - 6y' + 9y = 0, \ y(0) = -1, \ y'(0) = 1. \]

\[ r^2 - 6r + 9 = 0 \Rightarrow r = 3, 3. \quad \text{So} \quad y = Ce^{3t} + De^{3t}. \]

\[ y(0) = -1 \Rightarrow C = -1, \ y'(0) = 1 \Rightarrow 3C + D = 1 \Rightarrow D = 4. \]

So

\[ y = -e^{3t} + 4te^{3t}. \]

8. Suppose a tank contains 5 lb of salt dissolved in a 100 gallon water. Suppose a pipe containing 2 lb of salt per gallon is flowing into the tank at the rate 2 gallon per minute and the well-stirred mixture is flowing out from the tank at the same rate.

(a) Set up the differential equation and initial condition.

(b) Solve the differential equation with the initial value from (a).

\[ \frac{dy}{dt} = (2)(100) - (2) \left( \frac{y}{100} \right), \quad y(0) = 5. \]

\[ y' = 4 - 0.02y \]

\[ y' + 0.02y = 4 \quad \Rightarrow \quad \left( e^{0.02t} y \right)' = 4e^{0.02t} \]

\[ \Rightarrow \quad e^{0.02t} y = 200 e^{0.02t} + C \]

\[ \Rightarrow \quad y = 200 + Ce^{-0.02t} \]

\[ y(0) = 5 \Rightarrow C = -195. \]

So

\[ y = 200 - 195e^{-0.02t}. \]

Making sense, since \( y \) should \( \to 200 \) as \( t \to \infty \).